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RESEARCH ARTICLE

Lattices of partial sums

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Abstract. In this paper we introduce and study a class of partially ordered sets that can be interpreted as partial sums of indeterminate real numbers. An important example of these partially ordered sets, is the classical Young lattice $\mathbb{Y}$ of the integer partitions. In this context, the sum function associated to a specific assignment of real values to the indeterminate variables becomes a valuation on a distributive lattice.

Introduction

In the present paper we build a formal order context within which we examine the analogies that arise between combinatorial sum problems on multisets of real numbers and combinatorial set problems. Our construction provides a class of distributive lattices that we will call *partial sums lattices*. The elements of these lattices correspond to indeterminate partial sums of real numbers and any effective assignment of real values to such indeterminate becomes a valuation on these lattices (classical results concerning the valuations on a distributive lattice and their links with Euler characteristic and Mobius function can be found in [10–12, 20]). For a wide range of extremal combinatorial problems concerning partial sums on real numbers we refer to [2, 14–18]. In particular, a beautiful example of similarity between combinatorial sum problems and combinatorial set

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problems is a still unsolved conjecture raised in [16], which has a dual formulation (in terms of partial sums on multi-sets of real numbers) of the famous theorem of Erdös-Ko-Rado [9]. This conjecture and related problems have been studied in [2–8, 14, 17–19, 21, 23]. In [16] have been raised several combinatorial problems concerning partial sums on multi-sets of n real numbers, and it is interesting to observe as such problems have a strict analogy with corresponding problems concerning families of subsets of the set of indexes $\{1, \ldots, n\}$ (see [3, 4, 16, 23] for a more detailed description of some such analogies). If n is a positive integer we set $[n] = \{1, \ldots, n\}$. We denote by $\mathcal{P}([n])$ the power set of $[n]$ and we set $\mathcal{P}_k([n]) = \{A \in \mathcal{P}([n]) : |A| = k\}$, for each $k = 0, 1, \ldots, n$. In this context we call *indexes* the element of $[n]$. Let $x_n, \ldots, x_1$ be n indeterminate real numbers (not necessarily distinct) such that

$$x_n \geqslant x_{n-1} \geqslant \cdots \geqslant x_2 \geqslant x_1 \tag{1}$$

Our basic motivation is to investigate the analogies between order properties on the subsets of indexes $A \in \mathcal{P}([n])$ and on the partial sums $\sum_{i\in A} x_i$, where each x_i ($i \in A$) satisfies (1). For example, it is obviously false that if $A, B \in \mathcal{P}([n])$ and $A \subseteq B$ then $\sum_{i\in A} x_i \leqslant \sum_{i\in B} x_i$, since in (1) can also be negative numbers. We refine then the previous question as follows. Let us suppose that our indeterminate real numbers satisfy the condition

$$x_n \geqslant x_{n-1} \geqslant \cdots \geqslant x_2 \geqslant x_1 \geqslant 0 \tag{2}$$

Then, under the hypothesis (2), it is true that if $A, B \in \mathcal{P}([n])$ and $A \subseteq B$ then $\sum_{i\in A} x_i \leqslant \sum_{i\in B} x_i$. But the reverse implication is not true, i.e. if for each assignment of real values in (2) we have that $\sum_{i\in A} x_i \leqslant \sum_{i\in B} x_i$, then not necessarily $A \subseteq B$. For example, for each assignment that satisfies (2) we have $x_2 \geqslant x_1$, but $\{2\} \not\supseteq \{1\}$. We ask therefore the following question: does there exist a partial order $\sqsubseteq$ on $\mathcal{P}([n])$ such that $A \sqsubseteq B$ iff $\sum_{i\in A} x_i \leqslant \sum_{i\in B} x_i$ for each assignment of n real values that satisfies (2)? More generally, if

$$x_n \geqslant \cdots \geqslant x_{n-r+1} \geqslant 0 > x_{n-r} \geqslant \cdots \geqslant x_1 \geqslant 0 \tag{3}$$

does there exist a partial order $\sqsubseteq$ on $\mathcal{P}([n])$ such that $A \sqsubseteq B$ iff $\sum_{i\in A} x_i \leqslant \sum_{i\in B} x_i$ for each assignment of n real values that satisfies (3)?

In this paper we reformulate and study the previous question in a more general form, when i.e. the set $[n]$ of the indexes is substituted with an arbitrary total ordered set I and with an its fixed partition (I^+, I^-)

in two blocks: the elements of I^+ will be thought of as "positive" indexes and that of I^- of as "negative" indexes. In such an abstract context we obtain a class of posets, which we call *partial sums posets* on I. On such posets we define a particular class of order-preserving maps which are the natural generalizations of the partial sums $\sum_{i\in A} x_i$, when A is a subset of $[n]$ and we study some basic properties of such maps. The class of the partial sums posets includes, as particular cases, the classical Young lattice $\mathbb{Y}$ of all the the integer partitions ordered by means of the inclusion of the corresponding Young diagrams, and other sublattices of $\mathbb{Y}$. For example, the lattice $M(n)$ of all the integer partitions having all distinct parts and maximum part at most n, or, also, the lattice $L(m, n)$ of all the integer partitions with at most m parts and maximum part not exceeding n (both these lattices were introduced by Stanley in [22]).

