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RESEARCH ARTICLE

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Units of some group algebras of groups of order 12 over any finite field of characteristic 3

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Abstract. The structure of the unit groups of the group algebra of the groups D_{12} and $C_3 \rtimes C_4$ over any finite field of characteristic 3 is established.

1. Introduction

Let $\mathcal{U}(RG)$ be the group of units of the group ring RG of the group G over the commutative ring R. It is well known that

$$\mathcal{U}(RG) = V(RG) \times U(R),$$

where $V(RG) = \{\sum_{g \in G} \alpha_g g \in \mathcal{U}(RG) \mid \sum_{g \in G} \alpha_g = 1\}$ is the group of normalized units of RG and $\mathcal{U}(R)$ is the group of units of R. For further details and background see Polcino Milies and Sehgal [9].

We are interested in the structure of $\mathcal{U}(KG)$ when of order ap^k where p is a prime, $a, k \in \mathbb{N}_0$ and $(a,p) = 1$. It is well known that $|V(KG)| = |K|^{|G|-1}$ if G is a finite p-group and K is a finite field of characteristic p. A basis for $V(\mathbb{F}_p G)$ is determined in [10], where $\mathbb{F}_p$ is the Galois field of p elements and G is an abelian p-group. Note that several results were obtained in the case where K is a field of characteristic p and G is a non-abelian p-group, see [1, 2, 3] for further detials.

In [6], the order of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})$ is determined where $\mathbb{F}_{p^k}$ is the Galois field of p^k-elements, $D_{2p^m} = \langle a, b \mid a^{p^n} = 1, b^2 = 1, a^b = a^{-1} \rangle$ is the

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dihedral group of order $2p^m$ and p is an odd prime. In [4], the structure of $\mathcal{U}(\mathbb{F}_{3^k}D_6)$ was determined in terms of split extensions of elementary abelian groups. Additionally in [8], it is shown that $Z(V_1)$ and $V_1/Z(V_1)$ are elementary abelian 3-groups where $V_1 = 1+J(\mathbb{F}_{3^k}D_6)$, $J(\mathbb{F}_{3^k}D_6)$ is the Jacobson Radical of $\mathbb{F}_{3^k}D_6$ and $Z(V_1)$ is the center of V_1. The structure of FA_4 is established in [11] where F is any finite field and A_4 is the alternating group of degree 4. Our main result is the following:

Theorem 1. *Let $V(FG)$ be the group of normalized units of the group algebra FG of a group G over a finite field F of 3^k elements. The following conditions hold:*

(i) *If $G = \langle x, y \mid x^6 = y^2 = 1, \quad x^y = x^{-1}\rangle \cong D_{12}$, then*

$$V(FG) \cong ({C_3}^{6k} \rtimes {C_3}^{2k}) \rtimes C^3_{3^k-1}.$$

(ii) *If $G = \langle x, y \mid x^4 = y^3 = 1, \quad yxy = x\rangle \cong C_3 \rtimes C_4$, then*

$$V(FG) \cong \begin{cases} ({C_3}^{6k} \rtimes {C_3}^{2k}) \rtimes C^3_{3^k-1} & \text{if } 4 \mid (3^k-1) \\ ({C_3}^{6k} \rtimes {C_3}^{2k}) \rtimes (C_{3^k-1} \times C_{3^{2k}-1}) & \text{if } 4 \nmid (3^k-1). \end{cases}$$

The next two results can be found in [7].

Theorem 2. *Let F be an arbitrary field of characteristic $p > 0$, let G be a p-solvable group and c be the sum of all p-elements of G including 1, then*

$$J(FG) = lann(c)$$

where $J(FG)$ is the Jacobson Radical of FG and $lann(c)$ is the left annihilator of c.

