Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 15 **(2013)**. Number 1. pp. 23 – 36

On normalizers in fuzzy groups

L. A. Kurdachenko, K. O. Grin, N. A. Turbay

Abstract. In an arbitrary fuzzy group we define the normalizer of fuzzy subgroup and study some its properties. In particular, the characterization of nilpotent fuzzy group has been obtained.

Let G be a group with a multiplicative binary operation denoted by juxtaposition and identity e. We recall that a fuzzy subset $\gamma : G \to [0,1]$ is said to be a *fuzzy group on G* (see, for example, [1, S 1.2]), if it satisfies the following conditions:

(FSG 1) $\gamma(xy) \geq \gamma(x) \wedge \gamma(y)$ for all $x, y \in G$,

(FSG 2) $\gamma(x^{-1}) \geq \gamma(x)$ for every $x \in G$.

Here and everywhere we adopt the usual convention on the operator *wedge* $\wedge$(and on the operator *vee* $\vee$). If W is a subset of [0, 1], then denote by $\bigwedge W$ the greatest lower bound of W and denote by $\bigvee W$ the least upper bound of W. If $W = \{a, b\}$, then, as usual, instead of $\bigwedge W$ we will write $a \wedge b$, and instead of $\bigvee W$ we will write $a \vee b$. We assume that the least upper bound of the empty set is 0, and the greatest lower bound of the empty set is 1.

However we remark that we deliberately replace the standard expression a fuzzy subgroup of G by a fuzzy group on G in order to avoid

2010 MSC: Primary 20N25, 08A72, 03E72.

Key words and phrases: Fuzzy group; fuzzy subgroup; lower central series; nilpotent fuzzy group; normalizer condition; ascendant fuzzy subgroup; characteristic function.

possible misunderstanding in the sequel and to emphasize that a fuzzy group is in fact a function defined on a group G. For example, if γ, κ are the fuzzy groups on G and $\gamma \subseteq \kappa$, occurs, we will say that γ *is a fuzzy subgroup of* κ and denote this by $\gamma \preccurlyeq \kappa$.If γ is a fuzzy subgroup of κ, then $\gamma(e) \leq \kappa(e)$. Fuzzy subgroup γ of κ is called *unitary*, if $\gamma(e) = \kappa(e)$. If γ is an arbitrary fuzzy subgroup of κ, then clearly $\gamma^* = \gamma \cup \chi(e, \kappa(e))$ is a fuzzy subgroup of κ. Moreover, $\gamma^*(x) = \gamma(x)$ for all $x \neq e$ and $\gamma^*(e) = \kappa(e)$. Therefore, in the future we will focus on the unitary fuzzy subgroups of κ only. More precisely, when referring to that γ is a fuzzy subgroup of κ we will assume that the mentioned term means γ is an unitary fuzzy subgroup of κ.

Recall the following definition. If X is a set, for every subset Y of X and every $a \in [0, 1]$ we define a fuzzy subset $\chi(Y, a)$ as follows:

$$\chi(Y, a) = \begin{cases} a, x \in Y, \\ 0, x \notin Y. \end{cases}$$

Clearly $\chi(H, a)$ is a fuzzy group on G for every subgroup H of G. If $Y = \{y\}$, then instead of $\chi(\{y\}, a)$ we will write shorter $\chi(y, a)$. A fuzzy subset $\chi(y, a)$ is called a *fuzzy point* (or *fuzzy singleton*).

Fuzzy group theory, as well as other fuzzy algebraic structures, was introduced very soon after the beginning of fuzzy set theory. Many basic results of this theory were collected in the book [1]. But they are not systemized since it just a set of particular results that belong to different objects. There are a lot results on the structure of the largest fuzzy group $\chi(G, 1)$ on G. However one of the main goals of fuzzy group theory is the study of algebraic properties of an arbitrary fuzzy group defined on an abstract group G. There is essential difference with the case of fuzzy group $\chi(G, 1)$. Lets specify the next case. Consider arbitrary fuzzy group γ as an union the fuzzy point $\chi(x, \gamma(x))$, $x \in G$. By multiplication it is a semigroup. A fuzzy group $\chi(G, 1)$ has many invertible elements (all fuzzy points $\chi(g, 1)$, $g \in G$, are invertible), and this makes possible to use essentially impact on $\chi(G, 1)$ the group G. At the same time, an arbitrary fuzzy group defined on a group G may have very few invertible elements, and as a consequence, we have very little tangible results on arbitrary fuzzy group defined on a group G.

Our goal is to begin a systematic study of the properties of an arbitrary fuzzy group defined on a group G. One of the important concept not only in group theory, but also in the whole algebra is the notion of nilpotency. It was introduced for fuzzy groups too (see, [1, Chapters 3.2] and the papers [2], [3], [4]). In the paper [5] has been introduced the concept of upper central series in fuzzy group and considered some properties of hypercentral fuzzy group. In this paper we continue the investigation of generalized nilpotent fuzzy group. This study based on the concept of normalizer of fuzzy subgroup, which we introduce here. We consider also some properties of the class of fuzzy group satisfying the normalizer condition.