1. A quasi-order for partial sums

Let I a totally ordered set with order $\preceq$. If $\lambda, \lambda' \in I$, we set $\lambda \prec \lambda'$ if $\lambda \preceq \lambda'$ and $\lambda \neq \lambda'$; moreover, we write in an equivalent way $\lambda' \succ \lambda$ instead of $\lambda \prec \lambda'$ and $\lambda' \succeq \lambda$ instead of $\lambda \preceq \lambda'$. We fix a set partition (I^+, I^-) of I with two disjoint subsets I^+ and I^- such that $\lambda \succ \gamma$ if $\lambda \in I^+$ and $\gamma \in I^-$ (we admit the possibility that one of such subsets is empty). The elements of I are called *indexes*, those of I^+ *positive indexes* and those of I^- *negative indexes*. Let $\{x_\lambda : \lambda \in I^+\} \cup \{y_\gamma : \gamma \in I^-\}$ a family of indeterminate real numbers with running indexes in I. We assume that

$$x_\lambda \geqslant x_{\lambda'} \geqslant 0 > y_\gamma \geqslant y_{\gamma'} \tag{4}$$

if $\lambda, \lambda' \in I^+$ with $\lambda \succ \lambda'$ and if $\gamma, \gamma' \in I^-$ with $\gamma \succ \gamma'$. We refer to (4) as to the (I^+, I^-)-*initial conditions*. We denote by $\mathcal{M}(I)$ the family of the finite multisets of I.

Definition 1. We say that a function $f : I \to \mathbb{R}$ *realizes* the (I^+, I^-)-initial conditions if: the assignment $x_\lambda = f(\lambda)$ when $\lambda \in I^+$ and $y_\gamma = f(\gamma)$ when $\gamma \in I^-$, satisfies the inequalities in (4). We denote by $\Phi(I^+, I^-)$ the set of all the functions that realize the (I^+, I^-)-initial conditions.

We introduce now the concept of *partial sums order on I*.

Definition 2. A partial sums order on I that realizes the (I^+, I^-)-initial conditions (briefly a $(I|I^+, I^-)$-partial sums order) is a partial order $\sqsubseteq$ on $\mathcal{M}(I)$ such that if $A, B \in \mathcal{M}(I)$ then

$$A \sqsubseteq B \Longleftrightarrow \sum_{\alpha\in A} f(\alpha) \leqslant \sum_{\beta\in B} f(\beta) \text{ for each } f \in \Phi(I^+, I^-) \tag{5}$$

A partial sums poset on I that realizes the initial conditions (I^+, I^-) (briefly a $(I|I^+, I^-)$-partial sums poset) is an induced sub-poset $(T, \sqsubseteq)$ of $(\mathcal{M}(I), \sqsubseteq)$, for some subset T of $\mathcal{M}(I)$, where $\sqsubseteq$ is a $(I|I^+, I^-)$-partial sums order. If $(T, \sqsubseteq)$ is a lattice, we say that $(T, \sqsubseteq)$ is a $(I|I^+, I^-)$-partial sums lattice. If the partition (I^+, I^-) of I is clear from the context, we only say that $(T, \sqsubseteq)$ is a I-partial sums poset.

The aim of this section is to build explicitly a quasi-order which will induce a $(I|I^+, I^-)$-partial sums order on an arbitrary totally ordered set I. We now introduce a new symbol o and we formally assume that $\lambda \succ o$ if $\lambda \in I^+$ and $o \succ \gamma$ if $\gamma \in I^-$. We set $I^o = I \cup \{o\}$. Of course I^o is also a totally ordered set with the obvious extension of the relation $\succ$ to I^o.

Definition 3. Let q and p be two non-negative integers. A I^o-string w with balance (q, p) is a finite sequence of the form $(\lambda_q, \ldots, \lambda_1|\gamma_1, \ldots, \gamma_p)$, where $\lambda_q, \ldots, \lambda_1 \in I^+ \cup \{o\}$, $\gamma_p, \ldots, \gamma_1 \in I^- \cup \{o\}$ and $\lambda_q \succeq \cdots \succeq \lambda_1$, $\gamma_1 \succeq \cdots \succeq \gamma_p$. The elements $\lambda_q, \ldots, \lambda_1, \gamma_1, \ldots, \gamma_p$ are called *indexes of* w. A I^o-string w is a I^o-string having balance (q, p), for some non-negative integers q and p.

When I^+ is a subset of the real interval $(0, +\infty)$ and I^- is a subset of the real interval $(-\infty, 0)$, we assume the following convention. If $w = (\lambda_q, \ldots, \lambda_1|\gamma_1, \ldots, \gamma_p)$ is a I^o-string, we also write w in the form $\lambda_q \ldots \lambda_1|\mu_1 \ldots \mu_p$, where μ_j is the absolute value of γ_j, for $j = 1, \ldots, p$. For example, with $44o|oo11$ we mean $(4, 4, o|o, o, -1, -1)$, whereas the I^o-string $(44, o|o, o, -11)$ will be write as $(44)o|oo(11)$.

We denote by $\mathfrak{M}(I^o)$ the set of all the I^o-strings. We call $\lambda_q, \ldots, \lambda_1$ the *non-negative indexes of* w and $\gamma_1, \ldots, \gamma_p$ the *non-positive indexes of* w; also, we call *positive indexes of* w the elements λ_i with $\lambda_i \succ o$ and *negative indexes of* w the elements γ_j with $\gamma_j \prec o$. We assume that $q = 0$ $[p = 0]$ iff there are no non-negative [non-positive] indexes of w; in particular, it results that $p = 0$ and $q = 0$ iff w is the empty string, that we denote by $(|)$. If $(w = \lambda_q, \ldots, \lambda_1|\gamma_1, \ldots, \gamma_p)$, we set $w_+ = \lambda_q \ldots \lambda_1|$ and $w_- = |\mu_1 \ldots \mu_p$.

If w is a I^o-string, we denote by $\{w \succeq o\}$ $[\{w \succ o\}]$ the multi-set of all the non-negative [positive] indexes of w, by $\{w \preceq o\}$ $[\{w \prec o\}]$ the multi-set of all the non-positive [negative] indexes of w and by $\{w\}$ the multi-set of all the indexes of w (i.e. $\{w\} = \{w \succeq o\} \cup \{w \preceq o\}$). We denote respectively with $|w|_\succeq$, $|w|_\preceq$, $|w|_\succ$, $|w|_\prec$ the cardinality of $\{w \succeq o\}$, $\{w \preceq o\}$, $\{w \succ o\}$, $\{w \prec o\}$. We call the ordered couple $(|w|_\succ, |w|_\prec)$ the *signature* of w and we note that $(|w|_\succeq, |w|_\preceq)$ is exactly the balance of w .

