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Matrix approach to noncommutative stably free modules and Hermite rings

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Abstract. In this paper we present a matrix-constructive proof of an Stafford's Theorem about stably free modules over noncommutative rings. Matrix characterizations of noncommutative Hermite and projective-free rings are exhibit. Quotients, products and localizations of Hermite and some other classes of rings close related to Hermite rings are also considered.

1. Introduction

Finitely generated projective modules, stably free modules, projective-free rings, Bézout rings and Hermite rings have recently encountered interesting applications in algebraic control theory and algebraic analysis. For example, in [19], internal stabilization of coherent control systems over a commutative domain S is characterized in terms of conditions on S as being a Prüfer domain or an Hermite commutative ring (see [19], Theorems 2 and 4). In [20], the concept of internal stabilizability of stable time-invariant linear system over S is equivalent to the fact that certain S-modules are projective, whereas the existence of a doubly coprime factorization corresponds to the freeness of the same modules. In noncommutative algebraic analysis these homological objects have been applied to describe functional linear system of differential equations (SLF) over Ore algebras using matrix interpretations ([5], [6], [7], [18]).

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From a computational approach, it is very useful to study projective modules, stably free modules, projective-free rings and Hermite rings from a matrix-constructive point of view. This was done in [15] for the commutative case, the present paper can be considered as a generalization from commutative to noncommutative rings of some results presented in [15], Chapter 6 (see also [14]). The first section, needed for the rest of the paper, is about elementary facts of linear algebra over noncommutative rings and stably free modules (see also [8] and [17]). The second section is dedicated to give a matrix-constructive proof of a theorem due Stafford about stably free modules. The proof has been adapted from [18] but we can avoid the involution used in [18]. Matrix characterizations of Hermite and projective-free rings are presented in Sections 3 and 4. A matrix proof of a Kaplansky theorem about finitely generated projective modules over local rings is also included. Some remarkable classes of rings closed related to Hermite rings are considered in Section 5, in particular, we prove that for rings without nontrivial idempotents projective-free rings coincide with $\mathcal{ID}$ rings (i.e., rings for which each idempotent matrix is similar to a Smith normal diagonal matrix). Moreover, we will see that every $\mathcal{ID}$ ring S is Hermite when all idempotents of S are central. Products, quotients and localizations of Hermite rings are studied in Section 6, we proved that if S is a left Noetherian ring and P is a prime (completely prime) left localizable ideal of S, then S_P is Hermite (projective-free).

1.1. Some topics from linear algebra

We start recalling some notations and well known elementary properties of linear algebra for left modules. All rings are noncommutative and modules will be considered on the left; S will represent an arbitrary noncommutative ring; S^r is the left S-module of columns of size $r \times 1$; if $S^s \xrightarrow{f} S^r$ is an S-homomorphism then there is a matrix associated to f in the canonical bases of S^r and S^s, denoted $F := m(f)$, and disposed by columns, i.e., $F \in M_{r\times s}(S)$. In fact, if f is given by $S^s \xrightarrow{f} S^r$, $\boldsymbol{e}_j \mapsto \boldsymbol{f}_j$, where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_s\}$ is the canonical basis of S^s, f can be represented by a matrix, i.e., if $\boldsymbol{f}_j := \begin{bmatrix} f_{1j} & \dots & f_{rj} \end{bmatrix}^T$, then the matrix of f in the canonical bases of S^s and S^r is

$$F := \begin{bmatrix} \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_s \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1s} \\ \vdots & & \vdots \\ f_{r1} & \cdots & f_{rs} \end{bmatrix} \in M_{r\times s}(S).$$

Observe that $Im(f)$ is the column module of F, i.e., the left S-module generated by the columns of F, denoted by $\langle F\rangle$:

$$Im(f) = \langle f(\boldsymbol{e}_1), \dots, f(\boldsymbol{e}_s)\rangle = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s\rangle = \langle F\rangle.$$

We recall also that

$$Syz(\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}) := \{\boldsymbol{a} := \begin{bmatrix} a_1 & \cdots & a_s \end{bmatrix}^T \in S^s | a_1\boldsymbol{f}_1 + \cdots + a_s\boldsymbol{f}_s = \boldsymbol{0}\},$$

and

$$Syz(\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}) = \ker(f). \tag{1}$$

Moreover, observe that if $\boldsymbol{a} := (a_1, \dots, a_s)^T \in S^s$, then

$$f(\boldsymbol{a}) = (\boldsymbol{a}^T F^T)^T. \tag{2}$$

Note that function $m : Hom_S(S^s, S^r) \to M_{r\times s}(S)$ is bijective; moreover, if $S^r \xrightarrow{g} S^p$ is a homomorphism, then the matrix of gf in the canonical bases is $m(gf) = (F^T G^T)^T$. Thus, $f : S^r \to S^r$ is an isomorphism if and only if $F^T \in GL_r(S)$. Finally, let $C \in M_r(S)$; the columns of C conform a basis of S^r if and only if $C^T \in GL_r(S)$. With this notation, a matrix characterization of f.g. projective modules can be formulated in the following way.

Proposition 1. *Let S be an arbitrary ring and M a S-module. Then, M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F^T is idempotent and $M = \langle F\rangle$.*

Proof. $\Rightarrow$): If $M = 0$, then $F = 0$; let $M \neq 0$, there exists $s \geqslant 1$ and a M' such that $S^s = M \oplus M'$; let $f : S^s \to S^s$ be the projection on M and F the matrix of f in the canonical basis of S^s. Then, $f^2 = f$ and $(F^T F^T)^T = F$, so $F^T F^T = F^T$; note that $M = Im(f) = \langle F\rangle$.

$\Leftarrow$): Let $f : S^s \to S^s$ be the homomorphism defined by F (see (2)); from $F^T F^T = F^T$ we get that $f^2 = f$, moreover, since $M = \langle F\rangle$, then $Im(f) = M$ and hence M is direct summand of S^s, i.e., M is f.g. projective (observe that the complement M' of M is $\ker(f)$ and f is the projection on M). $\square$

Remark 1. (i) When S is commutative, or when we consider right modules instead of left modules, (2) says that $f(\boldsymbol{a}) = F\boldsymbol{a}$. Moreover, the matrix of a compose homomorphism gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^r \to S^r$ is an isomorphism if and only if $F \in GL_r(S)$; moreover, $C \in GL_r(S)$ if and only if its columns conform a basis of S^r. In addition, Proposition 1 says that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F \rangle$.

(ii) When the matrices of homomorphisms of left modules are disposed by rows instead of by columns, i.e., if $S^{1\times s}$ is the left free module of rows vectors of length s and the matrix of the homomorphism $S^{1\times s} \xrightarrow{f} S^{1\times r}$ is defined by

$$F' = \begin{bmatrix} f'_{11} & \cdots & f'_{1r} \\ \vdots & & \vdots \\ f'_{s1} & \cdots & f'_{sr} \end{bmatrix} := \begin{bmatrix} f_{11} & \cdots & f_{r1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{rs} \end{bmatrix} \in M_{s\times r}(S),$$

then

$$f(a_1, \dots, a_s) = (a_1, \dots, a_s)F', \tag{3}$$

i.e., $f(\boldsymbol{a}^T) = \boldsymbol{a}^T F^T$. Thus, the values given by (3) and (2) agree since $F' = F^T$. Moreover, the composed homomorphism gf means that g acts first and then acts f, and hence, the matrix of gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^{1\times r} \to S^{1\times r}$ is an isomorphism if and only if $m(f) \in GL_r(S)$; moreover, $C \in GL_r(S)$ if and only if its rows conform a basis of $S^{1\times r}$. This left-row notation is also used in [8] and [18]. Observe that with this notation, the proof of Proposition 1 says that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F \rangle$, but in this case $\langle F \rangle$ represents the module generated by the rows of F. Note that Proposition 1 could has been formulated this way: In fact, the set of idempotents matrices of $M_s(S)$ coincides with the set $\{F^T | F \in M_s(S), F^T \text{ idempotent}\}$.

Definition 1 ([12]). Let S be a ring.

(i) S satisfies the rank condition ($\mathcal{RC}$) if for any integers $r, s \geqslant 1$, given an epimorphism $S^r \xrightarrow{f} S^s$, then $r \geqslant s$.

(ii) S is an $\mathcal{IBN}$ ring (Invariant Basis Number) if for any integers $r, s \geqslant 1$, $S^r \cong S^s$ if and only if $r = s$.

Proposition 2. *Let S be a ring.*

(i) *S is $\mathcal{RC}$ if and only if given any matrix $F \in M_{s\times r}(S)$ the following condition holds:*

if F has a right inverse then $r \geqslant s$.

(ii) *S is $\mathcal{RC}$ if and only if given any matrix $F \in M_{s\times r}(S)$ the following condition holds:*

if F has a left inverse then $s \geqslant r$.

Proof. (i) $\Rightarrow$): Let G be a right inverse of F, $FG = I_s$; let $f : S^r \to S^s$ and $g : S^s \to S^r$ such that $m(f) = F$ and $m(g) = G$. Then, $((F^T)^T(G^T)^T)^T = I_s$; let $f^T : S^s \to S^r$ and $g^T : S^r \to S^s$ such that $m(f^T) = F^T$ and $m(g^T) = G^T$, then $m(g^T f^T) = m(i_{S^s})$ and hence $g^T f^T = i_{S^s}$, i.e., g^T is surjective. Since S is $\mathcal{RC}$, then $r \geqslant s$.

$\Leftarrow$): Let $S^r \xrightarrow{f} S^s$ be an epimorphism, there exists $S^s \xrightarrow{g} S^r$ such that $fg = i_{S^s}$; let $F := m(f) \in M_{s\times r}(S)$ and $G := m(g) \in M_{r\times s}(S)$, then $m(fg) = (G^T F^T)^T = I_s$, so $G^T F^T = I_s$, i.e., G^T has right inverse, and by hypothesis $r \geqslant s$. This means that S is $\mathcal{RC}$.

(ii) $\Rightarrow$): Let $G \in M_{r\times s}(S)$ a left inverse of F, then G has right inverse, and by (i), $s \geqslant r$.

$\Leftarrow$): Let $S^r \xrightarrow{f} S^s$ be an epimorphism; as in (i), $G^T F^T = I_s$, so $F^T \in M_{r\times s}(S)$ has a left inverse and by the hypothesis $r \geqslant s$. Thus, S is $\mathcal{RC}$. □

Proposition 3. $\mathcal{RC} \Rightarrow \mathcal{IBN}$.