We will start from following useful

Proposition 1. *Let G be a group and γ, κ be a fuzzy subsets of G. Then*

$$(\gamma \circledcirc \kappa) = \cup_{y \in \boldsymbol{Supp}(\gamma), z \in \boldsymbol{Supp}(\kappa)} \chi(y, \gamma(y)) \circledcirc \chi(z, \kappa(z)).$$

Proof. By definition we have

$$(\gamma \circledcirc \kappa)(x) = \bigvee_{y,z \in G, yz=x} (\gamma(y) \wedge \kappa(z)).$$

If $y \notin \mathbf{Supp}(\gamma)$, then $\gamma(y) = 0$ and $\gamma(y) \wedge \kappa(z) = 0$. Similarly, if $z \notin \mathbf{Supp}(\kappa)$, then $\kappa(z) = 0$ and again $\gamma(y) \wedge \kappa(z) = 0$. It follows that

$$(\gamma \circledcirc \kappa)(x) = \bigvee_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa), yz=x} (\gamma(y) \wedge \kappa(z)).$$

On the other hand, consider a fuzzy subset

$$\xi = \cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)} \chi(y, \gamma(y)) \circledcirc \chi(z, \kappa(z)).$$

By Proposition 1 of paper [5] $\chi(y, \gamma(y)) \circledcirc \chi(z, \kappa(z)) = \chi(yz, \gamma(y) \wedge \kappa(z))$. If $x \in G$ and $x = yz$, then $\chi(yz, \gamma(y) \wedge \kappa(z))(x) = \gamma(y) \wedge \kappa(z)$, otherwise $\chi(yz, \gamma(y) \wedge \kappa(z))(x) = 0$. Therefore

$$\xi(x) = \bigvee_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)} (\chi(yz, \gamma(y) \wedge \kappa(z)))(x) =$$
$$= \bigvee_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa), yz=x} (\gamma(y) \wedge \kappa(z)) = (\gamma \circledcirc \kappa)(x).$$

Since it is true for each $x \in G$,

$$(\gamma \circledcirc \kappa) = \cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)} \chi(y, \gamma(y)) \circledcirc \chi(z, \kappa(z)). \qquad \square$$

Corollary 1. *Let G be a group.*

(i) *If γ, λ, κ are the fuzzy subsets of G such that $\lambda \subseteq \kappa$, then $\gamma \circledcirc \lambda \subseteq \gamma \circledcirc \kappa$ and $\lambda \circledcirc \gamma \subseteq \kappa \circledcirc \gamma$;*

(ii) *If γ, λ_a are the fuzzy subsets of G, $a \in A$, then $\gamma \circledcirc (\cup_{a \in A} \lambda_a) = \cup_{a \in A} (\gamma \circledcirc \lambda_a)$ and $(\cup_{a \in A} \lambda_a) \circledcirc \gamma = \cup_{a \in A} (\lambda_a \circledcirc \gamma)$.*

Proof. (i) We have

$$\lambda = \cup_{x \in \mathbf{Supp}(\lambda)} \chi(x, \lambda(x)), \kappa = \cup_{x \in \mathbf{Supp}(\kappa)} \chi(x, \kappa(x)).$$

Since $\lambda \subseteq \kappa$, $\lambda(x) \leq \kappa(x)$ for each $x \in G$. We have $\chi(x, \lambda(x))(x) = \lambda(x) \leq \kappa(x) = \chi(x, \kappa(x))(x)$ and $\chi(x, \lambda(x))(y) = 0 \leq 0 = \chi(x, \kappa(x))(y)$ whenever $y \neq x$. It follows that $\chi(x, \lambda(x)) \subseteq \chi(x, \kappa(x))$. By Proposition 1

$$\gamma \circledcirc \lambda = \cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda)} \chi(y, \gamma(y)) \circledcirc \chi(z, \lambda(z)) \subseteq \\ \cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)} \chi(y, \gamma(y)) \circledcirc \chi(z, \kappa(z)) = \gamma \circledcirc \kappa.$$

The second inclusion proved in a similar way.

(ii) Put $\lambda = \cup_{a \in A} \lambda_a$. By Proposition 1

$$\gamma \circledcirc \lambda = \cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda)} \chi(y, \gamma(y)) \circledcirc \chi(z, \lambda(z)).$$

Clearly $L = \mathbf{Supp}(\cup_{a \in A} \lambda_a) = \cup_{a \in A} \mathbf{Supp}(\lambda_a)$. We have

$$\cup_{a \in A}(\gamma \circledcirc \lambda_a) = \cup_{a \in A}(\cup_{y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda_a)} \chi(y, \gamma(y)) \circledcirc \chi(z, \lambda_a(z))) = \\ \cup_{y \in \mathbf{Supp}(\gamma), z \in L}(\cup_{a \in A} \chi(y, \gamma(y)) \circledcirc \chi(z, \lambda_a(z))).$$

Using Proposition 1 of paper [5] we obtain $\chi(y, \gamma(y)) \circledcirc \chi(z, \lambda_a(z)) = \chi(yz, \gamma(y) \wedge \lambda_a(z))$. By the definition of the union of fuzzy subsets,

$$(\cup_{a \in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(g) = \bigvee_{a \in A}(\chi(yz, \gamma(y) \wedge \lambda_a(z))(g))$$

for every $g \in G$. In particular,

$$(\cup_{a \in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(yz) = \bigvee_{a \in A}(\chi(yz, \gamma(y) \wedge \lambda_a(z))(yz)) = \\ \bigvee_{a \in A}(\gamma(y) \wedge \lambda_a(z)) = \gamma(y) \wedge (\bigvee_{a \in A} \lambda_a(z)) = \gamma(y) \wedge \lambda(z),$$

$(\cup_{a \in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(g) = \bigvee_{a \in A}(\chi(yz, \gamma(y) \wedge \lambda_a(z))(g)) = 0$ whenever $g \neq yz$.