For example, if $I = \mathbb{Z}\setminus\{0\}$, $I^+ = \mathbb{Z}_+$, $I^- = \mathbb{Z}_-$ and $w = 444221ooo|o11333$, then $\{w \succeq o\} = \{4^3, 2^2, 1^1, o^3\}$, $\{w \preceq o\} = \{o^1, (-1)^2, (-3)^3\}$, $\{w \succ o\} = \{4^3, 2^2, 1^1\}$, $\{w \preceq o\} = \{(-1)^2, (-3)^3\}$, w has balance $(|w|_{\succeq}, |w|_{\preceq}) = (9, 6)$ and signature $(|w|_{\succ}, |w|_{\prec}) = (6, 5)$. In the sequel, two I^o-strings $w = (\lambda_q, \ldots, \lambda_1|\gamma_1, \ldots, \gamma_p)$ and $w' = (\lambda'_{q'}, \ldots, \lambda'_1|\gamma'_1, \ldots, \gamma'_{p'})$ are considered equals (and we shall write $w = w'$) if and only if $q = q'$, $p = p'$ and $\lambda_i = \lambda'_i$, $\mu_j = \mu'_j$ for $i = 1, \ldots, q$ and $j = 1, \ldots, p$. Therefore, for example, the two I^o-strings $(\lambda_2, \lambda_1, o, o, o|\gamma_1, \gamma_2)$ and $(\lambda_2, \lambda_1, o, o|o, \gamma_1, \gamma_2)$ are considered different between them in our context. If w is a I^o-string having signature (t, s) and balance (q, p), then w has the form

$$w = (\lambda_q, \ldots, \lambda_{q-t+1}, \lambda_{q-t}, \ldots \lambda_1|\gamma_1, \ldots, \gamma_{p-s}, \gamma_{p-s+1}, \ldots, \gamma_p) \qquad (6)$$

where $\lambda_q \succeq \cdots \succeq \lambda_{q-t+1} \succ 0 \succ \gamma_{p-s+1} \succeq \cdots \succeq \gamma_p$, $\lambda_{q-t} = \ldots = \lambda_1 = o$ and $\gamma_1 = \ldots = \gamma_{p-s} = o$. We also write w in (6) in the following form:

$$w = (\lambda_q, \ldots, \lambda_{q-t+1}, o_{q-t}|o_{p-s}, \gamma_{p-s+1}, \ldots, \gamma_p) \qquad (7)$$

If w is a I^o-string as in (7), we call *reduced I^o-string of w* the following I^o-string:

$$w_* = (\lambda_q, \ldots, \lambda_{q-t+1}|\gamma_{p-s+1} \cdots \gamma_p) \qquad (8)$$

If W is a subset of $\mathfrak{M}(I^o)$, we set $W_* = \{w_* : w \in W\}$. Then it is obvious that we can identify the set $\mathcal{M}(I)$ with the set $\mathfrak{M}(I^o)_*$. Let us note that $\{w \succ o\} = \{w_* \succ o\}$ and $\{w \prec o\} = \{w_* \prec o\}$.

If U is a subset I^o-strings, we say that U is *uniform* if all the I^o-strings in U have the same balance; in particular, if all the I^o-strings in U have balance (q, p), we also say that U is (q, p)-*uniform*. If $v_1, \ldots, v_k$ are I^o-strings, we say that they are *uniform* [(q, p)-*uniform*] if the subset $U = \{v_1, \ldots, v_k\}$ is uniform [(q, p)-*uniform*]. If F is a finite subset of I^o-strings, we define a way to make uniform all the I^o-strings in F: we set $q_F = \max\{|v|_{\succeq} : v \in F\}$, $p_F = \max\{|v|_{\preceq} : v \in F\}$; moreover, if $v = (\lambda_q, \ldots, \lambda_1|\gamma_1, \ldots, \gamma_p) \in F$ we also set

$$\overline{v}^F = (\lambda_q, \ldots, \lambda_1, o_{q_F - q}|o_{p_F - p}, \gamma_1, \ldots, \gamma_p)$$

and $\overline{F} = \{\overline{v}^F : v \in F\}$. Then $\overline{F}$ is (q_F, p_F)-uniform and $|\overline{F}| \leqslant |F|$. If F is uniform we note that $\overline{v}^F = v$ for each $v \in F$, hence $\overline{F} = F$. We call $\overline{F}$ the *uniform closure of F*. When F is clear from the context we simply write $\overline{v}$ instead of $\overline{v}^F$. In particular, if v and w are two I^o-strings, when we write $\overline{v}$ and $\overline{w}$ without further specification, we always mean $\overline{v}^F$ and $\overline{w}^F$, where $F = \{v, w\}$. We observe that if v and w are two uniform I^o-strings then $\overline{v} = v$ and $\overline{w} = w$; moreover, for each finite subset $F \subseteq P$ such that $w \in F$ we have $\{w \succ 0\} = \{\overline{w}^F \succ 0\}$ and $\{w \prec 0\} = \{\overline{w}^F \prec 0\}$.

If $u = (\lambda_q, \ldots, \lambda_1 | \gamma_1 \ldots \gamma_p)$ and $u' = (\lambda'_q, \ldots, \lambda'_1 | \gamma'_1, \ldots, \gamma'_p)$ are two uniform I^o-strings, we set:

$$u \leqslant u' \Longleftrightarrow \lambda_i \preceq \lambda'_i \text{ and } \gamma_j \preceq \gamma'_j \tag{9}$$

for all $i = 1, \ldots, q$ and $j = 1, \ldots, p$.