Proof. Let $S^r \xrightarrow{f} S^s$ be an isomorphism, then f is an epimorphism, and hence $r \geqslant s$; considering f^{-1} we get that $s \geqslant r$. □

Remark 2. Most of rings are $\mathcal{RC}$, and hence, $\mathcal{IBN}$. For example, commutative rings and left Noetherian rings are $\mathcal{RC}$ (see [17] and [9]). The condition $\mathcal{IBN}$ for rings is independent of the side we are considering the modules (see [8]). The same is true for the $\mathcal{RC}$ property. From now on we will assume that all rings considered in the present paper are $\mathcal{RC}$.

1.2. Stably free modules

Definition 2. Let M be an S-module and $t \geqslant 0$ an integer. M is stably free of rank $t \geqslant 0$ if there exist an integer $s \geqslant 0$ such that $S^{s+t} \cong S^s \oplus M$.

The rank of M is denoted by $\operatorname{rank}(M)$. Note that any stably free module M is finitely generated and projective. Moreover, as we will show in the next proposition, $\operatorname{rank}(M)$ is well defined, i.e., $\operatorname{rank}(M)$ is unique for M.

Proposition 4. *Let $t, t', s, s' \geqslant 0$ integers such that $S^{s+t} \cong S^s \oplus M$ and $S^{s'+t'} \cong S^{s'} \oplus M$. Then, $t' = t$.*

Proof. We have $S^{s'} \oplus S^{s+t} \cong S^{s'} \oplus S^s \oplus M$ and $S^s \oplus S^{s'+t'} \cong S^s \oplus S^{s'} \oplus M$, then since S is an $\mathcal{IBN}$ ring, $s' + s + t = s + s' + t'$, and hence $t' = t$. □

Corollary 1. *M is stably free of rank $t \geqslant 0$ if and only if there exist integers $r, s \geqslant 0$ such that $S^r \cong S^s \oplus M$, with $r \geqslant s$ and $t = r - s$.*

Proof. If M is stably free of rank t, then $S^{s+t} \cong S^s \oplus M$ for some integers $s, t \geqslant 0$, then taking $r := s + t$ we get the result. Conversely, if there exist integers $r, s \geqslant 0$ such that $S^r \cong S^s \oplus M$, with $r \geqslant s$, then $S^{s+r-s} \cong S^s \oplus M$, i.e., M is stably free of rank $r - s$. □

Proposition 5. *Let M be an S-module and let $r, s \geqslant 0$ integers such that $S^r \cong S^s \oplus M$. Then $r \geqslant s$.*

Proof. The canonical projection $S^r \to S^s$ is an epimorphism, but since we are assuming that S is $\mathcal{RC}$, then $r \geqslant s$. □

Corollary 2. *M is stably free if and only if there exist integers $r, s \geqslant 0$ such that $S^r \cong S^s \oplus M$.*

Proof. This is a direct consequence of Corollary 1 and Proposition 5. □

Proposition 6. *Let M be an S-module. Then, the following conditions are equivalent*

(i) *M is stably free.*

(ii) *M has a minimal presentation, i.e., M has a finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_1 has a left inverse.*

Proof. See [17], Chapter 11. □

From this proposition we get a matrix characterization of stably free modules (compare with [18], Lemma 16).

Corollary 3. *Let M be an S-module. Then the following conditions are equivalent:*

(i) *M is stably free.*

(ii) *M is projective and has a finite system of generators $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r$ such that $Syz\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r\}$ is the module generated by the columns of a matrix F_1 of size $r \times s$ such that F_1^T has a right inverse.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 6, M is projective and has a finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where f_1 has a left inverse. Let $\boldsymbol{f}_i = f_0(\boldsymbol{e}_i)$, where $\{\boldsymbol{e}_i\}_{1 \leqslant i \leqslant r}$ is the canonical basis of S^r. Then $M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_r \rangle$ and $Im(f_1) = \ker(f_0) = Syz\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r\}$, but $Im(f_1)$ is the module generated by the columns of the matrix F_1 defined by f_1 in the canonical bases. Thus, let $g_1 : S^r \to S^s$ be a left inverse of f_1, then $g_1 f_1 = i_{S^s}$ and the matrix of $g_1 f_1$ in the canonical bases is $I_s = (F_1^T G_1^T)^T$, so $I_s = F_1^T G_1^T$.

(ii) $\Rightarrow$ (i) Let $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r$ be a set of generators of M such that $Syz\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r\}$ is the module generated by the columns of a matrix F_1 of size $r \times s$ such that F_1^T has a right inverse. We have the exact sequence $0 \to \ker(f_0) \xrightarrow{\iota} S^r \xrightarrow{f_0} M \to 0$, where ι is the canonical injection and f_0 is defined as above. We have $\ker(f_0) = Syz\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_r\} = \langle F_1 \rangle$, and thus we get the finite presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, where $f_1(\boldsymbol{e}_j)$ is the j^{th} column of F_1, $1 \leqslant j \leqslant s$. By hypothesis F_1^T has a right inverse, $F_1^T G_1^T = I_s$, so $I_s = (F_1^T G_1^T)^T$. Let $g_1 : S^r \to S^s$ be the homomorphism defined by $G_1 \in M_{s \times r}(S)$ in the canonical bases, then $g_1 f_1 = i_{S^s}$ and f_1 is injective, this implies that the sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ is exact. By Proposition 6, M is stably free. □

2. Stafford's theorem: a constructive proof

A well known result due Stafford says that any left ideal of the Weyl algebras $D := A_n(K)$ or $B_n(K)$, where K is a filed with $\mathrm{char}(K) = 0$, is generated by two elements. Recall that the Weyl algebra is defined by $A_n(K) := K[t_1, \ldots, t_n][x_1; \partial/\partial t_1] \cdots [x_n; \partial/\partial t_n]$ and $B_n(K)$ is the extended Weyl algebra defined as $K(t_1, \ldots, t_n)[x_1; \partial/\partial t_1] \cdots [x_n; \partial/\partial t_n]$ (see [22] and [18]). From the Stafford's Theorem follows that any stably free left module M over D with $\mathrm{rank}(M) \geqslant 2$ is free. In [18] is presented a constructive proof of this result that we want to study for arbitrary $\mathcal{RC}$ rings. Actually, we will consider the generalization given in [18] staying that any stably free left S-module M with $\mathrm{rank}(M) \geqslant \mathrm{sr}(S)$ is free, where $\mathrm{sr}(S)$ denotes the stable rank of the ring S. Our proof have been adapted from [18], however we do not need the involution of ring S used in [18] because of our left notation for modules and column representation for homomorphism. This could justify our special left-column notation.

Definition 3. Let F be a matrix over S of size $r \times s$. Then,

(i) Let $r \geqslant s$. F is unimodular if and only if F has a left inverse.

(ii) Let $s \geqslant r$. F is unimodular if and only if F has a right inverse.

In particular, the set of unimodular column matrices of size $r \times 1$ is denoted by $Um_c(r, S)$. $Um_r(s, S)$ is the set of unimodular row matrices of size $1 \times s$.

Note that a column matrix is unimodular if and only if the left ideal generated by its entries coincides with S, and a row matrix is unimodular if and only if the right ideal generated by its entries is S.

Definition 4. Let S be a ring and $\mathbf{v} := \begin{bmatrix} v_1 & \dots & v_r \end{bmatrix}^T \in Um_c(r, S)$ an unimodular column vector. $\mathbf{v}$ is called stable (reducible) if there exists $a_1, \dots, a_{r-1} \in S$ such that $\mathbf{v}' := \begin{bmatrix} v_1 + a_1 v_r & \dots & v_{r-1} + a_{r-1} v_r \end{bmatrix}^T$ is unimodular. It says that the left stable rank of S is $d \geqslant 1$, denoted $\mathrm{sr}(S) = d$, if d is the least positive integer such that every unimodular column vector of length $d+1$ is stable. It says that $\mathrm{sr}(S) = \infty$ if for every $d \geqslant 1$ there exits a non stable unimodular column vector of length $d+1$.

In a similar way is defined the right stable rank of S, however, both ranks coincide ([2]). Two preliminary results are needed for the main theorem of this section.

Proposition 7. *Let S be a ring and $\boldsymbol{v} := \begin{bmatrix} v_1 & \dots & v_r \end{bmatrix}^T$ an unimodular stable column vector over S, then there exists $U \in E_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$.*

Proof. See [18], Proposition 38. □

Lemma 1. *Let S be a ring and M a stably free S-module given by a minimal presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. Let $g_1 : S^r \to S^s$ such that $g_1 f_1 = i_{S^s}$. Then the following conditions are equivalent:*

(i) *M is free of dimension $r - s$.*

(ii) *There exists a matrix $U \in GL_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, where G_1 is the matrix of g_1 in the canonical bases. In such case, the last $r - s$ columns of U^T conform a basis for M. Moreover, the first s columns of U^T conform the matrix F_1 of f_1 in the canonical bases.*

(iii) *There exists a matrix $V \in GL_r(S)$ such that G_1^T coincides with the first s columns of V, i.e., G_1^T can be completed to an invertible matrix V of $GL_r(S)$.*

Proof. By the hypothesis, the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ splits, so F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$, i.e., $F_1^T G_1^T = I_s$. Moreover, there exists $g_0 : M \to S^r$ such that $f_0 g_0 = i_M$.

From this we get also the split sequence $0 \to M \xrightarrow{g_0} S^r \xrightarrow{g_1} S^s \to 0$. Note that $M \cong \ker(g_1)$.

(i) $\Rightarrow$ (ii): We have $S^r = \ker(g_1) \oplus Im(f_1)$; by the hypothesis $\ker(g_1)$ is free. If $s = r$ then $\ker(g_1) = 0$ and hence f_1 is an isomorphism, so $f_1 g_1 = i_{S^s}$, i.e., $G_1^T F_1^T = I_s$. Thus, we can take $U := F_1^T$.
Let $r > s$; if $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\}$ is the canonical basis of S^s, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s\}$ is a basis of $Im(f_1)$ with $\boldsymbol{u}_i := f_1(\boldsymbol{e}_i)$, $1 \leqslant i \leqslant s$; let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be a basis of $\ker(g_1)$ with $p = r - s$. Then, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_s\}$ is a basis of S^r. We define $S^r \xrightarrow{h} S^r$ by $h(\boldsymbol{e}_i) := \boldsymbol{u}_i$ for $1 \leqslant i \leqslant s$, and $h(\boldsymbol{e}_{s+j}) := \boldsymbol{v}_j$ for $1 \leqslant j \leqslant p$. Clearly h is bijective; moreover, $g_1 h(\boldsymbol{e}_i) = g_1(\boldsymbol{u}_i) = g_1 f_1(\boldsymbol{e}_i) = \boldsymbol{e}_i$ and $g_1 h(\boldsymbol{e}_{s+j}) = g_1(\boldsymbol{v}_j) = \boldsymbol{0}$, i.e., $H^T G_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$. Let $U := H^T$, so we observe that the last p columns of U^T conform a basis of $\ker(g_1) \cong M$ and the first s columns of U^T conform F_1.