Theorem 3. *Let N be a normal subgroup of G such that G/N is p-solvable. If $|G/N| = np^a$ where $(n,p) = 1$, then*

$$J(FG)^{p^a} \subseteq FG \cdot J(FN) \subseteq J(FG)$$

where F is a field of characteristic $p > 0$. In particular, if G is p-solvable of order np^a where $(n,p) = 1$, then

$$J(FG)^{p^a} = 0.$$

Denote by $\widehat{g} = \sum_{h \in \langle g \rangle} h \in RG$.

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2. Proof of Main Theorem

Case (i). Let $G = D_{12}$. Consider the ring homomorphism $\theta : \mathbb{F}_{3^k}D_{12} \to \mathbb{F}_{3^k}(C_2 \times C_2)$ given by

$$\sum_{i=0}^{2} x^{2i}(a_i + a_{i+3}x^3 + a_{i+6}y + a_{i+9}x^3y) \mapsto$$
$$\sum_{i=0}^{2}(a_i + a_{i+3}\bar{x} + a_{i+6}\bar{y} + a_{i+9}\bar{x}\bar{y}).$$

where $C_2 \times C_2 = \langle \overline{x}, \overline{y} \,|\, \overline{x}^2 = \overline{y}^2 = 1,\, \overline{x}\,\overline{y} = \overline{y}\,\overline{x}\rangle$ and $\alpha_i \in \mathbb{F}_{3^k}$.

We can now construct the group homomorphism $\theta' : \mathcal{U}(\mathbb{F}_{3^k}D_{12}) \to \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$, since units gets mapped to units. We define the group epimorphism $\phi : \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2)) \to \mathcal{U}(\mathbb{F}_{3^k}D_{12})$ given by $a + b\bar{x} + c\bar{y} + d\bar{x}\bar{y} \mapsto a + bx^3 + cy + dx^3y$ where $a, b, c, d \in \mathbb{F}_{3^k}$. Clearly $\theta' \circ \phi = 1$ and therefore $\mathcal{U}(\mathbb{F}_{3^k}D_{12}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$ where $H = \ker(\theta')$. Let

$$\alpha = \sum_{i=0}^{2} x^{2i}(a_i + a_{i+3}x^3 + a_{i+6}y + a_{i+9}x^3y) \in \mathbb{F}_{3^k}D_{12},$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $\alpha \in H$ iff

$$\sum_{i=0}^{2} a_i = 1, \sum_{j=0}^{2} a_{j+3} = \sum_{k=0}^{2} a_{k+6} = \sum_{l=0}^{2} a_{l+9} = 0.$$

Lemma 1. *H has exponent* 3.

Proof. D_{12} is solvable and hence p-solvable. Clearly $|D_{12}| = 4.3$ and by Theorem 3, $J(\mathbb{F}_{3^k}D_{12})^3 = 0$ and $1 + J(\mathbb{F}_{3^k}D_{12})$ has exponent 3. Now by Theorem 2, $J(\mathbb{F}_{3^k}D_{12}) = \{\alpha \in \mathbb{F}_{3^k}D_{12} | \widehat{\alpha x^2} = 0\}$. Let

$$\alpha = \sum_{i=0}^{2} x^{2i}\left[\alpha_{i+1} + \alpha_{i+4}x^3 + \alpha_{i+7}y + \alpha_{i+10}x^3y\right] \in \mathcal{U}(\mathbb{F}_{3^k}D_{12}),$$

then

$$\alpha \in J(\mathbb{F}_{3^k}D_{12}) \Longleftrightarrow \sum_{i=1}^{3} \alpha_i = \sum_{j=1}^{3} \alpha_{j+3} = \sum_{l=1}^{3} \alpha_{l+6} = \sum_{m=1}^{3} \alpha_{m+9} = 0$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Therefore $H \cong 1 + J(\mathbb{F}_{3^k}D_{12})$. □

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Lemma 2. *Every element of the centralizer of x^2 in H ($C_H(x^2)$) has the form* $\sum\limits_{i=0}^{2} x^{2i}(\alpha_i+\alpha_{i+3}x^3)+(\alpha_6+\alpha_7x^3)\widehat{x^2}y$ *where* $\sum\limits_{i=0}^{2}\alpha_i=1$, $\sum\limits_{i=0}^{2}\alpha_{i+3}=0$ *and* $\alpha_i\in\mathbb{F}_{3^k}$. *Also* $C_H(x^2)\cong {C_3}^{6k}$.