We also set

$$u \bigtriangleup u' = (\min\{\lambda_q, \lambda'_q\}, \ldots, \min\{\lambda_1, \lambda'_1\} | \min\{\gamma_1, \gamma'_1\}, \ldots, \min\{\gamma_p, \gamma'_p\})$$

and

$$u \bigtriangledown u' = (\max\{\lambda_q, \lambda'_q\}, \ldots, \max\{\lambda_1, \lambda'_1\} | \max\{\gamma_1, \gamma'_1\}, \ldots, \max\{\gamma_p, \gamma'_p\}).$$

Let us introduce now a quasi-order $\sqsubseteq$ (i.e. a reflexive and transitive binary relation) on the set $\mathfrak{M}(I^o)$ of all the I^o-strings.

Definition 4. If $v, w \in \mathfrak{M}(I^o)$, we set

$$v \sqsubseteq w \text{ if } \overline{v} \leqslant \overline{w}. \tag{10}$$

In particular, if v and w are uniform, then $v \sqsubseteq w$ iff $v \leqslant w$.

The following result is simple but useful:

Proposition 1. *$\sqsubseteq$ is a quasi-order on the set $\mathfrak{M}(I^o)$. Moreover, if $v, w \in \mathfrak{M}(I^o)$ the following conditions are equivalent:*

(i) $v \sqsubseteq w$.

(ii) *There exists a finite subset F of $\mathfrak{M}(I^o)$ containing v and w such that $\overline{v}^F \leqslant \overline{w}^F$.*

(iii) *For each finite subset F of $\mathfrak{M}(I^o)$ containing v and w we have $\overline{v}^F \leqslant \overline{w}^F$.*

(iv) $v_* \sqsubseteq w_*$.

Proof. It is immediate to verify that the relation $\sqsubseteq$ is a quasi-order. The unique implication that requires some comment is (ii) $\Rightarrow$ (iii). For this, it is enough to observe that if $\{v, w\} \subseteq H \subseteq F$, with H and F both finite, then $\overline{v}^H \leqslant \overline{w}^H$ if and only if $\overline{v}^F \leqslant \overline{w}^F$. □

2. Abstract partial sums posets

We consider now the equivalence relation on $\mathfrak{M}(I^o)$ induced from the quasi-order $\sqsubseteq$, i.e. if v and w are two I^o-strings in $\mathfrak{M}(I^o)$, we set

$$v \sim w \Longleftrightarrow v \sqsubseteq w \text{ and } w \sqsubseteq v \tag{11}$$

The proof of the following result is immediate from the above definitions.

Proposition 2. (i) *If v and w are two I^o-strings, then $v = w$ if and only if v and w are uniform and $v \sim w$.*

(ii) *If F is a finite subset of $\mathfrak{M}(I^o)$ such that $v, w \in F$, then $v \sim w$ if and only if $\overline{v}^F = \overline{w}^F$.*

(iii) *If F is a finite subset of $\mathfrak{M}(I^o)$ such that $v \in F$ then $v \sim \overline{v}^F$.*

(iv) *If v is a I^o-string then $v \sim v_*$.*

If $\mathcal{F}$ is any subset of $\mathfrak{M}(I^o)$, we can consider on the quotient set $\mathcal{F}/\sim$ the usual partial order induced by $\sim$, which will be denoted by $\lesssim$. We recall that $\lesssim$ is defined as follows: if $Z, Z' \in \mathcal{F}/\sim$ then

$$Z \lesssim Z' \Longleftrightarrow v \sqsubseteq v' \tag{12}$$

for any/all $v, w \in \mathcal{F}$ such that $v \in Z$ and $v' \in Z'$.

If $w \in \mathcal{F}$, in some case we set $[w]_{\sim}^{\mathcal{F}} = \{v \in \mathcal{F} : v \sim w\}$, that is the equivalence class of w in $\mathcal{F}/\sim$.

Remark 1. If $\mathcal{F} \subseteq \mathcal{H} \subseteq \mathfrak{M}(I^o)$ we can consider $\mathcal{F}/\sim$ as a subset of $\mathcal{H}/\sim$ through the identification of $[v]_{\sim}^{\mathcal{F}}$ with $[v]_{\sim}^{\mathcal{H}}$, for each $v \in \mathcal{F}$. Therefore, if $\mathcal{F} \subseteq \mathcal{H} \subseteq \mathfrak{M}(I^o)$ we always can assume that $(\mathcal{F}/\sim, \lesssim)$ is a sub-poset of $(\mathcal{H}/\sim, \lesssim)$.

The next definition describes a type of subsets of I^o-strings which shall permit us to find several partial sums lattices.

Definition 5. We say that a subset $\mathcal{F} \subseteq \mathfrak{M}(I^o)$ is lattice-inductive if for each finite subset $F \subseteq \mathcal{F}$ it results that:

(i) $\overline{F} \subseteq \mathcal{F}$;

(ii) if $v, w \in F$, then $\overline{v}^F \bigtriangleup \overline{w}^F \in \mathcal{F}$ and $\overline{v}^F \bigtriangledown \overline{w}^F \in \mathcal{F}$.

Let us note that obviously $\mathfrak{M}(I^o)$ is lattice-inductive. The relevance of the lattice-inductive subsets of P is established in the following result.

Theorem 1. *Let $\mathcal{F}$ a lattice-inductive subset of $\mathfrak{M}(I^o)$. Then $(\mathcal{F}/\sim, \lesssim)$ is a distributive lattice.*