(ii) $\Rightarrow$ (i): Let $U_{(k)}$ the k-th row of U, then we have that $UG_1^T = [U_{(1)} \cdots U_{(s)} \cdots U_{(r)}]^T G_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, so $U_{(i)} G_1^T = \boldsymbol{e}_i^T$, $1 \leqslant i \leqslant s$, $U_{(s+j)} G_1^T = \boldsymbol{0}$, $1 \leqslant j \leqslant p$ with $p := r - s$. This means that $(U_{(s+j)})^T \in \ker(g_1)$ and hence $\langle (U_{(s+j)})^T | 1 \leqslant j \leqslant p \rangle \subseteq \ker(g_1)$. On the other hand, let $\boldsymbol{c} \in \ker(g_1) \subseteq S^r$, then $\boldsymbol{c}^T G_1^T = \boldsymbol{0}$ and $\boldsymbol{c}^T U^{-1} U G_1^T = \boldsymbol{0}$, thus $\boldsymbol{c}^T U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix} = \boldsymbol{0}$ and hence $(\boldsymbol{c}^T U^{-1})^T \in \ker(l)$, where $l : S^r \to S^s$ is the homomorphism with matrix $\begin{bmatrix} I_s & 0 \end{bmatrix}$. Let $\boldsymbol{d} = [d_1, \ldots, d_r]^T \in \ker(l)$, then $[d_1, \ldots, d_r] \begin{bmatrix} I_s \\ 0 \end{bmatrix} = \boldsymbol{0}$ and from this we conclude that $d_1 = \cdots = d_s = 0$, i.e., $\ker(l) = \langle \boldsymbol{e}_{s+1}, \boldsymbol{e}_{s+2}, \ldots, \boldsymbol{e}_{s+p} \rangle$. From $(\boldsymbol{c}^T U^{-1})^T \in \ker(l)$ we get that $(\boldsymbol{c}^T U^{-1})^T = a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p}$, so $\boldsymbol{c}^T U^{-1} = (a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p})^T$, i.e., $\boldsymbol{c}^T = (a_1 \cdot \boldsymbol{e}_{s+1} + \cdots + a_p \cdot \boldsymbol{e}_{s+p})^T U$ and from this we get that $\boldsymbol{c} \in \langle (U_{(s+j)})^T | 1 \leqslant j \leqslant p \rangle$. This proves that $\ker(g_1) = \langle (U_{(s+j)})^T | 1 \leqslant j \leqslant p \rangle$; but since U is invertible, then $\ker(g_1)$ is free of dimension p. We have proved also that the last p columns of U^T conform a basis for $\ker(g_1) \cong M$.

(ii) $\Leftrightarrow$ (iii): $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ if and only if $G_1^T = U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix}$, but the first s columns of $U^{-1} \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ coincides with the first s columns of U^{-1}; taking $V := U^{-1}$ we get the result. $\square$

Theorem 1. *Let S be a ring. Then any stably free S-module M with* $\operatorname{rank}(M) \geqslant \operatorname{sr}(S)$ *is free with dimension equals to* $\operatorname{rank}(M)$.

Proof. Since M is stably free it has a minimal presentation, and hence, it is given by an exact sequence

$$0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0;$$

moreover, note that $\operatorname{rank}(M) = r - s$. Since this sequence splits, F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$. The idea of the proof is to find a matrix $U \in GL_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ and then apply Lemma 1.

We have $F_1^T G_1^T = I_s$ and from this we get that the first column $\boldsymbol{g}_1$ of G_1^T is unimodular, but since $r > r - s \geqslant \operatorname{sr}(S)$, then $\boldsymbol{g}_1$ is stable, and by Proposition 7, there exists $U_1 \in E_r(S)$ such that $U_1\boldsymbol{g}_1 = \boldsymbol{e}_1$. If $s = 1$, we finish since $G_1^T = \boldsymbol{g}_1$.

Let $s \geqslant 2$; we have

$$U_1 G_1^T = \begin{bmatrix} 1 & * \\ 0 & F_2 \end{bmatrix}, \; F_2 \in M_{(r-1)\times(s-1)}(S).$$

Note that $U_1 G_1^T$ has a left inverse (for instance $F_1^T U_1^{-1}$), and the form of this left inverse is

$$L = \begin{bmatrix} 1 & * \\ 0 & L_2 \end{bmatrix}, \; L_2 \in M_{(s-1)\times(r-1)}(S),$$

and hence $L_2 F_2 = I_{s-1}$. The first column of F_2 is unimodular and since $r - 1 > r - s \geqslant \operatorname{sr}(S)$ we apply again Proposition 7 and we obtain a matrix $U_2' \in E_{r-1}(S)$ such that

$$U_2' F_2 = \begin{bmatrix} 1 & * \\ 0 & F_3 \end{bmatrix}, \; F_3 \in M_{(r-2)\times(s-2)}(S).$$

Let

$$U_2 := \begin{bmatrix} 1 & 0 \\ 0 & U_2' \end{bmatrix} \in E_r(S),$$

then we have

$$U_2 U_1 G_1^T = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & F_3 \end{bmatrix}.$$

By induction on s and multiplying on the left by elementary matrices we get a matrix $U \in E_r(S)$ such that

$$UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}. \qquad \square$$

From this result we get automatically Stafford's Theorem.

Corollary 4 (Stafford)**.** *Let* $D := A_n(K)$ *or* $B_n(K)$*, with* $\operatorname{char}(K) = 0$*. Then, any stably free left* D*-module* M *satisfying* $\operatorname{rank}(M) \geqslant 2$ *is free.*

Proof. The results follows from Theorem 1 since $\operatorname{sr}(D) = 2$. $\square$

3. Matrix descriptions of Hermite rings

Rings for which all stably free modules are free have occupied special attention in homological algebra. In this section we will extend the matrix-constructive interpretation of commutative Hermite rings given in [15] to the more general case of noncommutative rings.

Definition 5. Let S be a ring.

(i) S is a projective-free (PF) ring if every f.g. projective S-module is free.

(ii) S is a PSF ring if every f.g. projective S-module is stably free.

(iii) S is a Hermite ring, property denoted by H, if any stably free S-module is free.

The right versions of the above rings (i.e., for right modules) are defined in a similar way and denoted by PF_r, PSF_r and H_r, respectively. We say that S is a $\mathcal{PF}$ ring if S is PF and PF_r simultaneously; similarly, we define the properties $\mathcal{PSF}$ and $\mathcal{H}$. However, we will prove below later that these properties are left-right symmetric, i.e., they can be denoted simply by $\mathcal{PF}$, $\mathcal{PSF}$ and $\mathcal{H}$. For domains we will write $\mathcal{PFD}$, $\mathcal{PSFD}$ and $\mathcal{HD}$.

From Definition 5 and Theorem 6 we get that

$$H \cap \ PSF = PF. \tag{4}$$

The following theorem gives a matrix description of H rings (see [8] and compare with [15] for the particular case of commutative rings. In [4] is presented a different and independent proof of this theorem for right modules).

Theorem 2. *Let S be a ring. Then, the following conditions are equivalent.*

(i) *S is H.*

(ii) *For every $r \geqslant 1$, any unimodular row matrix $\boldsymbol{u}$ over S of size $1 \times r$ can be completed to an invertible matrix of $GL_r(S)$ adding $r-1$ new rows.*

(iii) *For every $r \geqslant 1$, if $\boldsymbol{u}$ is an unimodular row matrix of size $1 \times r$, then there exists a matrix $U \in GL_r(S)$ such that $\boldsymbol{u}U = (1, 0, \ldots, 0)$.*

(iv) *For every $r \geqslant 1$, given an unimodular matrix F of size $s \times r$, $r \geqslant s$, there exists $U \in GL_r(S)$ such that*

$$FU = \begin{bmatrix} I_s & | & 0 \end{bmatrix}.$$

Proof. (i) $\Rightarrow$) (ii): Let $\boldsymbol{u} := [u_1 \cdots u_r]$ and $\boldsymbol{v} := [v_1 \cdots v_r]^T$ such that $\boldsymbol{uv} = 1$, i.e., $u_1v_1 + \cdots + u_rv_r = 1$; we define

$$\begin{aligned} S^r &\xrightarrow{\alpha} S \\ \boldsymbol{e}_i &\mapsto v_i \end{aligned}$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ is the canonical basis of the left free module S^r of columns vectors. Observe that $\alpha(\boldsymbol{u}^T) = 1$; we define the homomorphism $\beta : S \to S^r$ by $\beta(1) := \boldsymbol{u}^T$, then $\alpha\beta = i_S$. From this we get that $S^r = Im(\beta) \oplus \ker(\alpha)$, β is injective, $\langle \boldsymbol{u}^T \rangle = Im(\beta) \cong S$ and $Im(\beta)$ is free with basis $\{\boldsymbol{u}^T\}$. This implies that $S^r \cong S \oplus \ker(\alpha)$, i.e., $\ker(\alpha)$ is stably free of rank $r-1$, so by hypothesis, $\ker(\alpha)$ is free of dimension $r-1$; let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r-1}\}$ be a basis of $\ker(\alpha)$, then $\{\boldsymbol{u}^T, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r-1}\}$ is a basis of S^r. This means that $\begin{bmatrix} \boldsymbol{u}^T & \boldsymbol{x}_1 \cdots \boldsymbol{x}_{r-1} \end{bmatrix}^T \in GL_r(S)$, i.e., $\boldsymbol{u}$ can be completed to an invertible matrix of $GL_r(S)$ adding $r-1$ rows.

(ii) $\Rightarrow$) (i): Let M be an stably free S-module, then there exist integers $r, s \geqslant 0$ such that $S^r \cong S^s \oplus M$. It is enough to prove that M is free for the case when $s = 1$. In fact, $S^r \cong S^s \oplus M = S \oplus (S^{s-1} \oplus M)$ is free and hence $S^{s-1} \oplus M$ is free; repeating this reasoning we conclude that $S \oplus M$ is free, so M is free.