In particular, $\cup_{a \in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)) = \chi(yz, \gamma(y) \wedge \lambda(z))$. Hence

$$\cup_{a \in A}(\gamma \circledcirc \lambda_a) = \cup_{y \in \mathbf{Supp}(\gamma), z \in L}(\cup_{a \in A} \chi(y, \gamma(y)) \circledcirc \chi(z, \lambda_a(z))) = \\ \cup_{y \in \mathbf{Supp}(\gamma), z \in L} \chi(yz, \gamma(y) \wedge \lambda(z)) = \gamma \circledcirc \lambda.$$

Using the similar arguments, we can prove a second equation. □

Let γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. We define *a normalizer* $N_\gamma(\kappa)$ *of* κ *in* γ as an union of all fuzzy points $\chi(x, a) \subseteq \gamma$, satisfying the following condition $\chi(x^{-1}, a) \circledcirc \kappa \circledcirc \chi(x, a) = \kappa$.

Since $\kappa = \cup_{g \in G} \chi(g, \kappa(g))$, an application of Proposition 1 gives a following equation

$$\chi(x^{-1}, a) \circledcirc \kappa \circledcirc \chi(x, a) = \cup_{g \in G} \chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a),$$

which implies a following

Proposition 2. *Let G be a group and γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. Then a normalizer $N_\gamma(\kappa)$ includes a fuzzy point $\chi(x,a) \subseteq \gamma$ if and only if for each $g \in G$ there are the elements $z, u \in G$ such that*

$$\chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a) \subseteq \chi(z, \kappa(z))$$

and

$$\chi(g, \kappa(g)) \subseteq \chi(x^{-1}, a) \circledcirc \chi(u, \kappa(u)) \circledcirc \chi(x, a).$$

Lemma 1. *Let G be a group and γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. Then a normalizer $N_\gamma(\kappa)$ includes a fuzzy point $\chi(x,a) \subseteq \gamma$ if and only if for each $g \in G$ there are the elements $z, u \in G$ such that*

$$\chi(g, \kappa(g)) \circledcirc \chi(x, a) \subseteq \chi(x, a) \circledcirc \chi(z, \kappa(z))$$

and

$$\chi(x, a) \circledcirc \chi(g, \kappa(g)) \subseteq \chi(u, \kappa(u)) \circledcirc \chi(x, a).$$

Proof. Suppose that $\chi(x,a) \subseteq N_\gamma(\kappa)$ and g be an arbitrary element of G. By Proposition 2 there are the elements $z, u \in G$ satisfying the following inclusions

$$\chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a) \subseteq \chi(z, \kappa(z))$$

and

$$\chi(g, \kappa(g)) \subseteq \chi(x^{-1}, a) \circledcirc \chi(u, \kappa(u)) \circledcirc \chi(x, a).$$

First inclusion implies

$$\chi(x, a) \circledcirc \chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a) \subseteq \chi(x, a) \circledcirc \chi(z, \kappa(z)).$$

Further, using Proposition 1 of paper [5], we obtain

$$\chi(x, a) \circledcirc \chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a) =$$
$$\chi(e, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a) = \chi(gx, a \wedge \kappa(g)).$$

On the other hand,

$$\chi(g, \kappa(g)) \circledcirc \chi(x, a) = \chi(gx, a \wedge \kappa(g)) =$$
$$\chi(x, a) \circledcirc \chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \circledcirc \chi(x, a).$$

So we obtain that $\chi(g, \kappa(g)) \circledcirc \chi(x, a) \subseteq \chi(x, a) \circledcirc \chi(z, \kappa(z))$.
Using the similar arguments, we obtain an inclusion

$$\chi(x, a) \circledcirc \chi(g, \kappa(g)) \subseteq \chi(u, \kappa(u)) \circledcirc \chi(x, a).$$

We can prove inverse assertion also using the similar arguments. □

We can reformulate Lemma 1 in a following form

Lemma 2. *Let G be a group and γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. Then a normalizer $N_\gamma(\kappa)$ includes a fuzzy point $\chi(x, a) \subseteq \gamma$ if and only if for each $g \in G$ there are the elements $z, u \in G$ such that*

$$\chi(g, \kappa(g)) \circledcirc \chi(x^{-1}, a) \subseteq \chi(x^{-1}, a) \circledcirc \chi(z, \kappa(z))$$

and

$$\chi(x^{-1}, a) \circledcirc \chi(g, \kappa(g)) \subseteq \chi(u, \kappa(u)) \circledcirc \chi(x^{-1}, a).$$

Lemma 3. *Let G be a group and γ be a fuzzy group on G. If $\lambda, \kappa \subseteq \gamma$, then $\lambda \circledcirc \kappa \subseteq \gamma$, in particular, $\gamma \circledcirc \gamma \subseteq \gamma$.*

Proof. Let x be an arbitrary element of G. We have

$$(\lambda \circledcirc \kappa)(x) = \bigvee_{y,z \in G, yz=x} (\lambda(y) \wedge \kappa(z)).$$

The inclusions $\lambda, \kappa \subseteq \gamma$ imply $\lambda(y) \wedge \kappa(z) \leq \gamma(y) \wedge \gamma(z)$. Since γ is a fuzzy subgroup, $\gamma(y) \wedge \gamma(z) \leq \gamma(yz)$, thus

$$(\lambda \circledcirc \kappa)(x) = \bigvee_{y,z \in G, yz=x} (\lambda(y) \wedge \kappa(z)) \leq \bigvee_{y,z \in G, yz=x} \gamma(yz) = \gamma(x). \qquad \square$$

Thus we get a criterion of being a fuzzy group needed in the future.

Proposition 3. *Let G be a group and γ be a fuzzy subset of G. Then γ is a fuzzy group if and only if the following assertion holds:*

(FSG 3) $\chi(x, \gamma(x)) \circledcirc \chi(y, \gamma(y)) \subseteq \gamma$ *for all* $x, y \in \textbf{\textit{Supp}}(\gamma)$,

(FSG 4) $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$ *for all* $x \in \textbf{\textit{Supp}}(\gamma)$.