Proof. To prove that $(\mathcal{F}/\sim)$ is a lattice, we take two equivalence classes $[v]_\sim^\mathcal{F}$ and $[w]_\sim^\mathcal{F}$ in $(\mathcal{F}/\sim)$ and we define the following operations: $[v]_\sim^\mathcal{F} \wedge [w]_\sim^\mathcal{F} = [\overline{v} \vartriangle \overline{w}]_\sim^\mathcal{F}$ and $[v]_\sim^\mathcal{F} \vee [w]_\sim^\mathcal{F} = [\overline{v} \triangledown \overline{w}]_\sim^\mathcal{F}$. It easy to see that the operations $\wedge$ and $\vee$ are well defined because they do not depend on the choice of representatives in the respective equivalence classes and that $[v]_\sim^\mathcal{F} \wedge [w]_\sim^\mathcal{F} \lesssim [v]_\sim^\mathcal{F}$, $[v]_\sim^\mathcal{F} \wedge [w]_\sim^\mathcal{F} \lesssim [w]_\sim^\mathcal{F}$. Let now $[z]_\sim \in \mathcal{F}/\sim$ such that $[z]_\sim^\mathcal{F} \lesssim [v]_\sim^\mathcal{F}$ and $[z]_\sim^\mathcal{F} \lesssim [w]_\sim^\mathcal{F}$. We set $F = \{v, w, z\}$. Then: $([z]_\sim^\mathcal{F} \lesssim [v]_\sim^\mathcal{F}$ and $[z]_\sim^\mathcal{F} \lesssim [w]_\sim^\mathcal{F}) \iff$ (by definition of $\lesssim$) $(z \sqsubseteq v$ and $z \sqsubseteq w) \iff$ (by Proposition 1 (iii)) $\overline{z}^F \leqslant \overline{v}^F$ and $\overline{z}^F \leqslant \overline{w}^F \iff$ (by definition of $\leqslant$ and of $\vartriangle$) $\overline{z}^F \leqslant \overline{v}^F \vartriangle \overline{w}^F \iff$ (since the I^o-strings are uniform) $\overline{z}^F \sqsubseteq \overline{v}^F \vartriangle \overline{w}^F \iff$ (by definition of $\lesssim$ and by Proposition 2 (iii)) $[z]_\sim^\mathcal{F} = [\overline{z}^F]_\sim^\mathcal{F} \lesssim [\overline{v}^F \vartriangle \overline{w}^F]_\sim^\mathcal{F}$. Now, since $[\overline{v}^F \vartriangle \overline{w}^F]_\sim^\mathcal{F} =$ (by definition of $\wedge$) $[\overline{v}^F]^\mathcal{F} \wedge [\overline{w}^F]^\mathcal{F} =$ (by Proposition 2 (iii)) $[v]_\sim^\mathcal{F} \wedge [w]_\sim^\mathcal{F}$, it follows that $[z]_\sim^\mathcal{F} \lesssim [v]_\sim^\mathcal{F} \wedge [w]_\sim^\mathcal{F}$. This proves that the operation $\wedge$ defines effectively the inf in $(\mathcal{F}/\sim, \lesssim)$. In the same way we can proceed for the operation $\vee$ in the sup-case. Finally, let us note that the distributivity holds because the operations $\vartriangle$ and $\triangledown$ are defined on the components of the uniform I^o-strings. □

Since $\mathfrak{M}(I^o)$ is obviously lattice-inductive, it follows that:

Corollary 1. $(\mathfrak{M}(I^o)/\sim, \lesssim)$ *is a distributive lattice.*

When $\mathcal{F}$ is finite and uniform is simple to verify if $\mathcal{F}$ is lattice-inductive:

Corollary 2. *Let $\mathcal{F}$ a finite uniform subset of $\mathfrak{M}(I^o)$, then $\mathcal{F}$ is lattice-inductive if and only if whenever $v, w \in \mathcal{F}$ also $v \vartriangle w \in \mathcal{F}$ and $v \triangledown w \in \mathcal{F}$*

Let us observe that if $v \in \mathcal{F}$, by Proposition 2 (iv) we can always choose v_* as a representative for the equivalence class $[v]_\sim^\mathcal{F}$. In such a way we identify the quotient set $\mathcal{F}/\sim$ with the subset $\mathcal{F}_*$ of $\mathfrak{M}(I^o)$, and we shall write $\mathcal{F}/\sim \equiv \mathcal{F}_*$. Therefore, if $\mathcal{F} \subseteq \mathfrak{M}(I^o)$, Z, Z' are two any equivalence classes in $\mathcal{F}/\sim$ and v, v' are two any elements in $\mathcal{F}$ such that $v \in Z$, $v' \in Z'$, the order (12) will have the following equivalent form (by Proposition 1 (iv)):

$$Z \lesssim Z' \iff v_* \sqsubseteq {v'}_* \tag{13}$$

Hence, taking into account the conventions established in (13), we can always think that on each quotient set $\mathcal{F}/\sim$ the partial order $\lesssim$ is identified with $\sqsubseteq$, therefore:

Remark 2. If $\mathcal{F} \subseteq \mathfrak{M}(I^o)$ we shall identify the quotient poset $(\mathcal{F}/\sim, \lesssim)$ with $(\mathcal{F}_*, \sqsubseteq)$, and this means that we shall consider $(\mathcal{F}/\sim, \lesssim)$ as a sub-poset of $(\mathfrak{M}(I^o)_*, \sqsubseteq) \equiv (\mathcal{M}(I), \sqsubseteq)$.

In particular, if $\mathcal{F}$ coincides with $\mathfrak{M}(I^o)$ then

$$(\mathfrak{M}(I^o)/\sim, \lesssim) \equiv (\mathfrak{M}(I^o)_*, \sqsubseteq) \equiv (\mathcal{M}(I), \sqsubseteq) \tag{14}$$

If $f \in \Phi(I^+, I^-)$ we define the function $\tilde{f} : I^o \to R$ setting

$$\tilde{f}(\alpha) = \begin{cases} f(\alpha) & \text{if} \quad \alpha \in I \\ 0 & \text{if} \quad \alpha = o \end{cases}$$

Definition 6. If $f \in \Phi(I^+, I^-)$, we denote by $\sum_f$ the function $\sum_f : \mathfrak{M}(I^o) \to \mathbb{R}$ such that $\sum_f(w) = \sum_{\alpha \in \{w\}} \tilde{f}(\alpha)$ for each $w \in \mathfrak{M}(I^o)$, and we call $\sum_f$ the sum function of f.

Let us note that if $w \in \mathfrak{M}(I^o)$ then $\sum_f(w) = \sum_{\alpha \in \{w_*\}} f(\alpha)$.

The next is the main result of this section.