Let $r \geqslant 1$ such that $S^r \cong S \oplus M$, let $\pi : S^r \longrightarrow S$ be the canonical projection with kernel isomorphic to M and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ be the canonical basis of S^r; there exists $\mu : S \longrightarrow S^r$ such that $\pi\mu = i_S$ and $S^r = \ker(\pi) \oplus Im(\mu)$. Let $\mu(1) := \boldsymbol{u}^T := [u_1 \cdots u_r]^T \in S^r$, then $\pi(\boldsymbol{u}^T) = 1 = u_1\pi(\boldsymbol{e}_1) + \cdots + u_r\pi(\boldsymbol{e}_r)$, i.e., $\boldsymbol{v} := [\pi(\boldsymbol{e}_1) \cdots \pi(\boldsymbol{e}_r)]^T$ is such that $\boldsymbol{u}\boldsymbol{v} = 1$, moreover, $S^r = \ker(\pi) \oplus \langle \boldsymbol{u}^T \rangle$. By hypothesis, there exists $U \in GL_r(S)$ such that $\boldsymbol{e}_1^T U = \boldsymbol{u}$.

Let $f^T : S^r \longrightarrow S^r$ be the homomorphism defined by U^T, then $f^T(\boldsymbol{e}_1) = \boldsymbol{u}^T$ and $f^T(\boldsymbol{e}_i) = \boldsymbol{u}_i$ for $i \geqslant 2$, where $\boldsymbol{u}_2, \dots, \boldsymbol{u}_r$ are the others columns of U^T (i.e., the transpose of the other rows of U). Since $U = (U^T)^T$ then f^T is an isomorphism. If we prove that $f^T(\boldsymbol{e}_i) \in \ker(\pi)$ for each $i \geqslant 2$, then $\ker(\pi)$ is free, and consequently, M is free. In fact, let f' be the restriction of f^T to $\langle \boldsymbol{e}_2, \dots, \boldsymbol{e}_r \rangle$, i.e., $f' : \langle \boldsymbol{e}_2, \dots, \boldsymbol{e}_r \rangle \longrightarrow \ker(\pi)$. Then f' is bijective: of course f' is injective; let $\boldsymbol{w}$ be any vector of S^r, then there exists $\boldsymbol{x} \in S^r$ such that $f^T(\boldsymbol{x}) = \boldsymbol{w}$, we write $\boldsymbol{x} := [x_1 \cdots x_r]^T = x_1\boldsymbol{e}_1 + \boldsymbol{z}$, with $\boldsymbol{z} = x_2\boldsymbol{e}_2 + \cdots + x_r\boldsymbol{e}_r$. We have $f^T(\boldsymbol{x}) = f^T(x_1\boldsymbol{e}_1 + \boldsymbol{z}) = x_1 f^T(\boldsymbol{e}_1) + f^T(\boldsymbol{z}) = x_1\boldsymbol{u}^T + f^T(\boldsymbol{z}) = \boldsymbol{w}$. In particular, if $\boldsymbol{w} \in \ker(\pi)$, then $\boldsymbol{w} - f^T(\boldsymbol{z}) \in \ker(\pi) \cap \langle \boldsymbol{u}^T \rangle = 0$, so $\boldsymbol{w} = f^T(\boldsymbol{z})$ and hence $\boldsymbol{w} = f'(\boldsymbol{z})$, i.e., f' is surjective.

In order to conclude the proof we will show that $f^T(\boldsymbol{e}_i) \in \ker(\pi)$ for each $i \geqslant 2$. Since f^T was defined by U^T, the idea is to change U^T in a such way that its first column was $\boldsymbol{u}^T$ and for the others columns were $\boldsymbol{u}_i \in \ker(\pi)$, $2 \leqslant i \leqslant r$. Let $\pi(\boldsymbol{u}_i) := r_i \in S$, $i \geqslant 2$, and $\boldsymbol{u}_i' := \boldsymbol{u}_i - r_i\boldsymbol{u}^T$; then adding to column i of U^T the first column multiplied by $-r_i$ we get a new matrix U^T such that its first column is again $\boldsymbol{u}^T$ and for the others we have $\pi(\boldsymbol{u}_i') = \pi(\boldsymbol{u}_i) - r_i\pi(\boldsymbol{u}^T) = r_i - r_i = 0$, i.e., $\boldsymbol{u}_i' \in \ker(\pi)$.

(ii) $\Leftrightarrow$ (iii): $\boldsymbol{u}$ can be completed to an invertible matrix of $GL_r(S)$ if and only if there exists $V \in GL_r(S)$ such that $(1, 0, \dots, 0)V = \boldsymbol{u}$ if and only if $(1, 0, \dots, 0) = \boldsymbol{u}V^{-1}$; thus $U := V^{-1}$.

(iii) $\Rightarrow$) (iv): The proof will be done by induction on s. For $s = 1$ the result is trivial. We assume that (iv) is true for unimodular matrices with $l \leqslant s - 1$ rows. Let F be an unimodular matrix of size $s \times r, r \geqslant s$, then there exists a matrix B such that $FB = I_s$. This implies that the first row $\boldsymbol{u}$ of F is unimodular; by (iii) there exists $U' \in GL_r(S)$ such that $\boldsymbol{u}U' = (1, 0, \dots, 0) = \boldsymbol{e}_1^T$, and hence $FU' = F''$,

$$F'' = \begin{bmatrix} \boldsymbol{e}_1^T \\ F' \end{bmatrix},$$

with F' a matrix of size $(s-1) \times r$. Since $FB = I_s$, then $I_s = F''(U'^{-1}B)$, i.e., F'' is an unimodular matrix; let F''' be the matrix obtained eliminating the first column of F', then F''' is unimodular of size $(s-1) \times (r-1)$, with $r - 1 \geqslant s - 1$, since the right inverse of F'' has the form $\begin{bmatrix} * & 0 \\ * & G''' \end{bmatrix}$.

By induction, there exists a matrix $C \in GL_{r-1}(S)$ such that $F'''C = \begin{bmatrix} I_{s-1} & | & 0 \end{bmatrix}$. From this we get,

$$FU' = F'' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a'_{11} & a'_{12} & \cdots & a'_{1r} \\ \vdots & \vdots & & \vdots \\ a'_{s-11} & a'_{s-12} & \cdots & a'_{s-1r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix},$$

and hence

$$FU' \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & I_{s-1} & 0 \end{bmatrix}.$$

Multiplying the last matrix on the right by elementary matrices we get (iv).

(iv) $\Rightarrow$) (iii): Taking $s = 1$ and $F = \boldsymbol{u}$ in (iv) we get (iii). □

From the proof of the previous theorem we get the following result.

Corollary 5. *Let S be a ring. Then, S is H if and only if any stably free S-module M of type $S^r \cong S \oplus M$ is free.*

Remark 3. (a) If we consider right modules and the right S-module structure on the module S^r of columns vectors, the conditions of the previous theorem can be formulated in the following way:

(i)$^{\mathrm{r}}$ S is H_r.

(ii)$^{\mathrm{r}}$ For every $r \geqslant 1$, any unimodular column matrix $\boldsymbol{v}$ over S of size $r \times 1$ can be completed to an invertible matrix of $GL_r(S)$ adding $r - 1$ new columns.

(iii)$^{\mathrm{r}}$ For every $r \geqslant 1$, given an unimodular column matrix $\boldsymbol{v}$ over S of size $r \times 1$ there exists a matrix $U \in GL_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$.

(iv)$^{\mathrm{r}}$ For every $r \geqslant 1$, given an unimodular matrix F of size $r \times s$, $r \geqslant s$, there exists $U \in GL_r(S)$ such that

$$UF = \begin{bmatrix} I_s \\ 0 \end{bmatrix}.$$

The proof is as in the commutative case, see [15]. Corollary 5 can be formulated in this case as follows: S is H_r if and only if any stably free right S-module M of type $S^r \cong S \oplus M$ is free.

(b) Considering again left modules and disposing the matrices of homomorphisms by rows and composing homomorphisms from the left to the right (see Remark 1), we can repeat the proof of Theorem 2 and obtain the equivalence of conditions (i)-(iv). With this notation we do not need to take transposes in the proof of Theorem 2.

(c) If S is a commutative ring, of course, left and right conditions are equivalent, see [15]. This follows from the fact that $(FG)^T = G^T F^T$ for any matrices $F \in M_{r\times s}(S), G \in M_{s\times r}(S)$. However, as we remarked before, the Hermite condition is left-right symmetric for general rings (Proposition 9). Another independent proof of this fact can be found in [4], Theorem 11.4.4.

4. Matrix characterization of PF rings

In [8] are given some matrix characterizations of projective-free rings; in this section we present another matrix interpretation of this important class of rings. The main result presented here (Corollary 7) extends Theorem 6.2.2 in [15]. This result has been proved independently also in [4], Proposition 11.4.9.

Theorem 3. *Let S be a Hermite ring and M a f.g. projective module given by the column module of a matrix $F \in M_s(S)$, with F^T idempotent. Then, M is free with $dim(M) = r$ if and only if there exists a matrix $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and*

$$(U^T)^{-1}F^TU^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T . \tag{5}$$

In such case, a basis of M is given by the last r rows of $(U^T)^{-1}$.

Proof. $\Rightarrow$): As in the proof of Proposition 1, let $f : S^s \to S^s$ be the homomorphism defined by F and $S^s = M \oplus M'$ with $Im(f) = M$ and $M' = \ker(f)$; by the hypothesis M es free with dimension r, so $r \leqslant s$ (recall that S is $\mathcal{RC}$). Let $h : M \to S^r$ an isomorphism and $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_r\} \subset M$ such that $h(\boldsymbol{z}_i) = \boldsymbol{e}_i$, $1 \leqslant i \leqslant r$, then $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_r\}$ is a basis of M. Since S is an Hermite ring, M' is free, let $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_{s-r}\}$ be a basis of M' (recall that S is $\mathcal{IBN}$). Then $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_{s-r}; \boldsymbol{z}_1, \dots, \boldsymbol{z}_r\}$ is a basis for S^s. With this we define u in the following way:

$$u(\boldsymbol{w}_j) := \boldsymbol{e}_j, \text{ for } 1 \leqslant j \leqslant s-r,$$
$$u(\boldsymbol{z}_i) := \boldsymbol{e}_{s-r+i}, \text{ for } 1 \leqslant i \leqslant r.$$

Note that u is an isomorphism and we get that $uf = t_0 u$, where t is given by $t_0(\boldsymbol{e}_j) := \boldsymbol{0}$ if $1 \leqslant j \leqslant s-r$, and $t_0(\boldsymbol{e}_{s-r+i}) = \boldsymbol{e}_{s-r+i}$ if $1 \leqslant i \leqslant r$; thus, the matrix of t_0 in the canonical basis is

$$T_0 = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}.$$

Thus, $F^T U^T = U^T T_0^T$; note that $(U^T)^{-1}$ exists since u is an isomorphism, hence $(U^T)^{-1} F^T U^T = T_0^T$. From $u(\boldsymbol{z}_i) := \boldsymbol{e}_{s-r+i}$ we get that $(\boldsymbol{z}_i^T U^T)^T = \boldsymbol{e}_{s-r+i}$, so $\boldsymbol{z}_i^T U^T = \boldsymbol{e}_{s-r+i}^T$ and hence $\boldsymbol{z}_i^T = \boldsymbol{e}_{s-r+i}^T (U^T)^{-1}$, i.e., the basis of M coincides with the last r rows of $(U^T)^{-1}$.