Proof. Suppose first that γ is a fuzzy group on G. Since γ includes the function $\chi(x, \gamma(x))$ and $\chi(y, \gamma(y))$ for every elements $x, y \in \mathbf{Supp}(\gamma)$, using Lemma 3, we obtain that $\chi(x, \gamma(x)) \circledcirc \chi(y, \gamma(y)) \subseteq \gamma$.

Let x be an arbitrary element of $\mathbf{Supp}(\gamma)$. So $(\chi(x^{-1}, \gamma(x))(x^{-1}) = \gamma(x)$. Since γ is a fuzzy group, $\gamma(x) \leq \gamma(x^{-1})$. We note that if $y \neq x^{-1}$, then $(\chi(x^{-1}, \gamma(x))(y) = 0$, so that $(\chi(x^{-1}, \gamma(x))(y) \leq \gamma(y)$ for every $y \in G$. This means that $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$.

Conversely, suppose that γ satisfies the both conditions (**FSG 3**), (**FSG 4**). Let x,y be the arbitrary elements of G. If, for example, $x \notin \mathbf{Supp}(\gamma)$, then $\gamma(x) = 0$. It follows that $\gamma(x) \wedge \gamma(y) = 0$, and hence $\gamma(xy) \geq 0 = \gamma(x) \wedge \gamma(y)$. Therefore assume that $x, y \in \mathbf{Supp}(\gamma)$. Then (**FSG 3**) shows that $\chi(x, \gamma(x)) \circledcirc \chi(y, \gamma(y)) \subseteq \gamma$. By Proposition 1 of paper [5]

$$\gamma(x) \wedge \gamma(y) = (\chi(x, \gamma(x)) \circledcirc \chi(y, \gamma(y)))(xy) \leq \gamma(xy),$$

thus we obtain $\gamma(x) \wedge \gamma(y) \leq \gamma(xy)$, and γ satisfies (**FSG 1**).

Let $x \in G$. Since $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$, $(\chi(x^{-1}, \gamma(x)))(y) \leq \gamma(y)$ for every $y \in G$. In particular,

$$(\chi(x^{-1}, \gamma(x)))(x^{-1}) = \gamma(x) \leq \gamma(x^{-1}),$$

so that γ satisfies (**FSG 2**). □

Theorem 1. *Let G be a group, γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. Then a normalizer $N_\gamma(\kappa)$ is a fuzzy subgroup of γ.*

Proof. Put $\nu = N_\gamma(\kappa)$. Let x, y are the arbitrary elements of G. Consider a product $\chi(x, \nu(x)) \circledcirc \chi(y, \nu(y))$. By Proposition 1 of paper [5] $\chi(x, \nu(x)) \circledcirc \chi(y, \nu(y)) = \chi(xy, \nu(x) \wedge \nu(y))$. Let $g \in G$. Consider the products $\chi(g, \kappa(g)) \circledcirc \chi(xy, \nu(x) \wedge \nu(y))$ and $\chi(xy, \nu(x) \wedge \nu(y)) \circledcirc \chi(g, \kappa(g))$. We have

$$\chi(g, \kappa(g)) \circledcirc \chi(xy, \nu(x) \wedge \nu(y)) = \chi(g, \kappa(g)) \circledcirc (\chi(x, \nu(x)) \circledcirc \chi(y, \nu(y))) =$$
$$(\chi(g, \kappa(g)) \circledcirc \chi(x, \nu(x))) \circledcirc \chi(y, \nu(y)).$$

Since $\chi(x, \nu(x)) \subseteq N_\gamma(\kappa)$, Lemma 1 shows that there is an element $z \in G$ such that $\chi(g, \kappa(g)) \circledcirc \chi(x, \nu(x)) \subseteq \chi(x, \nu(x)) \circledcirc \chi(z, \kappa(z))$,so that

$$(\chi(g, \kappa(g)) \circledcirc \chi(x, \nu(x))) \circledcirc \chi(y, \nu(y)) \subseteq$$
$$(\chi(x, \nu(x)) \circledcirc \chi(z, \kappa(z))) \circledcirc \chi(y, \nu(y)) = \chi(x, \nu(x)) \circledcirc (\chi(z, \kappa(z)) \circledcirc \chi(y, \nu(y))).$$

Using again Lemma 1, we obtain the existence of element $w \in G$ such that

$$\chi(z, \kappa(z)) \circledcirc \chi(y, \nu(y)) \subseteq \chi(y, \nu(y)) \circledcirc \chi(w, \kappa(w)),$$

so that

$$\chi(g, \kappa(g)) \circledcirc \chi(xy, \nu(x) \wedge \nu(y)) \subseteq \chi(x, \nu(x)) \circledcirc \chi(y, \nu(y)) \circledcirc \chi(w, \kappa(w)) =$$
$$\chi(xy, \nu(x) \wedge \nu(y)) \circledcirc \chi(w, \kappa(w)).$$

Using again Lemma 1 and similar arguments we obtain a following inclusion

$$\chi(xy, \nu(x) \wedge \nu(y)) \circledcirc \chi(g, \kappa(g)) \subseteq \chi(u, \kappa(u)) \circledcirc \chi(xy, \nu(x) \wedge \nu(y))$$

for some element $u \in G$. These both inclusion together with Lemma 1 prove that $\chi(xy, \nu(x) \wedge \nu(y)) \subseteq N_\gamma(\kappa)$. As we have seen above

$$\chi(xy, \nu(x) \wedge \nu(y)) = \chi(x, \nu(x)) \circledcirc \chi(y, \nu(y)),$$

thus $\chi(x,\nu(x)) \circledcirc \chi(y,\nu(y)) \subseteq N_\gamma(\kappa)$ and $N_\gamma(\kappa)$ satisfies (**FSG 3**).