Theorem 2. (i) *The partial order $\sqsubseteq$ on $\mathcal{M}(I)$ in (14) is a $(I|I^+, I^-)$-partial sums order.*

(ii) *If $\mathcal{F}$ is a subset of $\mathfrak{M}(I^o)$, then $(\mathcal{F}_*, \sqsubseteq)$ is a $(I|I^+, I^-)$-partial sums poset.*

(iii) *If $\mathcal{F}$ is a lattice-inductive subset of $\mathfrak{M}(I^o)$, then $(\mathcal{F}_*, \sqsubseteq)$ is a distributive $(I|I^+, I^-)$-partial sums lattice.*

Proof. (i) By Proposition 1 (iii) it is sufficient to prove that if w and w' are two uniform I^o-strings then

$$w \sqsubseteq w' \Longleftrightarrow \sum_f(w) \leqslant \sum_f(w') \tag{15}$$

Let $w = (\lambda_q, \dots, \lambda_1 | \gamma_1, \dots, \gamma_p)$ and $w' = (\lambda'_q, \dots, \lambda'_1 | \gamma'_1, \dots, \gamma'_p)$ two uniform I^o-strings. If $w \sqsubseteq w'$, by (9) we have $\lambda_i \preceq \lambda'_i$ and $\gamma_j \preceq \gamma'_j$ for all $i = 1, \dots, q$ and $j = 1, \dots, p$. Therefore, if $f \in \Phi(I^+, I^-)$, from the hypothesis that f realizes the (I^+, I^-)-initial conditions and by definition of $\sum_f$, we obtain $\sum_f(w) \leqslant \sum_f(w')$.

We assume now that $\sum_f(w) \leqslant \sum_f(w')$ for each $f \in \Phi(I^+, I^-)$ and that the condition $w \sqsubseteq w'$ is false. This means that there exists some $i \in \{1, \dots, q\}$ such that $\lambda_i \succ \lambda'_i$ or some $j \in \{1, \dots, p\}$ such that $\gamma_j \succ \gamma'_j$. Let us suppose first that there exists $i \in \{1, \dots, q\}$ such that $\lambda_i \succ \lambda'_i$ and we assume that i is maximal among all the positive integers $l \in \{1, \ldots, q\}$ such that $\lambda_l \succ \lambda'_l$, therefore

$$\lambda_q \succeq \cdots \succeq \lambda_{i+1} \succeq \lambda_i \succ \lambda'_i \succeq \lambda'_{i-1} \succeq \cdots \succeq \lambda'_1 \; ; \; \lambda'_q \succeq \lambda_q, \ldots, \lambda'_{i+1} \succeq \lambda_{i+1} \tag{16}$$

We consider now the following function :

$$f(\alpha) := \begin{cases} -1 & \text{if} \quad \alpha \in I^- \\ 0 & \text{if} \quad \alpha \in I^+ \text{ and } \alpha \prec \lambda_i \\ +1 & \text{if} \quad \alpha \in I^+ \text{ and } \alpha \succeq \lambda_i \end{cases}$$

Then $f \in \Phi(I^+, I^-)$ and by (16) it follows that $\sum_f(w) \geqslant (q-i+1) + \sum_{1 \leqslant j \leqslant p} \tilde{f}(\gamma_j) > (q-i) + \sum_{1 \leqslant j \leqslant p} \tilde{f}(\gamma_j) = (q-i) + \sum_{1 \leqslant j \leqslant p} \tilde{f}(\gamma'_j) = \sum_f(w')$, that is a contradiction.

We can suppose then $\lambda_i \preceq \lambda'_i$ for all $i = 1, \ldots, q$, so there exists $j \in \{1, \ldots, p\}$ such that $\gamma_j \succ \gamma'_j$ and we assume that j is minimal among all the positive integers $l \in \{1, \ldots, p\}$ such that $\gamma_l \succ \gamma'_l$, therefore

$$\gamma_1 \succeq \cdots \succeq \gamma_{j-1} \succeq \gamma_j \succ \gamma'_j \succeq \gamma'_{j+1} \succeq \cdots \succeq \gamma'_p \; ; \; \gamma'_1 \succeq \gamma_1, \ldots, \gamma'_{j-1} \succeq \gamma_{j-1} \tag{17}$$

We must now distinguish two cases. First we suppose that $\gamma_j = o$. In this case we consider the following function:

$$h(\alpha) := \begin{cases} 0 & \text{if} \quad \alpha \in I^+ \\ -1 & \text{if} \quad \alpha \in I^- \end{cases}$$

Then $h \in \Phi(I^+, I^-)$ and by (17) it follows that $\sum_h(w) \geqslant (-1)(p-j) > (-1)(p-j+1) = \sum_h(w')$, that is a contradiction. We assume now that $\gamma_j \prec o$. In this case we consider the following function:

$$g(\alpha) := \begin{cases} +1 & \text{if} \quad \alpha \in I^+ \\ -1 & \text{if} \quad \alpha \in I^- \text{ and } \alpha \succeq \gamma_j \\ -2 & \text{if} \quad \alpha \in I^- \text{ and } \alpha \prec \gamma_j \end{cases}$$

Then $g \in \Phi(I^+, I^-)$ and by (17) we have:

$$\begin{aligned} \sum_g(w) &\geqslant \sum_{1 \leqslant l \leqslant j-1} \tilde{g}(\gamma_l) + (-1) + \sum_{j+1 \leqslant l \leqslant p} \tilde{g}(\gamma_l) \\ &> \sum_{1 \leqslant l \leqslant j-1} \tilde{g}(\gamma_l) + (-2) + \sum_{j+1 \leqslant l \leqslant p} \tilde{g}(\gamma_l) \\ &= \sum_{1 \leqslant l \leqslant j-1} \tilde{g}(\gamma'_l) + \tilde{g}(\gamma'_j) + \sum_{j+1 \leqslant l \leqslant p} \tilde{g}(\gamma_l) \\ &\geqslant \sum_{1 \leqslant l \leqslant j-1} \tilde{g}(\gamma'_l) + (-2) + (-2)(p-j) = \sum_g(w') \end{aligned}$$

that is a contradiction. This complete the proof of (i). Parts (ii) and (iii) are direct consequence of Remark 2 and Theorem 1. □

We establish now some direct consequences of the previous theorem.