Proof. $C_H(x^2)=\{h\in H \mid hx^2=x^2h\}$. Let $h=\sum_{i=0}^{2}x^{2i}(\alpha_i+\alpha_{i+3}x^3+\alpha_{i+6}y+\alpha_{i+9}x^3y)\in H$ where $\alpha_i\in\mathbb{F}_{3^k}$, $\sum_{i=0}^{2}\alpha_i=1$ and $\sum_{i=0}^{2}\alpha_{i+3}=\sum_{i=0}^{2}\alpha_{i+6}=\sum_{i=0}^{2}\alpha_{i+9}=0$.

$$\begin{aligned}(hx^2-x^2h)&=\left(\sum_{i=0}^{2}\alpha_{i+6}x^{2i}y+\sum_{i=0}^{2}\alpha_{i+9}x^{2i+3}y\right)x^2-\\&\quad-x^2\left(\sum_{i=0}^{2}\alpha_{i+6}x^{2i}y+\sum_{i=0}^{2}\alpha_{i+9}x^{2i+3}y\right)\\&=\sum_{i=0}^{2}\alpha_{i+6}x^{2i+4}y+\sum_{i=0}^{2}\alpha_{i+9}x^{2i+1}y-\sum_{i=0}^{2}\alpha_{i+6}x^{2i+2}y-\sum_{i=0}^{2}\alpha_{i+9}x^{2i+5}y\\&=\sum_{i=0}^{2}\left[\alpha_{i+6}x^{2i+4}y-\alpha_{i+6}x^{2i+2}y\right]+\sum_{i=0}^{2}\left[\alpha_{i+9}x^{2i+1}y-\alpha_{i+9}x^{2i+5}y\right]\\&=(\alpha_6-\alpha_7)\,\widehat{x^2}y+(\alpha_9-\alpha_{10})\,x^3\widehat{x^2}y.\end{aligned}$$

Therefore $h\in C_H(x^2)$ if and only if $\alpha_6=\alpha_7=\alpha_8$ and $\alpha_9=\alpha_{10}=\alpha_{11}$. Hence every element of $C_H(x^2)$ has the form $\sum_{i=0}^{2}x^{2i}(\alpha_i+\alpha_{i+3}x^3)+(\alpha_6+\alpha_7x^3)\widehat{x^2}y$ where $\sum\limits_{i=0}^{2}\alpha_i=1$, $\sum\limits_{i=0}^{2}\alpha_{i+3}=0$ and $\alpha_i\in\mathbb{F}_{3^k}$. It remains to show that $C_H(x^2)$ is abelian. Let $\alpha=\sum_{i=0}^{2}x^{2i}\left(\alpha_i+\alpha_{i+3}x^3\right)+(\alpha_6+\alpha_7x^3)\widehat{x^2}y\in C_H(x^2)$ and $\beta=\sum_{i=0}^{2}x^{2i}\left(\beta_i+\beta_{i+3}x^3\right)+(\beta_6+\beta_7x^3)\widehat{x^2}y\in C_H(x^2)$ where $\sum\limits_{i=0}^{2}\alpha_i=1$, $\sum\limits_{i=0}^{2}\alpha_{i+3}=0$, $\sum\limits_{i=0}^{2}\beta_i=1$, $\sum\limits_{i=0}^{2}\beta_{i+3}=0$ and $\alpha_i,\beta_j\in\mathbb{F}_{3^k}$. Note that $(\widehat{x^2})^2=0$. Then

$$\begin{aligned}\alpha\beta&=\sum_{i=0}^{2}x^{2i}\left(a_i+a_{i+3}x^3\right)\sum_{i=0}^{2}x^{2i}\left(b_i+b_{i+3}x^3\right)\\&\quad+\sum_{i=0}^{2}x^{2i}\left(a_i+a_{i+3}x^3\right)(b_6+b_7x^3)\widehat{x^2}y\\&\quad+(a_6+a_7x^3)\widehat{x^2}y\sum_{i=0}^{2}\left(b_ix^{2i}+b_{i+3}x^{2i+3}\right)\end{aligned}$$