Let $x \in G$ and consider fuzzy point $\chi(x,\nu(x))$. By Lemma 1 for every element $g \in G$ there exist the elements $z, u \in G$ such that

$$\chi(g,\kappa(g)) \circledcirc \chi(x^{-1},\nu(x)) \subseteq \chi(x^{-1},\nu(x)) \circledcirc \chi(z,\kappa(z))$$

and

$$\chi(x^{-1},\nu(x)) \circledcirc \chi(g,\kappa(g)) \subseteq \chi(u,\kappa(u)) \circledcirc \chi(x^{-1},\nu(x)).$$

But Lemma 1 shows that in this case $\chi(x^{-1},\nu(x)) \subseteq \nu$, so that $N_\gamma(\kappa)$ satisfies (**FSG 4**). Proposition 3 shows that $N_\gamma(\kappa)$ is a fuzzy group. $\square$

The concept of normalizer is connected with a concept of normal fuzzy subgroup. Recall that if γ, κ are the fuzzy groups on G and $\kappa \preccurlyeq \gamma$, then it is said that κ is a *normal fuzzy subgroup of* γ, if $\kappa(yxy^{-1}) \geq \kappa(x) \wedge \gamma(y)$ for every elements $x, y \in G$ [1, 1.4]. We denote this fact by $\kappa \trianglelefteq \gamma$. We need a following criteria of normality, which specifies Proposition 3 of paper [5].

Proposition 4. *Let G be a group, γ, κ be the fuzzy groups on G. Suppose that $\kappa \preccurlyeq \gamma$. Then κ is a normal fuzzy subgroup of γ, if and only if*

$$\chi(x,\gamma(x)) \circledcirc \kappa \circledcirc \chi(x^{-1},\gamma(x)) = \kappa$$

for every element $x \in G$.

Proof. Suppose first that κ is a normal fuzzy subgroup of γ. Let $y \in G$ and consider a product $\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y))$. Let x be an arbitrary element of G. From Lemma 2 of paper [5] we obtain

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(x) = \gamma(y) \wedge \kappa(y^{-1}xy).$$

Put $u = y^{-1}xy$, then $x = y(y^{-1}xy)y^{-1} = yuy^{-1}$, so that

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(yuy^{-1}) = \gamma(y) \wedge \kappa(u).$$

Since $\kappa(u) \wedge \gamma(y) \leq \kappa(yuy^{-1})$, we obtain

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(yuy^{-1}) \leq \kappa(yuy^{-1}),$$

that is

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(x) \leq \kappa(x).$$

Since it is valid for every element $x \in G$, $\chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)) \preccurlyeq \kappa$.

We have $\kappa = \cup_{g \in \mathbf{Supp}(\kappa)} \chi(g, \kappa(g))$. Since κ is a normal fuzzy subgroup of γ, $\mathbf{Supp}(\kappa)$ is a normal subgroup of $\mathbf{Supp}(\gamma)$ (see, for example, [1, Theorem 1.4.4]), so that there exists an element $x \in \mathbf{Supp}(\kappa)$ such that $g = yxy^{-1}$. Then $x = y^{-1}gy$ and $\kappa(x) \geq \kappa(y) \wedge \kappa(g)$. Consider a product

$$\chi(y, \gamma(y)) \circledcirc \chi(x, \kappa(x)) \circledcirc \chi(y^{-1}, \gamma(y)) = \chi(yxy^{-1}, \gamma(y) \wedge \kappa(x)) = \chi(g, \gamma(y) \wedge \kappa(x)).$$

Suppose first that $\kappa(x) \neq \kappa(y)$, then $\kappa(g) = \kappa(x) \wedge \kappa(y)$ (see, for example, [1, p.7]). If $\kappa(y) > \kappa(x)$, then $\kappa(g) = \kappa(x)$, $\gamma(y) \geq \kappa(y) > \kappa(x)$ and $\gamma(y) \wedge \kappa(x) = \kappa(x)$. Hence in this case

$$\chi(g, \kappa(g)) = \chi(g, \gamma(y) \wedge \kappa(x)) = \chi(y, \gamma(y)) \circledcirc \chi(x, \kappa(x)) \circledcirc \chi(y^{-1}, \gamma(y)) \subseteq \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)).$$

Assume that $\kappa(y) < \kappa(x)$, then $\kappa(g) = \kappa(y)$. If $\kappa(x) \leq \gamma(y)$, then $\gamma(y) \wedge \kappa(x) = \kappa(x) > \kappa(g)$ and

$$\chi(g, \kappa(g)) \subseteq \chi(g, \gamma(y) \wedge \kappa(x)) = \chi(y, \gamma(y)) \circledcirc \chi(x, \kappa(x)) \circledcirc \chi(y^{-1}, \gamma(y)) \subseteq \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)).$$

If $\kappa(x) > \gamma(y)$, then $\gamma(y) \wedge \kappa(x) = \gamma(y) \geq \kappa(y) = \kappa(g)$ and again

$$\chi(g, \kappa(g)) \subseteq \chi(g, \gamma(y) \wedge \kappa(x)) = \chi(y, \gamma(y)) \circledcirc \chi(x, \kappa(x)) \circledcirc \chi(y^{-1}, \gamma(y)) \subseteq \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)).$$

Suppose now that $\kappa(x) = \kappa(y)$, then $\kappa(x) = \kappa(x) \wedge \kappa(y) \leq \kappa(x) \wedge \gamma(y)$. Since $x = y^{-1}gy$, $\kappa(x) \geq \kappa(g)$, and we have again $\kappa(g) \leq \kappa(x) \wedge \gamma(y)$, which follows that

$$\chi(g, \kappa(g)) \subseteq \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)).$$

Hence we obtain an inclusion $\kappa \preccurlyeq \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y))$, so that

$$\kappa = \chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)).$$

Conversely, suppose that $\chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)) = \kappa$ for each $y \in G$. Let x be an arbitrary element of G. Put $z = yxy^{-1}$, then $x = y^{-1}zy$. We have

$$(\chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)))(z) \leq \kappa(z).$$

Lemma 2 of paper [5] shows that $(\chi(y, \gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1}, \gamma(y)))(z) = \gamma(y) \wedge \kappa(y^{-1}zy)$. Then $\gamma(y) \wedge \kappa(y^{-1}zy) \leq \kappa(z)$, that is $\gamma(y) \wedge \kappa(x) \leq \kappa(yxy^{-1})$. □

This Proposition shows that κ is a normal fuzzy subgroup of $N_\gamma(\kappa)$.