Corollary 3. *$(\mathcal{M}(I), \sqsubseteq)$ is a distributive $(I|I^+, I^-)$-partial sums lattice.*

Proof. We take $\mathcal{F} = \mathfrak{M}(I^o)$. Since $\mathcal{F}$ is lattice-inductive, by Theorem 2 (iii) it follows that $(\mathfrak{M}(I^o)_*, \sqsubseteq)$ is a distributive $(I|I^+, I^-)$-partial sums lattice. The result follows then by (14). □

Corollary 4. *The classical Young lattice $\mathbb{Y}$ of the integer partitions is a distributive partial sums lattice.*

Proof. In the particular case $I = \mathbb{N}$, $I^+ = \mathbb{N}$ and $I^- = \varnothing$ we have $\mathbb{Y} = \mathcal{M}(I)$ and the partial order on $\mathbb{Y}$ is exactly $\sqsubseteq$. Hence the thesis follows from the previous corollary. □

We recall now the definition of the lattice $L(m, n)$ introduced by Stanley in [22]: if m and n are two positive integers, $L(m, n)$ is the sub-lattice of $\mathbb{Y}$ of all the integer partitions with at most m parts and maximum part not exceeding n.

Corollary 5. *$L(m, n)$ is a distributive partial sums lattice.*

Proof. We take $I = \{n > n-1 > \cdots > 1\}$, $I^+ = I$ and $I^- = \varnothing$. We consider the subset $\mathcal{F}$ of $\mathfrak{M}(I^o)$ whose elements are the I^o-strings w such that $|w|_{\geqslant} = m$. Then $\mathcal{F}$ is lattice-inductive and $\mathcal{F}_* = L(m, n)$. Hence the thesis follows by (iii) of the previous theorem. □

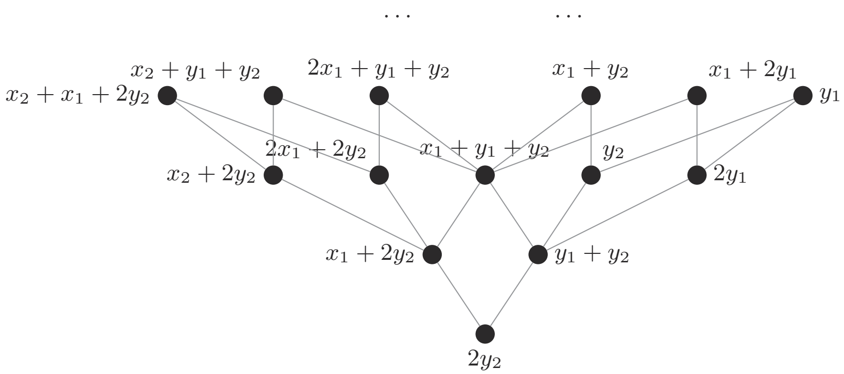
Corollary 6. *If $\mathcal{F}$ is a subset of $\mathfrak{M}(I^o)$ and $f \in \Phi(I^+, I^-)$, the restricted sum function $\sum_f : \mathcal{F}_* \to \mathbb{R}$ is order-preserving.*

Proof. It follows from the definition of $(I|I^+, I^-)$-partial sums poset and by Theorem 2 (ii). □

In terms of partial sums on a numerable quantity of real variables, we must take $I^- = \mathbb{Z}_- = \{-1, -2, -3, \dots\}$ and $I^+ = \mathbb{Z}_+ = \{1, 2, 3, \dots\}$. In this case, (4) becomes

$$\ldots \geqslant x_r \geqslant x_{r-1} \geqslant \ldots \geqslant x_1 \geqslant 0 > y_1 \geqslant y_2 \geqslant \ldots \tag{18}$$

In this case, if $w = (\lambda_t, \dots, \lambda_1 | \mu_1, \dots, \mu_s) \in \mathcal{M}(I)$, we set $\sum(w) = x_{\lambda_t} + \cdots + x_{\lambda_1} + y_{\mu_1} + \cdots + y_{\mu_s}$. Then, by the Theorem 2 we can to think the partial order $\sqsubseteq$ on $\mathcal{M}(I)$ as the natural order induced from the linear systems inequalities (18) on the partial sum of the real variables $\ldots x_r, x_{r-1}, \ldots, x_1, y_1, \ldots, y_q, \ldots$ In other terms, if we formally identify the signed partitions w and w' respectively with the indeterminate real partial sums $\sum(w)$ and $\sum(w')$, then the result of the Theorem 2 tell us that $w \sqsubseteq w'$ if and only if the real inequality $\sum(w) \leqslant \sum(w')$ holds and it can be deduced by using only the inequalities in (18). We can therefore use a more suggestive terminology and to think the signed partition lattice $(\mathcal{M}(I), \sqsubseteq)$ as a lattice of indeterminate partial sums take over a numerable quantity of real variables $\ldots x_r, x_{r-1}, \ldots, x_1, y_1, \ldots, y_q, \ldots$ that satisfy (18). Using this interpretation a finite piece of the Hasse diagram of $\mathcal{M}(I)$ is the following:



A particularly relevant lattice-inductive subset of $\mathfrak{M}(I^o)$ is the subset of all the I^o-strings without repeated negative indexes and without repeated positive indexes. We denote such a subset by $\mathfrak{S}(I^o)$.

