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On characteristic properties of semigroups

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Abstract. Let $\mathcal{K}$ be a class of semigroups and $\mathcal{P}$ be a set of general properties of semigroups. We call a subset Q of $\mathcal{P}$ characteristic for a semigroup $S \in \mathcal{K}$ if, up to isomorphism and anti-isomorphism, S is the only semigroup in $\mathcal{K}$, which satisfies all the properties from Q. The set of properties $\mathcal{P}$ is called char-complete for $\mathcal{K}$ if for any $S \in \mathcal{K}$ the set of all properties $P \in \mathcal{P}$, which hold for the semigroup S, is characteristic for S. We indicate a 7-element set of properties of semigroups which is a minimal char-complete set for the class of semigroups of order 3.

Introduction

All properties of semigroups are assumed to be invariant with respect to isomorphism and anti-isomorphism.

Let $\mathcal{K}$ be a class of semigroups and $\mathcal{P}$ be some set of general (qualitative and quantitative) properties of semigroups. For $S \in \mathcal{K}$, we denote by $\mathcal{P}(S)$ the set of all properties $P \in \mathcal{P}$ which hold for the semigroup S.

We say that a subset Q of $\mathcal{P}$ *characteristic for a semigroup* $S \in \mathcal{K}$ if, up to isomorphism and anti-isomorphism, S is the only semigroup in $\mathcal{K}$, which satisfies all the properties from Q; if $Q = \{q_1, \ldots, q_s\}$, then the properties $q_1, \ldots, q_s$ are called *characteristic for* S. Obviously, if $Q \subset Q' \subseteq \mathcal{P}$ and Q is characteristic for S then so is Q'. The set of properties $\mathcal{P}$ is called *char-complete for* $\mathcal{K}$ if for any $S \in \mathcal{K}$ the subset $\mathcal{P}(S)$ of $\mathcal{P}$ is characteristic

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for S. A char-complete set of properties is called *minimal* if it does not contain a proper char-complete subset of ones.

In this paper we indicate a 7 properties of semigroups which form a minimal char-complete set for the class of semigroups of order 3.

1. Formulation of the main result

We consider the following properties of a semigroup of order 3:

$P(C)$: commutativity;

$P(1)$: the existence of a unit element;

$P(0)$: the existence of a zero element;

$P^+(0)$: the existence of an added zero element;

$P_{id}(1)$: the number of idempotents is equal to 1;

$P_{id}(2)$: the number of idempotents is equal to 2;

$P_{gen}(2)$: the smallest number of generators is equal to 2.

The set of all these properties is denoted by $P_3(7)$.

Our aim is to prove the following theorem.

Theorem 1. *The set $P_3(7)$ is a minimal char-complete set of properties for the class of semigroups of order 3.*

2. Preliminaries

In this section we present results from the paper [1].

Let $S = \{\langle 1\rangle, \langle 2\rangle, \ldots, \langle n\rangle\}$ be a finite semigroup which is given by the Cayley table T. One wants to find some its minimal system of generators and the complete set of defining relations for these generators.

In the first step one chooses an element $\langle s\rangle$ of S that is (according to the table) the product of two elements $\langle i\rangle \neq \langle s\rangle$ and $\langle j\rangle \neq \langle s\rangle$; then in the Cayley table T (including the header row and the header column) one substitutes $\langle i\rangle\langle j\rangle$ instead of $\langle s\rangle$. The new table is denoted by T_1.

In the second step one chooses an element $\langle s^{(1)}\rangle$ of the set $S^{(1)} = S\setminus\{\langle s\rangle\}$ that is the product of two elements $i^{(1)}$ and $j^{(1)}$, where $i^{(1)} = \langle i_1\rangle$, $j^{(1)} = \langle j_1\rangle$, or $i^{(1)} = \langle i_1\rangle\langle i_2\rangle$, $j^{(1)} = \langle j_1\rangle$, or $i^{(1)} = \langle i_1\rangle$, $j^{(1)} = \langle j_1\rangle\langle j_2\rangle$, or $i^{(1)} = \langle i_1\rangle\langle i_2\rangle$, $j^{(1)} = \langle j_1\rangle\langle j_2\rangle$, with $\langle i_1\rangle, \langle i_2\rangle, \langle j_1\rangle, \langle j_2\rangle \neq \langle s^1\rangle$; then in the table T_1 (including the header row and header column) one substitutes $\langle i^1\rangle\langle j^1\rangle$ instead of $\langle s^1\rangle$. The new table is denoted by T_2.