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$$
\begin{aligned}
&+ (a_6 + a_7x^3)\widehat{x^2}y(b_6 + b_7x^3)\widehat{x^2}y \\
&= \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \\
&+ \sum_{i=0}^{2}\left(a_i + a_{i+3}x^3\right)\widehat{x^2}(b_6 + b_7x^3)y \\
&+ \sum_{i=0}^{2}\left(b_i + b_{i+3}x^3\right)\widehat{x^2}(a_6 + a_7x^3)y \\
&= \sum_{i=0}^{2} x^{2i}\left(a_i + a_3x^{2i+3}\right) \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \\
&+ \widehat{x^2}(b_6 + b_7x^3)y + \widehat{x^2}(a_6 + a_7x^3)y \\
&= \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(b_ix^{2i} + b_{i+3}x^{2i+3}\right) \\
&+ ((a_6 + b_6) + (a_7 + b_7)x^3)\widehat{x^2}y
\end{aligned}
$$

and

$$
\begin{aligned}
\beta\alpha &= \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \\
&+ \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right)(a_6 + a_7x^3)\widehat{x^2}y \\
&+ (b_6 + b_7x^3)\widehat{x^2}y\sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \\
&+ (b_6 + b_7x^3)\widehat{x^2}y(a_6 + a_7x^3)\widehat{x^2}y \\
&= \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \\
&+ \sum_{i=0}^{2}\left(b_i + b_{i+3}x^3\right)\widehat{x^2}(a_6 + a_7x^3)y \\
&+ \sum_{i=0}^{2}\left(a_i + a_{i+3}x^3\right)\widehat{x^2}(b_6 + b_7x^3)y \\
&= \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right)
\end{aligned}
$$

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$$
\begin{aligned}
&+ \widehat{x^2}(a_6 + a_7x^3)y + \widehat{x^2}(b_6 + b_7x^3)y \\
&= \sum_{i=0}^{2} x^{2i}\left(b_i + b_{i+3}x^3\right) \sum_{i=0}^{2} x^{2i}\left(a_i + a_{i+3}x^3\right) \\
&+ ((a_6 + b_6) + (a_7 + b_7)x^3)\widehat{x^2}y.
\end{aligned}
$$

Let $\gamma = \sum_{i=0}^{2} x^{2i}\left(\beta_i + \beta_{i+3}x^3\right) \in H$ and $\delta = \sum_{i=0}^{2} x^{2i}\left(\alpha_i + \alpha_{i+3}x^3\right) \in H$. Clearly $\gamma, \delta \in V(\mathbb{F}_{3^k}C_6)$ and $\gamma\delta = \delta\gamma$ since $V(\mathbb{F}_{3^k}C_6)$ is abelian. Therefore $C_H(x^2)$ is abelian. $\square$

Lemma 3. *Let S be the subgroup of H consisting of elements of the form $1 + \breve{x}(r + r_1x^3)(1 + y)$ where $\breve{x} = \sum_{i=0}^{2} ix^{2i}$ and $r, r_1 \in \mathbb{F}_{3^k}$. Then $S \cong {C_3}^{2k}$.*