Theorem 3. *$\mathfrak{S}(I^o)$ is lattice-inductive.*

Proof. Let F be a finite subset of $\mathfrak{S}(I^o)$ and let $w \in \overline{F}$, then, by definition of $\overline{F}$, there exists $v \in F$ such that $w = \overline{v}^F$. Since v is also an element of $\mathfrak{S}(I^o)$, it has no repeated negative or positive indexes, therefore also $\overline{v}^F$ has no repeated negative or positive indexes, since $\overline{v}^F$ is built by adding only eventual further o's to v. Hence $w \in \mathfrak{S}(I^o)$, and this shows that $\overline{F} \subseteq \mathfrak{S}(I^o)$.

Let now $v, v' \in F$, and $w = \overline{v}^F = (\lambda_q, \ldots, \lambda_1 | \gamma_1, \ldots, \gamma_p)$, $w' = \overline{v'}^F = (\lambda'_q, \ldots, \lambda'_1 | \gamma'_1, \ldots, \gamma'_p)$, then $w, w' \in \overline{F} \subseteq \mathfrak{S}(I^o)$. We must to prove that $w \bigtriangleup w' \in \mathfrak{S}(I^o)$ and $w \bigtriangledown w' \in \mathfrak{S}(I^o)$. We show only that $w \bigtriangleup w' \in \mathfrak{S}(I^o)$ because the proof that $w \bigtriangledown w' \in \mathfrak{S}(I^o)$ is similar. Let $w'' = w \bigtriangledown w'$, with

$w'' = (\lambda''_q, \dots, \lambda''_1 | \gamma''_1, \dots, \gamma''_p)$. Let us suppose by absurd that $w'' \notin \mathfrak{S}(I^o)$; by definition of $\mathfrak{S}(I^o)$ this means that there exist two positive indexes or two negative indexes in w'' that must be equal. We can assume that there exist λ''_k and λ''_l such that $k > l$ and $\lambda''_k = \lambda''_l \succ o$ (the case when $\lambda''_k = \lambda''_l \prec o$ is analogue). Since $\lambda''_k = \lambda''_l$ and $w'' \in \mathfrak{M}(I^o)$, we have

$$\lambda''_k = \lambda''_{k-1} = \cdots = \lambda''_{l+1} = \lambda''_l \tag{19}$$

We note that $\lambda_l \succ o$ and $\lambda'_l \succ o$ because $\lambda''_l \succ o$, therefore, since $w, w' \in \mathfrak{S}(I^o)$, by definition of $\mathfrak{S}(I^o)$ it follows that

$$\lambda_k \succ \lambda_{k-1} \succ \cdots \succ \lambda_{l+1} \succ \lambda_l \tag{20}$$

and

$$\lambda'_k \succ \lambda'_{k-1} \succ \cdots \succ \lambda'_{l+1} \succ \lambda'_l \tag{21}$$

We assume now that $\lambda''_k = \lambda_k$. Then, if $\lambda''_{k-1} = \lambda_{k-1}$, by (19) we have $\lambda_k = \lambda_{k-1}$, that contradicts (20). Therefore it must be $\lambda''_{k-1} = \lambda'_{k-1}$. Then by (19) it follows that $\lambda_k = \lambda'_{k-1}$ and hence, by (20), $\lambda'_{k-1} \succ \lambda_{k-1}$, but this contradicts the equality $\lambda''_{k-1} = \min\{\lambda_{k-1}, \lambda'_{k-1}\} = \lambda'_{k-1}$. On the other side, also the equality $\lambda''_k = \lambda'_k$ leads to an absurd if we use (19) and (21). This complete the proof. □

Obviously we can identify $\mathfrak{S}(I^o)_*$ with the subset of $\mathcal{M}(I)$ whose elements are the finite subsets (i.e. the finite multi-sets without repeated elements) of I, that we denote by $\mathcal{S}(I)$. Therefore, by the previous proposition it follows that $(\mathcal{S}(I), \sqsubseteq) \equiv (\mathfrak{S}(I^o)_*, \sqsubseteq)$ is a distributive $(I|I^+, I^-)$-partial sums lattice. We recall now the definition of the Stanley lattice $M(n)$ (see [22]): $M(n)$ is the sub-lattice of the Young lattice $\mathbb{Y}$ whose elements are the integer partitions having all distinct parts and whose maximum is at most n, where n is an arbitrary positive integer. We have then:

Corollary 7. *$M(n)$ is a distributive partial sums lattice.*

Proof. If we take $I = \{n, n-1, \dots, 1\}$, $I^+ = I$ and $I^- = \varnothing$, then $M(n)$ coincides exactly with $\mathfrak{S}(I^o)_*$. Hence the thesis is a direct consequences of the previous theorem. □

If $n \geqslant r \geqslant 0$ are two integers, we can take $I = \{r > \cdots > 1 > -1 > \cdots > -(n-r)\}$, $(I^+ = \{r, \dots, 1\}$ and $I^- = \{-1, \dots, -(n-r)\})$. In this case $\mathfrak{S}(I^o)_*$ coincides with the lattice $S(n, r)$ introduced in [3] and [4] in order to study some extremal combinatorial sum problems. As observed in [8], $S(n,r)$ is isomorphic to the direct product $M(r) \times M(n-r)^*$.

We recall now the definition of valuation on an arbitrary lattice X. If X is a lattice, a map $\eta : X \to \mathbb{R}$ is called a *valuation* on X if for all $a, b \in X$: $\eta(a \wedge b) + \eta(a \vee b) = \eta(a) + \eta(b)$.

Proposition 3. *Let $\mathcal{F}$ be a lattice-inductive subset of $\mathfrak{M}(I^o)$ and $f \in \Phi(I^+, I^-)$. Then the restricted sum function $\sum_f : \mathcal{F}_* \to \mathbb{R}$ is a valuation on $(\mathcal{F}_*, \sqsubseteq)$.*

Proof. Since the partial order $\sqsubseteq$ is made on the components of the I^o-strings, the result follows. □

In [20] was shown that a valuation on a distributive lattice is uniquely determined by the values that it takes on the join-irreducible elements of the lattice, therefore, in our case, this means that $\sum_f$ is uniquely determined by the values that it takes on the join-irreducible elements of the distributive lattice $\mathcal{F}_*$.

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