$\Leftarrow$): Let f, u be the homomorphisms defined by F and U, then $m(uf) = m(t_0 u)$, where t_0 is the homomorphism defined by T_0, this means that $uf = t_0 u$, but by the hypothesis U^T is invertible, so u is an isomorphism; from this we conclude that $Im(f) \cong Im(t_0)$, i.e., $M = Im(f) \cong Im(t_0) = \langle T_0 \rangle \cong S^r$. Note that this part of the proof does not use that S is an Hermite ring. $\square$

From the previous theorem we get the following matrix description of PF rings.

Corollary 6. *Let S be a ring. S is PF if and only if for each $s \geqslant 1$, given a matrix $F \in M_s(S)$, with F^T idempotent, there exists a matrix $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and*

$$(U^T)^{-1} F^T U^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T, \tag{6}$$

where $r = dim(\langle F \rangle)$, $0 \leqslant r \leqslant s$.

Proof. $\Rightarrow$): Let $F \in M_s(S)$, with F^T idempotent, and let M be the S-module generated by the columns of F. By Proposition 1, M is a f.g. projective module, and by the hypothesis, M is free. Since S is H, we can apply Theorem 3. If $r = \dim(M)$, then $r = dim(\langle F \rangle)$.

$\Leftarrow$): Let M be a finitely generated projective S-module, so there exists $s \geqslant 1$ such that $S^s = M \oplus M'$; let $S^s \xrightarrow{f} S^s$ be the canonical projection on M, so F^T is idempotent and, by the hypothesis, there exists $U \in M_s(S)$ such that $U^T \in GL_s(S)$ and (6) holds. From the second part of the proof of Theorem 3 we get that M is free. $\square$

Remark 4. (i) If we consider right modules instead of left modules, then the previous corollary can be reformulated in the following way: S is PF_r

if and only if for each $s \geqslant 1$, given an idempotent matrix $F \in M_s(S)$, there exists a matrix $U \in GL_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{7}$$

where $r = dim(\langle F \rangle)$, $0 \leqslant r \leqslant s$, and $\langle F \rangle$ represents the right S-module generated by the columns of F. The proof is as in the commutative case, see [15].

(ii) Considering again left modules and disposing the matrices of homomorphisms by rows and composing homomorphisms from the left to the right (see Remark 1), we can repeat the proofs of Theorem 3 and Corollary 6 and get the characterization (7) for the PF property; with this row notation we do not need to take transposes in the proofs. However, observe that in this case $\langle F \rangle$ represents the left S-module generated by the rows of F. Note that Corollary 6 could has been formulated this way: In fact,

$$\begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$

and we can rewrite (6) as (7) changing F^T by F (see Remark 1) and $(U^T)^{-1}$ by U.

(iii) If S is a commutative ring, of course $PF = PF_r = \mathcal{PF}$. However, we will prove in Corollary 8 that the projective-free property is left-right symmetric for general rings.

Corollary 7. *S is PF if and only if for each $s \geqslant 1$, given an idempotent matrix $F \in M_s(S)$, there exists a matrix $U \in GL_s(S)$ such that*

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{8}$$

where $r = dim(\langle F \rangle)$, $0 \leqslant r \leqslant s$, and $\langle F \rangle$ represents the left S-module generated by the rows of F.

Proof. This is the content of the part (ii) in the previous remark. □

Corollary 8. *Let S be a ring. S is PF if and only if S is PF_r, i.e., $PF = PF_r = \mathcal{PF}$.*

Proof. Let $F \in M_s(S)$ be an idempotent matrix, if S is PF, then there exists $U \in GL_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

where r is the dimension of the left S-module generated by the rows of F. Observe that UFU^{-1} is also idempotent, moreover, the matrices $X := UF$ and $Y := U^{-1}$ satisfy $UFU^{-1} = XY$ and $F = YX$, then from Proposition 0.3.1 in [8] we conclude that the left S-module generated by the rows of UFU^{-1} coincides with the left S-module generated by the rows of F, and also, the right S-module generated by the columns of UFU^{-1} coincides with the right S-module generated by the columns of F. This implies that the left S-module generated by the rows of F coincides with the right S-module generated by the columns of F. Thus, S is PF_r. The symmetry of the problem completes the proof. □

Another interesting matrix characterization of $\mathcal{PF}$ rings is given in [8], Proposition 0.4.7: a ring S is $\mathcal{PF}$ if and only if given an idempotent matrix $F \in M_s(S)$ there exist matrices $X \in M_{s\times r}(S), Y \in M_{r\times s}(S)$ such that $F = XY$ and $YX = I_r$. Note that from this characterization we get again that $PF = PF_r = \mathcal{PF}$.

A similar matrix interpretation can be given for PSF rings using Proposition 0.3.1 in [8] and Corollary 2.

Proposition 8. *Let S be a ring. Then,*

(i) *S is PSF if and only if given an idempotent matrix $F \in M_r(S)$ there exist $s \geqslant 0$ and matrices $X \in M_{(r+s)\times r}(S), Y \in M_{r\times(r+s)}(S)$ such that*

$$\begin{bmatrix} F & 0 \\ 0 & I_s \end{bmatrix} = XY \text{ and } YX = I_r.$$

(ii) *$PSF = PSF_r = \mathcal{PSF}$.*

Proof. Direct consequence of Proposition 0.3.1 in [8] and Corollary 2. □

For the H property we have a similar characterization that proves the symmetry of this condition.

Proposition 9. *Let S be a ring. Then,*

(i) *S is H if and only if given an idempotent matrix $F \in M_r(S)$ with factorization*

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} = XY \text{ and } YX = I_r, \text{ for some matrices}$$
$$X \in M_{(r+1)\times r}(S), Y \in M_{r\times(r+1)}(S),$$

there exist matrices $X' \in M_{r\times(r-1)}(S), Y' \in M_{(r-1)\times r}(S)$ *such that* $F = X'Y'$ *and* $Y'X' = I_{r-1}$.

(ii) $H = H_r = \mathcal{H}$.

Proof. Direct consequence of Propositions 0.3.1 and 0.4.7 in [8], and Corollary 5. □

We conclude this section given a matrix constructive proof of a well known Kaplansky's theorem.

Proposition 10. *Any local ring S is $\mathcal{PF}$.*

Proof. Let M a projective left S-module. By Remark 1, part (ii), there exists an idempotent matrix $F = [f_{ij}] \in M_s(S)$ such that the module generated by the rows of F coincides with M. According to Corollary 7, we need to show that there exists $U \in GL_s(S)$ such that the relation (8) holds. The proof is by induction on s.

$s = 1$: In this case $F = [f_{ij}] = [f]$; since S is local, its idempotents are trivial, then $f = 1$ or $f = 0$ and hence M is free.

$s = 2$: In view of fact that S is local, two possibilities arise:

f_{11} is invertible. One can find $G \in GL_2(S)$ such that $GFG^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}$, for some $f \in S$. In fact, in this case note that

$$G := \begin{bmatrix} 1 & f_{11}^{-1}f_{12} \\ -f_{21}f_{11}^{-1} & 1 \end{bmatrix} \in GL_2(S) \text{ with}$$
$$G^{-1} = \begin{bmatrix} f_{11} & -f_{12} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12} + 1 \end{bmatrix}.$$

Since F is idempotent, f so is; applying the case $s = 1$ we get the result.

$1 - f_{11}$ is invertible. In the same way, we find $H \in GL_2(S)$ such that $HFH^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}$, we take

$$H := \begin{bmatrix} 1 & -(1-f_{11})^{-1}f_{12} \\ f_{21} & -f_{21}(1-f_{11})^{-1}f_{12} + 1 \end{bmatrix} \in GL_2(S) \text{ with}$$

$$H^{-1} = \begin{bmatrix} 1-f_{11} & (1-f_{11})^{-1}f_{12} \\ -f_{21} & 1 \end{bmatrix}.$$

Note that g is an idempotent of S, then $g = 0$ or $g = 1$ and the statement follows.

Now suppose that the result holds for $s-1$; considering both possibilities for f_{11} we have:

If f_{11} is invertible, taking

$$G := \begin{bmatrix} 1 & f_{11}^{-1}f_{12} & f_{11}^{-1}f_{13} & \cdots & f_{11}^{-1}f_{1s} \\ -f_{21}f_{11}^{-1} & 1 & 0 & \cdots & 0 \\ -f_{31}f_{11}^{-1} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \\ -f_{s1}f_{11}^{-1} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

we have that $G \in GL_s(S)$ and its inverse is:

$$G^{-1} = \begin{bmatrix} f_{11} & -f_{12} & -f_{13} & \cdots & -f_{1s} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12}+1 & -f_{21}f_{11}^{-1}f_{13} & \cdots & -f_{21}f_{11}^{-1}f_{1s} \\ f_{31} & -f_{31}f_{11}^{-1}f_{12} & -f_{31}f_{11}^{-1}f_{13}+1 & \cdots & -f_{31}f_{11}^{-1}f_{1s} \\ \vdots & & & \vdots & \\ f_{s1} & -f_{s1}f_{11}^{-1}f_{12} & -f_{s1}f_{11}^{-1}f_{13} & \cdots & -f_{s1}f_{11}^{-1}f_{1s}+1 \end{bmatrix}.$$

Moreover, $GFG^{-1} = \begin{bmatrix} 1 & 0_{1,s-1} \\ 0_{s-1,1} & F_1 \end{bmatrix}$ where $F_1 \in M_{s-1}(S)$ is an idempotent matrix. Only remains to apply the induction hypothesis.