In the next step one chooses an element $s^{(2)}$ of the set $S^{(2)} = S \setminus \{\langle s\rangle, \langle s^{(1)}\rangle\}$, and so on. Upon completion of this process, say after m steps ($m \geqslant 0$), one has a minimal system of generators $S^{(m)}$ of the semigroup S (consisting of those elements of the header column of the last table that are of the form $\langle k\rangle$) and an appropriate set of defining relations in the form of the last table (which must be taken fully).

Note that the specified process is ambiguous and so one can get different final system of generators.

In the paper [1] this algorithm is applied to all semigroups of order 3 which are considered up to isomorphism and anti-isomorphism (if S be a semigroup, then a semigroup S' with multiplication $\circ$ is called anti-isomorphic to S if $S' = S$ as sets and $x \circ y = yx$). In each from 18 cases the algorithm has less than 3 steps. Under the transition from one table to another, the equality between the arrows specifies a replacement in the first table.

1)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$

$\Rightarrow (\langle 0\rangle = \langle 2\rangle^2) \Rightarrow$

	$\boldsymbol{\langle 2\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$
$\langle 1\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$
$\langle 2\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$

2)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$

$\Rightarrow (\langle 1\rangle = \langle 2\rangle^2) \Rightarrow$

	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$

$\Rightarrow$

$\Rightarrow (\langle 0\rangle = \langle 2\rangle^2 \cdot \langle 2\rangle = \langle 2\rangle^3) \Rightarrow$

	$\boldsymbol{\langle 2\rangle^3}$	$\langle 2\rangle^2$	$\langle 2\rangle$
$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$
$\langle 2\rangle^2$	$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$
$\langle 2\rangle$	$\boldsymbol{\langle 2\rangle^3}$	$\boldsymbol{\langle 2\rangle^3}$	$\langle 2\rangle^2$

3)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 2\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 2\rangle$

4)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 2\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$

5)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

6)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle^2) \Rightarrow$

	$\boldsymbol{\langle 1\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle^2}$	$\boldsymbol{\langle 1\rangle^2}$	$\boldsymbol{\langle 1\rangle^2}$	$\boldsymbol{\langle 1\rangle^2}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle^2}$	$\boldsymbol{\langle 1\rangle^2}$	$\langle 1\rangle$
$\langle 2\rangle$	$\boldsymbol{\langle 1\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$

7)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 2\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 2\rangle$

8)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$	$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle \cdot \langle 2\rangle}$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

9)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1\rangle$

$\Rightarrow (\langle 1\rangle = \langle 2\rangle^2) \Rightarrow$

	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$
$\langle 2\rangle$	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$

10)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$

11)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

12)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 2\rangle$	$\langle 1\rangle$

$\Rightarrow (\langle 1\rangle = \langle 2\rangle^2) \Rightarrow$

	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\langle 0\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 2\rangle$	$\boldsymbol{\langle 2\rangle^2}$

13)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

14)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$

15)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 2\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 0\rangle$

$\Rightarrow (\langle 0\rangle = \langle 2\rangle^2) \Rightarrow$

	$\boldsymbol{\langle 2\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 1\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\boldsymbol{\langle 2\rangle^2}$

16)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 2\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 0\rangle$

$\Rightarrow (\langle 0\rangle = \langle 2\rangle^2) \Rightarrow$

	$\boldsymbol{\langle 2\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle$
$\langle 1\rangle$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\boldsymbol{\langle 2\rangle^2}$

17)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 1\rangle$	$\langle 0\rangle$	$\langle 0\rangle$