Let γ be the fuzzy group on G, μ be a fuzzy set of G and suppose that $\mu \subseteq \gamma$. Recall that a fuzzy subgroup, *generated by* μ is an intersection of all fuzzy subgroups, including μ. We denote this subgroup by $< \mu >$. We have $\mu = \cup_{g \in \mathbf{Supp}(\mu)} \chi(g, \mu(g))$. Let κ be a fuzzy group on G, including μ. Then $\mu(g) \leq \kappa(g)$ for each element $g \in \mathbf{Supp}(\mu)$. It follows that $\chi(g, \mu(g)) \subseteq \kappa$ for each element $g \in \mathbf{Supp}(\mu)$. Proposition 3 shows that

$$\chi(g_1^{t_1}, \mu(g_1)) \circledcirc \chi(g_2^{t_2}, \mu(g_2)) \circledcirc \ldots \circledcirc \chi(g_k^{t_k}, \mu(g_k)) \subseteq \kappa$$

for every elements $g_1, \ldots, g_k \in \mathbf{Supp}(\mu)$ where $t_j \in \{1, -1\}, 1 \leq j \leq k$. Since it is true for each fuzzy group κ, including μ, $\chi(g_1^{t_1}, \mu(g_1)) \circledcirc \chi(g_2^{t_2}, \mu(g_2)) \circledcirc \ldots \circledcirc \chi(g_k^{t_k}, \mu(g_k)) \subseteq < \mu >$. Put

$$\lambda = \cup_{g_1, \ldots, g_k \in \mathbf{Supp}(\mu), k \in \mathbb{N}, t_1, \ldots, t_k \in \{1, -1\}} \chi(g_1^{t_1}, \mu(g_1)) \circledcirc \chi(g_2^{t_2}, \mu(g_2)) \circledcirc \ldots \circledcirc \chi(g_k^{t_k}, \mu(g_k)).$$

From an equation

$$\chi(g_1^{t_1}, \mu(g_1)) \circledcirc \chi(g_2^{t_2}, \mu(g_2)) \circledcirc \ldots \circledcirc \chi(g_k^{t_k}, \mu(g_k)) = \chi(g_1^{t_1} g_2^{t_2} \ldots g_k^{t_k}, \mu(g_1) \wedge \mu(g_2) \wedge \ldots \wedge \mu(g_k))$$

follows that λ satisfies the both conditions (**FSG 3**), (**FSG 4**) and Proposition 3 shows that λ is a fuzzy group. By definition of λ,

$$\chi(g, \mu(g)) \subseteq \lambda$$

for each element $g \in \mathbf{Supp}(\mu)$. It follows that $\mu \subseteq \lambda$ and therefore $< \mu > \preccurlyeq \lambda$. On the other hand, we could see above that $\lambda \preccurlyeq < \mu >$, so that $\lambda = < \mu >$. We can see that $\mathbf{Supp}(< \mu >) = < \mathbf{Supp}(\mu) >$.

Let G be a group, $x, y \in G$, $a, b \in [0, 1]$. Then a product $\chi(x^{-1}, a) \circledcirc \chi(y^{-1}, b) \circledcirc \chi(x, a) \circledcirc \chi(y, b)$ is called a *commutator of* $\chi(x, a)$ *and* $\chi(y, b)$ and will denoted by $[\chi(x, a), \chi(y, b)]$.

Let G be a group, γ, η be the fuzzy groups on G. Then a *fuzzy commutator subgroup* $[\gamma, \eta]$ is a fuzzy subgroup generated by all commutators $[\chi(x, \gamma(x)), \chi(y, \eta(y))]$ where $x \in \mathbf{Supp}(\gamma), y \in \mathbf{Supp}(\eta)$.

Let γ be a fuzzy group on a group G. We define the lower central series of γ by the following rule: put $\mathfrak{g}_1(\gamma) = \gamma$, $\mathfrak{g}_2(\gamma) = [\gamma, \gamma]$. Assume that we have already construct the terms $\mathfrak{g}_\beta(\gamma)$ for all ordinals $\beta < \alpha$. If α is a limit ordinal, then we put $\mathfrak{g}_\alpha(\gamma) = \cap_{\beta < \alpha} \mathfrak{g}_\beta(\gamma)$. Suppose now that α is a not limit ordinal, that is $\alpha - 1$ exists. Then put $\mathfrak{g}_\alpha(\gamma) = [\mathfrak{g}_{\alpha-1}(\gamma), \gamma]$. Thus, for every ordinal α we constructed the α^{th} term $\mathfrak{g}_\alpha(\gamma)$ of a lower central series of γ. The building of a lower central series of γ come to an end on some ordinal σ. In other words, this means that $\mathfrak{g}_\sigma(\gamma) = [\mathfrak{g}_\sigma(\gamma), \gamma]$. Then $\mathfrak{g}_\sigma(\gamma)$ is called the *lower hypocenter of* γ and will denoted by $\mathfrak{g}_\infty(\gamma)$.