If $1-f_{11}$ is invertible, taking

$$H := \begin{bmatrix} 1 & -(1-f_{11})^{-1}f_{12} & -(1-f_{11})^{-1}f_{13} & \cdots & -(1-f_{11})^{-1}f_{1s} \\ f_{21} & -f_{21}(1-f_{11})^{-1}f_{12}+1 & -f_{21}(1-f_{11})^{-1}f_{13} & \cdots & -f_{21}(1-f_{11})^{-1}f_{1s} \\ f_{31} & -f_{31}(1-f_{11})^{-1}f_{12} & -f_{31}(1-f_{11})^{-1}f_{13}+1 & \cdots & -f_{31}(1-f_{11})^{-1}f_{1s} \\ \vdots & & & \cdots & \\ f_{s1} & -f_{s1}(1-f_{11})^{-1}f_{12} & -f_{s1}(1-f_{11})^{-1}f_{13} & \cdots & -f_{s1}(1-f_{11})^{-1}f_{1s}+1 \end{bmatrix}$$

we have that $H \in GL_s(S)$ with inverse given by:

$$H^{-1} = \begin{bmatrix} 1-f_{11} & (1-f_{11})^{-1}f_{12} & (1-f_{11})^{-1}f_{13} & \cdots & (1-f_{11})^{-1}f_{1s} \\ -f_{21} & 1 & 0 & \cdots & 0 \\ -f_{31} & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \\ -f_{s1} & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and also $HFH^{-1} = \begin{bmatrix} 0 & 0_{1,s-1} \\ 0_{s-1,1} & F_2 \end{bmatrix}$ with $F_2 \in M_{s-1}(S)$ an idempotent matrix. One more time we apply the induction hypothesis. $\square$

5. Some important subclasses of Hermite rings

There are some other classes of rings close related to Hermite rings (see [8], [11], [12] and [25]) that were studied in [15] for commutative rings, now we will consider the general noncommutative situation. Some proofs are easy adaptable from [15] and hence we omit them.

Definition 6. Let S be a ring.

(i) S is an elementary divisor ring ($\mathcal{ED}$) if for any $r, s \geqslant 1$, given a rectangular matrix $F \in M_{r\times s}(S)$ there exist invertible matrices $P \in GL_r(S)$ and $Q \in GL_s(S)$ such that PFQ is a Smith normal diagonal matrix, i.e., there exist $d_1, d_2, \dots, d_l \in S$, with $l = \min\{r, s\}$, such that

$$PFQ = \operatorname{diag}(d_1, d_2, \dots, d_l), \text{ with } Sd_{i+1}S \subseteq Sd_i \cap d_iS \text{ for } 1 \leqslant i \leqslant l,$$

where SdS denotes the two-sided ideal generated by d.

(ii) S is an $\mathcal{ID}$ ring if for any $s \geqslant 1$, given an idempotent matrix $F \in M_s(S)$ there exists an invertible matrix $P \in GL_s(S)$ such that PFP^{-1} is a Smith normal diagonal matrix.

(iii) S is a left K-Hermite ring (KH) if given $a, b \in S$ there exist $U \in GL_2(S)$ and $d \in S$ such that $U\begin{bmatrix} a & b\end{bmatrix}^T = \begin{bmatrix} d & 0\end{bmatrix}^T$. S is a right K-Hermite ring (KH_r) if $\begin{bmatrix} a & b\end{bmatrix}U = \begin{bmatrix} d & 0\end{bmatrix}$. The ring S is $\mathcal{KH}$ if S is KH and KH_r.

(iv) S is a left Bézout ring (B) if every f.g. left ideal of S is principal. S is a right Bézout ring (B_r) if every f.g. right ideal of S is principal. S is a $\mathcal{B}$ ring if S is B and B_r.

(v) S is a left cancellable ring (C) if for any f.g. projective left S-modules P, P' holds: $P \oplus S \cong P' \oplus S \Leftrightarrow P \cong P'$. S is right cancellable (C_r) if for any f.g. projective right S-modules P, P' holds: $P \oplus S \cong P' \oplus S \Leftrightarrow P \cong P'$. S is cancellable ($\mathcal{C}$) if S is (C) and (C_r).

From Proposition 0.3.1 of [8] it is easy to give a matrix interpretation of C rings, and also, we can deduce that $C = C_r = \mathcal{C}$.

Proposition 11. *Let S be a ring. Then,*

(i) *S is C if and only if given idempotent matrices $F \in M_s(S)$, $G \in M_r(S)$ the following statement is true: The matrices*

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized as

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} = X'Y', \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} = Y'X', \text{ for some matrices}$$
$$X' \in M_{(s+1)\times(r+1)}(S),\ Y' \in M_{(r+1)\times(s+1)}(S)$$

if and only if $F = XY$, $G = YX$, *for some matrices* $X \in M_s(S)$, $Y \in M_r(S)$.

(ii) $C = C_r = \mathcal{C}$.

Proof. Direct consequence of Proposition 0.3.1 in [8]. □

For domains, the above classes of rings are denoted by $\mathcal{EDD}$, $\mathcal{IDD}$, KHD, KHD_r, $\mathcal{KHD}$, BD, BD_r, $\mathcal{BD}$ and $\mathcal{CD}$, respectively.

Proposition 12. (i) $\mathcal{ED} \subseteq KH \subseteq B$.

(ii) $KHD = BD \subseteq \mathcal{PFD}$.

(iii) $\mathcal{PF} \subseteq \mathcal{ID}$

(iv)$\mathcal{ID} = \mathcal{PF}$ *for rings without nontrivial idempotents. In particular,* $\mathcal{IDD} = \mathcal{PFD}$.

(v) $\mathcal{PF} \subseteq \mathcal{C} \subseteq \mathcal{H}$.

Similar relations are valid for $KH_r, \mathcal{KH}, B_r$ *and* $\mathcal{B}$.

Proof. (i) It is clear that $\mathcal{ED} \subseteq KH$. Let $a, b \in S$, we want to proof that any left ideal $Sa + Sb$ is principal. There exist $U \in GL_2(S)$ and $d \in S$ such that $U \begin{bmatrix} a & b \end{bmatrix}^T = \begin{bmatrix} d & 0 \end{bmatrix}^T$, this implies that $Sd \subseteq Sa + Sb$, but since $\begin{bmatrix} a & b \end{bmatrix}^T = U^{-1} \begin{bmatrix} d & 0 \end{bmatrix}^T$, then $Sa + S \subseteq Sd$. This proved that $KH \subseteq B$.

(ii) The equality $KHD = BD$ was proved by Amitsur in [1], Theorem 1.4. The proof of the inclusion $BD \subseteq \mathcal{PFD}$ is as in the commutative case, see [15], Example 6.1.2.

(iii) Using permutation matrices it is clear that $\mathcal{PF} \subseteq \mathcal{ID}$ (see Corollary 7).

(iv) Let S be an $\mathcal{ID}$ ring and let $F = [f_{ij}] \in M_s(S)$ be an idempotent matrix over S; by the hypothesis, there exists $P \in GL_s(S)$ such that PFP^{-1} is diagonal, let $D := PFP^{-1} = \operatorname{diag}(d_1, d_2, \ldots, d_s)$; since PFP^{-1}

is idempotent, then each d_i is idempotent, so $d_i = 0$ or $d_i = 1$ for each $1 \leqslant i \leqslant s$. By permutation matrices we can assume that

$$PFP^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

in addition, note that r is the dimension of the left S-module generated by the rows of F. Then, S is $\mathcal{PF}$.

(v) Let P, P' be f.g. S-modules such that $P \oplus S \cong P' \oplus S$; since S is $\mathcal{PF}$ there exists n, n' such that $P \cong S^n$, $P' \cong S^{n'}$ and hence $S^n \oplus S \cong S^{n'} \oplus S$, so $n + 1 = n' + 1$, i.e., $P \cong P'$.

Let now M be a stably free module, $M \oplus S^s \cong S^r$, since $r \geqslant s$ and S is left cancellable, then $M \cong S^{r-s}$. $\square$

Remark 5. (a) $\mathbb{Z}_6$ shows that $\mathcal{PF} \neq \mathcal{H}$ and also that $\mathcal{PSF} \neq \mathcal{H}$ (see [15], Example 6.1.2). On the other hand, note that if K is a field, then $K[x, y]$ is $\mathcal{PFD}$ but is not $\mathcal{BD}$. Thus, $B \neq \mathcal{PF}$, and consequently, $B \neq \mathcal{H}$, $KH \neq \mathcal{H}$, $\mathcal{ED} \neq \mathcal{H}$. $\mathbb{Z}[\sqrt{-5}]$ shows that $\mathcal{ID} \neq \mathcal{H}$, see [15], Example 6.6.1 and Remark 6.7.14.

(b) In [8] P.M. Cohn asks if there exist examples of $\mathcal{H}$ rings that are not $\mathcal{C}$ rings, in other words, $\mathcal{C} \neq \mathcal{H}$ is probably still an open problem.

(c) It is well known that $B \neq B_r$, a classical example is given by the skew polynomial ring $T[x; \sigma]$, where T is a division ring a σ is an endomorphism of T that is not automorphism. Every left ideal of this ring is principal, hence, it is a left Bézout ring; but if $a \notin \sigma(T)$, then the right ideal generated by x and ax is not principal. This example shows also that $KH \neq KH_r$.

(d) From proposition 12 we conclude that for domains the following inclusions hold:

$$\mathcal{EDD} \subseteq KHD = BD \subseteq \mathcal{PFD} = \mathcal{IDD} \subseteq \mathcal{CD} \subseteq \mathcal{HD}. \tag{9}$$

Similar relations are valid for the right side.

(e) Many authors have considered additional relations between these classes of rings in some particular cases, for example, for commutative rings $\mathcal{KH} \subseteq \mathcal{H}$ and $\mathcal{ID} \subseteq \mathcal{H}$ ([12], [16], [23]). In [25] and [26] are analyzed many cases for which B coincides with KH, see also [10] and [21]. Kaplansky has conjectured that for commutative domains, $\mathcal{BD} = \mathcal{EDD}$ (see [11]). This conjecture probably has not been solved yet.

Next we will see that $\mathcal{ID} \subseteq \mathcal{H}$ in rings for which all idempotents are central.