Proof. Let $s_1 = 1+\breve{x}(r+r_1x^3)(1+y) \in S$ and $s_2 = 1+\breve{x}(s+s_1x^3)(1+y) \in S$ where $r, s, r_1, s_1 \in \mathbb{F}_{3^k}$, then

$$
\begin{aligned}
s_1s_2 &= 1 + \breve{x}(s + s_1x^3)(1 + y) + \breve{x}(r + r_1x^3)(1 + y) \\
&+ \breve{x}(r + r_1x^3)(1 + y)\breve{x}(s + s_1x^3)(1 + y) \\
&= 1 + \breve{x}\left((s + r) + (s_1 + r_1)x^3\right)(1 + y) \\
&+ \breve{x}(r + r_1x^3)(1 + y)(1 - y)\breve{x}(s + s_1x^3) \\
&= 1 + \breve{x}\left((s + r) + (s_1 + r_1)x^3\right)(1 + y) \\
&+ \breve{x}(r + r_1x^3)(1 - 1)\breve{x}(s + s_1x^3) \\
&= 1 + \breve{x}\left((s + r) + (s_1 + r_1)x^3\right)(1 + y).
\end{aligned}
$$

Therefore S is closed. $\square$

Lemma 4. *$H \cong C_H(x^2) \rtimes S$.*

Proof. Clearly $H = C_H(x^2).S$ since $C_H(x^2) \cap S = \{1\}$. It remains to show that S normalizes $C_H(x^2)$. Let $s = 1 + \breve{x}(r + r_1x^3)(1 + y)$, then $s^{-1} = 1 - \breve{x}(r + r_1x^3)(1 + y)$ where $r, r_1 \in \mathbb{F}_{3^k}$. Also let $c = \sum_{i=0}^{2}x^{2i}(\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y \in C_H(x^2)$ where $\sum_{i=0}^{2}\alpha_i = 1$, $\sum_{i=0}^{2}\alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Note that $\breve{x}^2 = \widehat{x^2}\breve{x} = 0$. Then

$$
\begin{aligned}
c^s = &\left(1 - \breve{x}(r + r_1x^3)(1 + y)\right)\left(\sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y\right) \\
&\times \left(1 + \breve{x}(r + r_1x^3)(1 + y)\right)
\end{aligned}
$$

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$$
\begin{aligned}
&= \big(1 - \breve{x}(r + r_1x^3)(1+y)\big)\left(\sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)\right.\\
&\left.+ \sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\breve{x}(r + r_1x^3)(1+y)) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y\right)\\
&= \sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3) + \sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\breve{x}(r + r_1x^3)(1+y))\\
&+ (\alpha_6 + \alpha_7x^3)\widehat{x^2}y - \breve{x}(r + r_1x^3)(1+y)\sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)\\
&= \sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3) + \sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\breve{x}(r + r_1x^3)(y))\\
&+ (\alpha_6 + \alpha_7x^3)\widehat{x^2}y - \breve{x}(r + r_1x^3)(y)\sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3)\\
&= \left(\sum_{i=0}^{2} x^{2i}(\alpha_i + \alpha_{i+3}x^3) + ((\alpha_6 + \delta) + (\alpha_7 + \delta)x^3)\widehat{x^2}y\right) \in C_H(x^2)
\end{aligned}
$$

where $\delta = r(\alpha_1 - \alpha_2) + r_1(\alpha_4 - \alpha_5)$ and $\alpha_i, r, r_1 \in \mathbb{F}_{3^k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}D_{12}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$. Consider $\mathbb{F}_{3^k}(C_2 \times C_2)$.

$$
\begin{aligned}
\mathbb{F}_{3^k}(C_2 \times C_2) &\cong (\mathbb{F}_{3^k}C_2)C_2\\
&\cong (\mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k})C_2\\
&\cong \mathbb{F}_{3^k}C_2 \oplus \mathbb{F}_{3^k}C_2\\
&\cong \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k}.
\end{aligned}
$$

Thus $\mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2)) \cong {C_{3^k-1}}^4$. Therefore

$$
\begin{aligned}
\mathcal{U}(\mathbb{F}_{3^k}D_{12}) &\cong [{C_3}^{6k} \times {C_3}^{2k}] \rtimes {C_{3^k-1}}^4\\
&\cong [({C_3}^{6k} \times {C_3}^{2k}) \rtimes {C_{3^k-1}}^3] \times \mathcal{U}(\mathbb{F}_{3^k})
\end{aligned}
$$