A fuzzy group γ is called *hypocentral*, if $\mathfrak{g}_\infty(\gamma) \preccurlyeq \chi(e, \gamma(e))$.

A fuzzy group γ is called *nilpotent*, if there exists a positive integer k such that $\mathfrak{g}_k(\gamma) \preccurlyeq \chi(e, \gamma(e))$. By Proposition 2 of paper [5] we obtain that $\mathbf{Supp}(\mathfrak{g}_j(\gamma))$ is the j^{th} term of a lower central series of $\mathbf{Supp}(\gamma)$. Hence if a fuzzy group γ is nilpotent, then $\mathbf{Supp}(\gamma)$ is nilpotent. And conversely, if $\mathbf{Supp}(\gamma)$ is a nilpotent group, then a fuzzy group γ is nilpotent (see, for example, [1, Theorem 3.2.24]).

A fuzzy group γ is called *locally nilpotent*, if for every finite set of fuzzy points $\chi(g_1, \gamma(g_1)), \chi(g_2, \gamma(g_2)), \ldots, \chi(g_k, \gamma(g_k))$ a fuzzy group, generated by

$$\mu = \chi(g_1, \gamma(g_1)) \cup \chi(g_2, \gamma(g_2)) \cup \ldots \cup \chi(g_k, \gamma(g_k))$$

is nilpotent.

Let L be a subgroup of G and γ be a fuzzy group on G. We define the function $L|^\gamma : G \to [0, 1]$ by the following rule:

$$L|^\gamma(x) = \begin{cases} \gamma(x), if\ x \in L, \\ 0, if\ x \notin L. \end{cases}$$

Let x, y be the arbitrary elements of G. Then it is easy to check that

$$L|^\gamma(xy) \geq L|^\gamma(x) \wedge L|^\gamma(y).$$

It follows that $L|^\gamma$ is a fuzzy group on G.

Proposition 5. *Let G be a group and γ be a fuzzy group on G. Then γ is a locally nilpotent if and only if* ***Supp***(γ) *is a locally nilpotent abstract group.*

Proof. Suppose first that γ is locally nilpotent. Consider an arbitrary finite set of fuzzy points $\chi(g_1, \gamma(g_1)), \chi(g_2, \gamma(g_2)), \ldots, \chi(g_k, \gamma(g_k))$. Let

$$\mu = \chi(g_1, \gamma(g_1)) \cup \chi(g_2, \gamma(g_2)) \cup \ldots \cup \chi(g_k, \gamma(g_k)).$$

Since $\mathbf{Supp}(\mu) = \{g_1, g_2, \ldots, g_k\}$ and $\mathbf{Supp}(< \mu >) =< \mathbf{Supp}(\mu) >$, we obtain by above remarked that every finitely generated subgroup of $\mathbf{Supp}(\gamma)$ is nilpotent. In other words, $\mathbf{Supp}(\gamma)$ is locally nilpotent.

Conversely, suppose that $\mathbf{Supp}(\gamma)$ is locally nilpotent and let $M = \{g_1, g_2, \ldots, g_k\}$ be an arbitrary finite subset of $\mathbf{Supp}(\gamma)$. Then a subgroup $L =< M >$ is nilpotent. Consider a function $L|^\gamma$. As we saw above $L|^\gamma$ is a fuzzy group on G and $\mathbf{Supp}(L|^\gamma) = L$. Then a fuzzy group $L|^\gamma$ is nilpotent (see, for example, [1, Theorem 3.2.24]). Let again

$$\mu = \chi(g_1, \gamma(g_1)) \cup \chi(g_2, \gamma(g_2)) \cup \ldots \cup \chi(g_k, \gamma(g_k))$$

and $\lambda =< \mu >$. As we saw above λ is an union of fuzzy points

$$\chi(g_1^{t_1} g_2^{t_2} \ldots g_k^{t_k}, \gamma(g_1) \wedge \gamma(g_2) \wedge \ldots \wedge \gamma(g_k)).$$

Since γ is a fuzzy group, $\gamma(g_1^{t_1} g_2^{t_2} \ldots g_k^{t_k}) \geq \gamma(g_1) \wedge \gamma(g_2) \wedge \ldots \wedge \gamma(g_k)$. It follows that $\chi(g_1^{t_1} g_2^{t_2} \ldots g_k^{t_k}, \gamma(g_1) \wedge \gamma(g_2) \wedge \ldots \wedge \gamma(g_k)) \subseteq L|^{\gamma}$, so that $< \mu > \preccurlyeq L|^{\gamma}$. Since $L|^{\gamma}$ is nilpotent, $< \mu >$ is also nilpotent (see, for example, [1, Theorem 3.2.26]). This means that γ is locally nilpotent. □

Let γ, κ be the fuzzy groups on G and $\kappa \preccurlyeq \gamma$. We say that γ *satisfies a normalizer condition* if $N_\gamma(\kappa) \neq \kappa$ for every subgroup κ of γ.