Proposition 13. *Let S be a ring such that all idempotents are central. Then the following conditions are equivalent:*

(i) *S is $\mathcal{ID}$.*

(ii) *Any idempotent matrix over S is similar to a diagonal matrix.*

(iii) *Given an idempotent matrix $F \in M_r(S)$ there exists an unimodular vector $\boldsymbol{v} = [v_1, \ldots, v_r]^T$ over S and an invertible matrix $U \in GL_r(S)$ such that $U\boldsymbol{v} = \boldsymbol{e}_1$ and $F\boldsymbol{v} = a\boldsymbol{v}$, for some $a \in S$.*

Proof. The proof is an easy adaptation of the commutative case, see [15], Proposition 6.3.2. □

Theorem 4. *Let S be a ring such that all idempotents are central. Then, $\mathcal{ID} \subseteq \mathcal{H}$.*

Proof. We start with the following remark: If $\boldsymbol{u}$ is an unimodular row of size $1 \times r$ over S and $P \in GL_r(S)$, then $\boldsymbol{u}$ is completable to an invertible matrix if and only if $\boldsymbol{u}P$ is completable.

Let $\boldsymbol{u} = [u_1 \ \cdots \ u_r]$ be an unimodular row matrix of size $1 \times r$, there exists $\boldsymbol{v} = [v_1 \ \cdots \ v_r]^T$ such that $u_1v_1 + \cdots + u_rv_r = 1$; we consider the matrix $F = [f_{ij}] \in M_r(S)$, with $f_{ij} := v_iu_j$, $1 \leqslant i, j \leqslant r$. Note that $F^2 = F$; by the hypothesis there exists $P \in GL_r(S)$ such that PFP^{-1} is diagonal, let $D := PFP^{-1} = \operatorname{diag}(d_1, d_2, \ldots, d_r)$; since PFP^{-1} is idempotent, then each d_i is idempotent. Let $\boldsymbol{w} := \boldsymbol{u}P^{-1}$ and $\boldsymbol{x} := P\boldsymbol{v}$, then $\boldsymbol{wx} = \boldsymbol{u}P^{-1}P\boldsymbol{v} = 1$ and $\boldsymbol{xw} = P\boldsymbol{vu}P^{-1} = PFP^{-1} = D$. By the above remark, $\boldsymbol{u}$ is completable if and only if $\boldsymbol{w}$ is. Thus, we will show that $\boldsymbol{w}$ is completable. From $\boldsymbol{xw} = D$ we obtain that $x_iw_i = d_i$ is idempotent for all $1 \leqslant i \leqslant r$ and $x_iw_j = 0$ for $i \neq j$. But $\sum_{k=1}^{r} w_ix_i = 1$, then $w_i = w_ix_iw_i$ and $x_i = x_iw_ix_i$. Let $f_i := w_ix_i$ for $1 \leqslant i \leqslant r$, hence each f_i is idempotent. By the hypothesis d_i, f_i are central, then $d_i = d_i^2 = x_if_iw_i = f_id_i$ and $f_i = f_i^2 = d_if_i$, so that $d_i = f_i$ and $x_iw_i = w_ix_i$ for $1 \leqslant i \leqslant r$. Therefore, $(\sum_{i=1}^{r} x_i)(\sum_{i=1}^{r} w_i) = 1$, hence $c := \sum_{i=1}^{r} w_i$ is left invertible, $c'c = 1$. Observe that cc' is idempotent, so central, and by the hypothesis there exists $x \in S^*$ such that $xcc'x^{-1} = d$, with $d \in S$ idempotent, from this we get that $cc' = d$ and $c' = c'd$, i.e., $c'(1-d) = 0$, so $(1-d)c' = 0$ and consequently $1 - d = 0$, i.e, $cc' = 1$. This means that c is invertible. Hence, the matrix

$$V := \begin{bmatrix} w_1 & w_2 & w_3 & \cdots & w_r \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

is invertible, i.e., $\boldsymbol{w}$ is completable. □

6. Products, quotients and localizations

Next we will study the properties introduced in Definition 6 with respect to some algebraic standard constructions. $Rad(S)$ represents the Jacobson radical of the ring S and S^* the group of units of S.

Proposition 14. *Let S be a ring and $I \subseteq Rad(S)$ an ideal of S. Let $\{S_i\}_{i\in\mathcal{C}}$ be a family of rings. Then,*

(i) *S is $\mathcal{H}$ if and only if S/I is $\mathcal{H}$.*

(ii) *$\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{H}$ if and only if each S_i is $\mathcal{H}$.*

(iii) *If $\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{PF}$, then each S_i is $\mathcal{PF}$.*

(iv) *If S is $\mathcal{ED}$, then S/I is $\mathcal{ED}$ for any proper ideal I of S.*

(v) *$\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{ED}$ if and only if each S_i is $\mathcal{ED}$.*

(vi) *If S is B, then S/I is B for any proper ideal I of S which is f.g. as left ideal.*

(vii) *$\prod_{i\in\mathcal{C}} S_i$ is B if and only if each S_i is B.*

(viii) *Suppose that in S all idempotents are central and I is a nilideal. If S/I is $\mathcal{ID}$, then S is $\mathcal{ID}$.*

(ix) *$\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{ID}$ if and only if each S_i is $\mathcal{ID}$.*

(x) *If S is KH, then S/I is KH for any proper ideal I of S.*

(xi) *$\prod_{i\in\mathcal{C}} S_i$ is KH if and only if each S_i is KH.*

(xii) *$\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{C}$ if and only if each S_i is $\mathcal{C}$.*

(xiii) *If $\prod_{i\in\mathcal{C}} S_i$ is $\mathcal{PSF}$, then each S_i is $\mathcal{PSF}$.*

Similar relations are valid for the right side.

Proof. Some proofs can be adapted from the commutative case (see [15]) or can be get directly from the definition. We include only the proof of (viii) and (xii): First note that if $\overline{S} := S/I$, then $U := [u_{ij}] \in GL_r(S)$

if and only if $\overline{U} = [\overline{u_{ij}}] \in GL_r(\overline{S})$. Moreover, let $B := \prod_{i\in\mathcal{C}} S_i$, then $M_s(B) \cong \prod_{i\in\mathcal{C}} M_s(S_i)$, where the isomorphism is defined by $F \mapsto (F^{(i)})$, with $F = [f_{uv}]$, $f_{uv} = (f_{uv}^{(i)})$, $F^{(i)} = [f_{uv}^{(i)}]$. From this we obtain that $M_s(B)^* = GL_s(B) \cong \prod_{i\in\mathcal{C}} GL_s(S_i) = \prod_{i\in\mathcal{C}} M_s(S_i)^*$.

(viii) Let $F \in M_s(S)$ be an idempotent matrix, then $\overline{F} \in M_s(\overline{S})$ is idempotent and there exists $\overline{P} \in GL_s(\overline{S})$ such that

$$\overline{D} = \overline{P}\ \overline{F}\ (\overline{P})^{-1} = \operatorname{diag}(\overline{d_1}, \ldots, \overline{d_r}), \text{ with } \overline{S}\ \overline{d_{i+1}}\ \overline{S} \subseteq \overline{S}\ \overline{d_i} \cap \overline{d_i}\ \overline{S}.$$

Note that $\overline{D}$ is idempotent, so each $\overline{d_i}$ is idempotent, $1 \leqslant i \leqslant r$; let $\overline{d} := \overline{d_1} \cdots \overline{d_r}$, then $\overline{d}^2 = \overline{d}$. Since I is nilideal we can assume that d is idempotent (see [13]), and hence, central; moreover since each $\overline{d_i}$ is central, $\overline{d_i} | \overline{d_{i+1}}$, and then $\overline{d} = \overline{d_r}$ (this can be easy prove by induction on r). Note that $\overline{D}\overline{\boldsymbol{e}_r} = \overline{d}\overline{\boldsymbol{e}_r}$, so $\overline{F}\overline{\boldsymbol{v}} = \overline{d}\overline{\boldsymbol{v}}$, with $\overline{\boldsymbol{v}} := (\overline{P})^{-1}\overline{\boldsymbol{e}_r}$ unimodular over $\overline{S}$, and hence, $\boldsymbol{v}$ is unimodular over S. Moreover, there exists $V \in GL_r(S)$ such that $V\boldsymbol{v} = \boldsymbol{e}_1$. In fact, we have $\boldsymbol{v} - P^{-1}\boldsymbol{e}_r = \boldsymbol{u} = [u_1, \ldots, u_r]^T$, with $u_i \in Rad(S)$, $1 \leqslant i \leqslant r$. Then, $\boldsymbol{v} = P^{-1}\boldsymbol{e}_r + \boldsymbol{u}$, and hence, $P\boldsymbol{v} = \boldsymbol{e}_r + P\boldsymbol{u}$ is a column matrix with the last component invertible, so multiplying by elementary and permutation matrices we get $V \in GL_r(S)$ such that $V\boldsymbol{v} = \boldsymbol{e}_1$.

We have $F\boldsymbol{v} = d\boldsymbol{v} + \boldsymbol{z}$, with $\boldsymbol{z} = [z_1, \ldots, z_r]^T$, $z_i \in Rad(S)$, $1 \leqslant i \leqslant r$. From this we get that $F^2\boldsymbol{v} = F\boldsymbol{v} = dF\boldsymbol{v} + F\boldsymbol{z}$, so $F\boldsymbol{z} = (1-d)F\boldsymbol{v} = (1-d)(d\boldsymbol{v} + \boldsymbol{z}) = (1-d)\boldsymbol{z}$ since $(1-d)d = 0$. Then, $F(\boldsymbol{v} + (2d-1)\boldsymbol{z}) = F\boldsymbol{v} + (2d-1)F\boldsymbol{z} = d\boldsymbol{v} + \boldsymbol{z} + (2d-1)(1-d)\boldsymbol{z} = d\boldsymbol{v} + d\boldsymbol{z} = d(\boldsymbol{v} + (2d-1)\boldsymbol{z})$. Thus, given the idempotent matrix F we have found a vector $\boldsymbol{w} := \boldsymbol{v} + (2d-1)\boldsymbol{z}$ and an element $d \in S$ such that $F\boldsymbol{w} = d\boldsymbol{w}$, moreover $\boldsymbol{w}$ is unimodular since $\boldsymbol{v}$ is unimodular and $z_i \in Rad(S)$, $1 \leqslant i \leqslant r$. In addition, the first component of the vector $V\boldsymbol{w} = \boldsymbol{e}_1 + V(2d-1)\boldsymbol{z}$ is invertible, so by elementary operations we found a matrix $W \in GL_r(S)$ such that $W\boldsymbol{w} = \boldsymbol{e}_1$. From Proposition 13 we get that S is an $\mathcal{ID}$ ring.