$\Rightarrow (\langle 1\rangle = \langle 0\rangle \cdot \langle 2\rangle) \Rightarrow$

$\Rightarrow$

	$\langle 0\rangle$	$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$
$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$	$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\boldsymbol{\langle 0\rangle \cdot \langle 2\rangle}$	$\langle 0\rangle$	$\langle 0\rangle$

$\Rightarrow$

$\Rightarrow (\langle 0\rangle = \langle 2\rangle^2) \Rightarrow$

	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle^3$	$\langle 2\rangle$
$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$	$\langle 2\rangle^3$	$\langle 2\rangle^3$
$\langle 2\rangle^3$	$\langle 2\rangle^3$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$
$\langle 2\rangle$	$\langle 2\rangle^3$	$\boldsymbol{\langle 2\rangle^2}$	$\boldsymbol{\langle 2\rangle^2}$

18)

	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 0\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$

$\Rightarrow (\langle 2\rangle = \langle 1\rangle^2) \Rightarrow$

	$\langle 0\rangle$	$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle^2}$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle^2}$
$\langle 1\rangle$	$\langle 1\rangle$	$\boldsymbol{\langle 1\rangle^2}$	$\langle 0\rangle$
$\boldsymbol{\langle 1\rangle^2}$	$\boldsymbol{\langle 1\rangle^2}$	$\langle 0\rangle$	$\langle 1\rangle$

$\Rightarrow$

$\Rightarrow (\langle 0\rangle = \langle 1\rangle \cdot \langle 1\rangle^2 = \langle 1\rangle^3) \Rightarrow$

	$\boldsymbol{\langle 1\rangle^3}$	$\langle 1\rangle$	$\langle 1\rangle^2$
$\boldsymbol{\langle 1\rangle^3}$	$\boldsymbol{\langle 1\rangle^3}$	$\langle 1\rangle$	$\langle 1\rangle^2$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle^2$	$\boldsymbol{\langle 1\rangle^3}$
$\langle 1\rangle^2$	$\langle 1\rangle^2$	$\boldsymbol{\langle 1\rangle^3}$	$\langle 1\rangle$

From the above, it easily follows the next statement (which was not formulated in [1]) .

Theorem 2. *Let S be a semigroup of order* 3*. Then any two its minimal systems of generators are of the same order, and coincide if S is not a group.*

Note that the group of order 3 is given by the case 18).

3. Proof of Theorem 1

The following table $\mathcal{T}$, which follows from the results of section 2, shows what the properties hold for the semigroups 1) – 18) ("+" means that the corresponding property holds, and its absence means that the corresponding property does not hold).

Since all rows of this table (without the header row and the header column) are mutually different, the set of properties $P_3(7)$ is char-complete.

To prove that the char-complete set $P_3(7)$ is minimal it suffices to verify that the table $\mathcal{T}$ without any fix column X (and, of course, without the header row and the header column) has two equal rows. It is easy to see that if X is equal to $C, P(1), P(0), P^+(0), P_{id}(1), P_{id}(2), P_{gen}(2)$, then, respectively, the following two rows are equal: 3 and 4, 13 and 14, 4 and 5, 3 and 9, 1 and 7, 5 and 8, 8 and 14.

Table 1. $\mathcal{T}$

	C	$P(1)$	$P(0)$	$P^+(0)$	$P_{id}(1)$	$P_{id}(2)$	$P_{gen}(2)$
1	+		+		+		+
2	+		+		+		
3	+		+			+	+
4			+			+	+
5						+	+
6	+	+	+			+	+
7	+		+				+
8							+
9	+		+	+		+	+
10	+	+	+	+			
11			+	+			
12	+	+	+	+		+	+
13		+					
14							
15	+				+		+
16	+	+				+	+
17	+				+		
18	+	+			+		

References

[1] V. M. Bondarenko, Y. V. Zaciha, *On defining relations for minimal generator systems of three-order semigroups*, Science Journal of National Pedagogical Dragomanov University, Series 1: Physics and Mathematics (2013), no. 14, 62-67 (in Ukrainian).

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