Theorem 2. *Let G be a group, γ be a fuzzy group on G. If γ satisfies a normalizer condition, then **Supp**(γ) satisfies normalizer condition.*

Proof. Let L be an arbitrary subgroup of **Supp**(γ). Put $\lambda = L|^{\gamma}$, $\nu = N_\gamma(\lambda)$. Then $\nu \neq \lambda$. Suppose that **Supp**$(\lambda) =$ **Supp**(ν). Then for every element $x \in$ **Supp**(λ) we have $\gamma(x) \geq \nu(x) \geq \lambda(x) = \gamma(x)$, in particular, $\nu(x) = \lambda(x)$. Hence if we assume that **Supp**$(\lambda) =$ **Supp**(ν), then $\lambda = \nu$, and we obtain a contradiction. This contradiction shows that **Supp**$(\lambda) \neq$ **Supp**(ν). By above remarked, λ is a normal fuzzy subgroup of ν. It follows that **Supp**(λ) is a normal subgroup of **Supp**(ν) (see, for example, [1, Theorem 1.4.4]). Then $N_{\mathbf{Supp}(\gamma)}(L) \geq$ **Supp**$(\nu) \neq L$. Thus **Supp**(γ) satisfies normalizer condition. □

Corollary 2. *Let G be a group, γ be a fuzzy group on G. If γ satisfies a normalizer condition, then γ is locally nilpotent.*

Proof. Indeed, Theorem 2 shows that **Supp**(γ) satisfies a normalizer condition. Then **Supp**(γ) is locally nilpotent [6]. An application of Proposition 5 shows that γ is locally nilpotent. □

Let G be a group, γ be a fuzzy group on G, and suppose that γ satisfies a normalizer condition. If κ is a proper fuzzy subgroup of γ, then $\kappa_1 = N_\gamma(\kappa) \neq \kappa$. By Proposition 4 κ is a normal fuzzy subgroup of κ_1. Suppose that $\kappa_1 \neq \gamma$. Since γ satisfies a normalizer condition, $\kappa_2 = N_\gamma(\kappa_1) \neq \kappa_1$, so that κ_1 is a proper normal fuzzy subgroup of κ_2. Using the same arguments, we construct an ascending series

$$\kappa = \kappa_0 \trianglelefteq \kappa_1 \trianglelefteq \ldots \trianglelefteq \kappa_\alpha \trianglelefteq \kappa_{\alpha+1} \trianglelefteq \ldots \trianglelefteq \kappa_\beta = \gamma$$

where $\kappa_{\alpha+1} = N_\gamma(\kappa_\alpha)$ and $\kappa_\sigma = \cup_{\tau<\sigma}\kappa_\tau$ whenever σ is a limit ordinal, for all $\alpha, \sigma < \beta$.

Let G be a group, γ be a fuzzy group on G. A fuzzy subgroup κ of γ is called an *ascendant subgroup* of γ, if there exists an ascending series

$$\kappa = \kappa_0 \trianglelefteq \kappa_1 \trianglelefteq \ldots \trianglelefteq \kappa_\alpha \trianglelefteq \kappa_{\alpha+1} \trianglelefteq \ldots \trianglelefteq \kappa_\beta = \gamma.$$

The above arguments shows that if γ satisfies a normalizer condition, then every fuzzy subgroup of γ is ascendant. We can obtain now a following characterization of fuzzy group satisfying a normalizer condition.

Theorem 3. *Let G be a group, γ be a fuzzy group on G. Then γ satisfies a normalizer condition if and only if every fuzzy subgroup of γ is ascendant.*

Proof. Let κ be an arbitrary fuzzy subgroup of γ and assume that κ is ascendant in γ. Let

$$\kappa = \kappa_0 \trianglelefteq \kappa_1 \trianglelefteq \ldots \trianglelefteq \kappa_\alpha \trianglelefteq \kappa_{\alpha+1} \trianglelefteq \ldots \trianglelefteq \kappa_\beta = \gamma.$$

be an ascending series between κ and γ. In particular, κ is normal in $\kappa_1 \neq \kappa$. It follows that there exists an element x such that $\kappa_1(x) > \kappa(x)$. Then κ does not include $\chi(x, \kappa_1(x))$. By Proposition 4 $\chi(x^{-1}, \kappa_1(x)) \circledcirc \kappa \circledcirc \chi(x, \kappa_1(x)) = \kappa$, so that $\chi(x, \kappa_1(x)) \subseteq N_\gamma(\kappa)$ and $N_\gamma(\kappa) \neq \kappa$. Conversely, if γ satisfies a normalizer condition, then we have seen above that every fuzzy subgroup of γ is ascendant in γ. □

References

[1] J.N. Mordeson, K.R. Bhutani, A. Rosenfeld, *Fuzzy Group Theory*, Springer: Berlin, 2005.

[2] K.C. Gupta, B.K. Sarma, *Nilpotent fuzzy groups*,Fuzzy Sets and Systems, N.**101**, 1999, pp.167-176.

[3] J.G. Kim, *Commutative fuzzy sets and nilpotent fuzzy groups*, Information Sciences, N.**83**, 1995, pp.161-174.

[4] M. Asaad, S. Abou - Zaid, *Fuzzy subgroups of nilpotent groups*, Fuzzy Sets and Systems, N.**24**, 1993, pp.321-323.

[5] L.A. Kurdachenko, K.O. Grin, N.A. Turbay, *On hypercentral fuzzy groups*, Algebra and Discrete Mathematics, Vol. 13, N.**1**, 2012, pp.92-106.

[6] B.I. Plotkin, *To the theory of locally nilpotent groups*, Doklady AN USSR, N.**76**, 1951, pp.639-642.

Contact information

L. A. Kurdachenko Department of Algebra, Oles Honchar Dnipropetrovsk National University, 72 Gagarin Av., Dnepropetrovsk, Ukraine 49010
E-Mail: `lkurdachenko@i.ua`

K. O. Grin Department of Algebra, Oles Honchar Dnipropetrovsk National University, 72 Gagarin Av., Dnepropetrovsk, Ukraine 49010
E-Mail: `k.grin@mail.ru`

N. A. Turbay Department of Algebra, Oles Honchar Dnipropetrovsk National University, 72 Gagarin Av., Dnepropetrovsk, Ukraine 49010
E-Mail: `nadezhda.turbay@gmail.com`

Received by the editors: 18.01.2013
and in final form 18.01.2013.