(xii) $\Rightarrow$): We will apply Proposition 11. Let $k \in \mathcal{C}$ and $F^{(k)} = [f_{uv}^{(k)}] \in M_s(S_k)$, $G^{(k)} = [g_{uv}^{(k)}] \in M_r(S_k)$ idempotent matrices, then $F \in M_s(B)$, $G \in M_r(B)$ are idempotent, where $F = [f_{uv}], G = [g_{uv}]$, with $f_{uv} = (f_{uv}^{(i)}), g_{uv} = (g_{uv}^{(i)})$ and $f_{uv}^{(i)} = 0 = g_{uv}^{(i)}$ for $i \neq k$. Since B is a $\mathcal{C}$ ring, the enlarged matrices

$$\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized as in Proposition 11 if and only if the matrices F, G can be factorized. This implies that the matrices

$$\begin{bmatrix} F^{(k)} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} G^{(k)} & 0 \\ 0 & 1 \end{bmatrix}$$

can be factorized if and only if the matrices $F^{(k)}, G^{(k)}$ can be factorized. This proves that S_k is a $\mathcal{C}$ ring.

$\Leftarrow$): Let $F = [f_{uv}] \in M_s(B), G = [g_{uv}] \in M_r(B)$ be idempotent matrices, with $f_{uv} = (f_{uv}^{(k)}), g_{uv} = (g_{uv}^{(k)})$, $f_{uv}^{(k)}, g_{uv}^{(k)} \in S_k$; since each ring S_k is $\mathcal{C}$, we can repeat the previous reasoning, but in the inverse order, and conclude that B is a $\mathcal{C}$ ring. $\square$

Now we will consider the localizations of some classes of rings introduced in Definition 6.

Proposition 15. *Let S be a ring and T a multiplicative system of S such that $T^{-1}S$ exits. If S is $\mathcal{ED}$ (KH, B), then $T^{-1}S$ is $\mathcal{ED}$ (KH, B). Similar properties are valid for the right side.*

Proof. Let S a $\mathcal{ED}$ ring and $F \in M_{r\times s}(T^{-1}S)$, then $F = [f_{ij}]$ with $f_{ij} = t_{ij}^{-1}s_{ij}$, where $t_{ij} \in T$ and $s_{ij} \in S$, for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. By Proposition 2.1.16 in [17], there exist $t \in T$ and $l_{ij} \in S$ such that $f_{ij} = t^{-1}l_{ij}$, then $tF = [l_{ij}] \in M_{r\times s}(S)$, hence tF admits a diagonal reduction, i.e., there exist $P \in GL_r(S)$ and $Q \in GL_s(S)$ such that $P(tF)Q = \operatorname{diag}(d_1, \dots, d_l)$, with $d_1, \dots, d_l \in S, l = \min\{r, s\}$ and $Sd_{i+1}S \subseteq Sd_i \cap d_iS$. Note that $Pt, Q \in GL_r(T^{-1}S)$. Thus, $(Pt)FQ = P(tF)Q = D$, moreover, $T^{-1}Sd_{i+1}T^{-1}S \subseteq T^{-1}Sd_i \cap d_iT^{-1}S$. This proves that $T^{-1}S$ is $\mathcal{ED}$.

The proof for KH is completely analogous.

Suppose now that S is a B ring and let J be a f.g. left ideal of $T^{-1}S$, then $J = \langle q_1, \dots, q_r\}$ where $q_i = t_i^{-1}s_i$ with $t_i \in T$ and $s_i \in S$ for $1 \leqslant i \leqslant r$. Let $t \in T$ and $a_i \in S$ such that $q_i = t^{-1}q_i$, then $tq_i = a_i$. Therefore, $J' := T^{-1}S\frac{a_1}{1} + \cdots + T^{-1}S\frac{a_r}{1} \subseteq J$; but $J \subseteq J'$: in fact, let $x = \frac{b_1}{t_1}q_1 + \cdots + \frac{b_r}{t_r}q_r \in J$, then $x = t_1^{-1}b_1t^{-1}\frac{a_1}{1} + \cdots + t_r^{-1}b_rt^{-1}\frac{a_r}{1}$; since $b_it^{-1} \in T^{-1}S$ exist, $b_i' \in S$ and $l_i \in T$ such that $b_it^{-1} = l_i^{-1}b_i'$, $1 \leqslant i \leqslant r$, hence $x = t_1^{-1}l_1^{-1}b_1'\frac{a_1}{1} + \cdots + t_r^{-1}l_r^{-1}b_r'\frac{a_r}{1} = (l_1t_1)^{-1}b_1'\frac{a_1}{1} + \cdots + (l_rt_r)^{-1}b_r'\frac{a_r}{1} \in J'$. Thus, $J = J'$.

Now note that $J' = T^{-1}I$, where $I := Sa_1 + \cdots + Sa_r$: clearly $T^{-1}I \subseteq J'$; let $y \in J'$, then $y = \frac{b_1}{s_1}\frac{a_1}{1} + \cdots + \frac{b_r}{s_r}\frac{a_r}{1} = \frac{b_1a_1}{s_1} + \cdots + \frac{b_ra_r}{s_r} = \frac{c_1b_1a_1+\cdots+c_rb_ra_r}{u}$ for some $c_i \in S$ and $u \in T$. Hence $y = u^{-1}(c_1b_1a_1 + \cdots + c_rb_ra_r) \in T^{-1}I$. But I is a f.g. left ideal of S, then $I = \langle a\}$ for some $a \in S$, and therefore $J = T^{-1}S\frac{a}{1}$, i.e., J is principal. $\square$

We observe that if S is $\mathcal{B}$ (or $\mathcal{KH}$) and T a multiplicative system of S such that $T^{-1}S$ and ST^{-1} exist, then $T^{-1}S$ is $\mathcal{B}$ ($\mathcal{KH}$) since $ST^{-1} \cong T^{-1}S$. On the other hand, if S is $\mathcal{H}$ ($\mathcal{PF}$, $\mathcal{PSF}$) not always $T^{-1}S$ has the correspondent property (see [8]).

For the localization by primes ideals we need to recall a definition. Let S be a left Noetherian ring and P a prime ideal of S. It says that P is *left localizable* if the set

$$S(P) := \{a \in S | \overline{a} \in S/P \text{ is not a zero divisor}\}$$

is a multiplicative system of S and $S(P)^{-1}S$ exists; we will write $S_P := S(P)^{-1}S$. Right localizable prime ideals are defined similarly (see [3]).

Theorem 5. *Let S be a left Noetherian ring.*

(i) *If P is a left (right) localizable prime ideal, then S_P is $\mathcal{H}$.*

(ii) *If P is a left (right) localizable completely prime ideal, then S_P is $\mathcal{PF}$, and hence, $\mathcal{C}$ and $\mathcal{PSF}$.*

Proof. (i) It is well known ([3]) that S_p has a unique maximal ideal $PS_P := \{\frac{a}{s} \mid a \in P, s \in S(P)\}$; moreover, $Rad(S_P) = PS_P$ and $S_p/PS_p \cong Q_l(S/P)$ is simple Artinian, where $Q_l(S/P)$ denotes the left quotient ring of S/P. Therefore, S_P is a semilocal ring and hence S_P is H (from Theorem 1 follows that any left Artinian ring S is H since $\mathrm{sr}(S) = 1$, so semilocal rings are $\mathcal{H}$).

(ii) If P is completely prime, S/P is a domain, so that $Q_l(S/P)$ is a division ring, and therefore, S_P is a local ring. From Proposition 10 we get that S_P is $\mathcal{PF} \subseteq \mathcal{C} \cap \mathcal{PSF}$. $\square$

References

[1] S.A. Amitsur, *Remarks of principal ideal rings*, Osaka Math. Journ. 15, (1963), 59-69.

[2] H. Bass, *Algebraic K-theory*, Benjamin, 1968.

[3] A. Bell, *Notes on Localizations in Noncommutative Noetherian Rings*, Departamento de Álgebra y Fundamentos, Universidad de Granada, España, 1989.

[4] H. Chen, *Rings Related to Stable Range Conditions*, World Scientific: Series in Algebra, Vol. 11, 2011.

[5] F. Chyzak, A. Quadrat, and D. Robertz, *Effective algorithms for parametrizing linear control systems over Ore algebras*, Appl. Algebra Engrg. Comm. Comput., 16, 2005, 319-376.

[6] T. Cluzeau and A. Quadrat, *Factoring and decomposing a class of linear functional systems*, Lin. Alg. And Its Appl., 428, 2008, 324-381.

[7] T. Cluzeau and A. Quadrat, *Serre's reduction of linear partial differential systems with holonomic adjoints*, J. of Symb. Comp., 47, 2012, 1192-1213.

[8] P. Cohn, *Free Ideal Rings and Localizations in General Rings*, Cambridge University Press, 2006.

[9] K. Goodearl and R. Jr. Warfield, *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society, ST 61, 2004.

[10] A.I. Hatalevych, *Right Bézout rings with waist is a right Hermite ring*, Ukr. Math. J., 62, 2010, 151-154.

[11] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. 66 (1949), 464-491.

[12] T.Y. Lam, *Serre's Problem on Projective Modules* , Springer Monographs in Mathematics, Springer, 2006.

[13] J. Lambeck, *Lectures on Rings and Modulea*, Chelsea Publishing Company, 1986.

[14] O. Lezama et. al., *Quillen-Suslin rings*, Extracta Mathematicae, 24, 2009, 55-97.

[15] O. Lezama, *Matrix and Gröbner Methods in Homological Algebra over Commutative Polynomial Rings*, Lambert Academic Publishing, 2011.

[16] B. MacDonald, *Linear Algebra over Commutative Rings*, Marcel Dekker, 1984.

[17] J. McConnell and J. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, AMS, 2001.

[18] A. Quadrat and D. Robertz, *Computation of bases of free modules over the Weyl algebras*, J. Symb. Comp., 42, 2007, 1113-1141.

[19] A. Quadrat, *Internal stabilization of coherent control systems*, IFAC, 2001.

[20] A. Quadrat, *"Stabilizing" the stabilizing controllers*, preprint.

[21] O. M. Romaniv, *Elementary reduction of matrices over right* 2*-Euclidean rings*, Ukr. Math. J., 56, 2004, 2028-2034.

[22] J.T. Stafford, *Module structure of Weyl algebras*, J. London Math. Soc. 18, 1978, 429-442.

[23] A. Steger, *Diagonability of idempotent matrices*, Pac. J. Math., 19, 1966, 535-542.

[24] C. A. Weibel, *Algebraic K-theory*, preprint, 2012.

[25] B. Zabavsky, *Diagonalizability theorems for matrices over rings with finite stable range*, Algebra and Discrete Mathematics, 1, (2005), 151-165.

[26] B. Zabavsky, *Reduction of matrices with stable range not exceeding 2*, Ukr. Math. Zh. (in Ucraine), 55, (2003), 550-554.

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