Case (ii). Let $G \cong C_3 \rtimes C_4$. Define the group epimorphism

$$\tau : \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \longrightarrow \mathcal{U}(\mathbb{F}_{3^k}C_4)$$

given by

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$$\sum_{i=0}^{2}(\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \mapsto \sum_{i=0}^{2}(\alpha_i + \alpha_{i+3}\overline{x} + \alpha_{i+6}\overline{x}^2 + \alpha_{i+9}\overline{x}^3)$$

where $\overline{x}$ generates C_4.

Define the group homomorphism $\phi : \mathcal{U}(\mathbb{F}_{3^k}C_4) \longrightarrow \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$ by $a + b\overline{x} + c\overline{x}^2 + d\overline{x}^3 \mapsto a + bx + cx^2 + dx^3$ where $a, b, c, d \in \mathbb{F}_{3^k}$. Then $\tau \circ \phi = 1$, therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}C_4)$ where $L \cong ker(\tau)$. Let

$$l = \sum_{i=0}^{2}(\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)),$$

then $l \in L$ if and only if $\sum_{i=0}^{2} \alpha_i = 1$ and

$$\sum_{j=0}^{2}\alpha_{j+3} = \sum_{l=0}^{2}\alpha_{l+6} = \sum_{m=0}^{2}\alpha_{m+9} = 0 \qquad (\alpha_i \in \mathbb{F}_{3^k})$$

Therefore $|L| = 3^{8k}$.

Lemma 5. *L has exponent 3.*

Proof. By Theorem 3, $J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))^3 = 0$ and $1 + J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$ has exponent 3. By Theorem 2, $J(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \rtimes C_4) | \alpha\widehat{y} = 0\}$. Therefore $L \cong 1 + J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$. □

Lemma 6. *Every element of the centralizer of y in L ($C_L(y)$) has the form $\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3 x + \alpha_7 x^3)\widehat{y}$ where $\sum_{i=0}^{2}\alpha_i = 1$, $\sum_{i=0}^{2}\alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Also $C_L(y) \cong {C_3}^{6k}$.*

Proof. $C_L(y) = \{l \in L \,|\, ly = yl\}$. Let

$$l = \sum_{i=0}^{2}(\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \in L.$$

where $\sum_{i=0}^{2}\alpha_i = 1$, $\sum_{i=0}^{2}\alpha_{i+3} = 0$, $\sum_{i=0}^{2}\alpha_{i+6} = 0$, $\sum_{i=0}^{2}\alpha_{i+9} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$.

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Then

$$ly - yl = \left(\sum_{i=0}^{2}(\alpha_{i+3}x + \alpha_{i+9}x^3)y^i\right) y - y\left(\sum_{i=0}^{2}(\alpha_{i+3}x + \alpha_{i+9}x^3)y^i\right)$$
$$= \sum_{i=0}^{2}(\alpha_{i+3}x + \alpha_{i+9}x^3)y^{i+1} - \sum_{i=0}^{2}(\alpha_{i+3}yx + \alpha_{i+9y}x^3)y^i$$
$$= (\alpha_5 - \alpha_4)x\widehat{y} + (\alpha_{11} - \alpha_{10})x^3\widehat{y}.$$

Therefore $l \in C_L(y)$ if and only if $\alpha_3 = \alpha_4 = \alpha_5$ and $\alpha_9 = \alpha_{10} = \alpha_{11}$. Thus every element of $C_L(y)$ is of the form

$$\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3 x + \alpha_7 x^3)\widehat{y}$$

where $\sum_{i=0}^{2}\alpha_i = 1, \sum_{i=0}^{2}\alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Let $\alpha = \sum_{i=0}^{2}(\alpha_i+\alpha_{i+4}x^2)y^i+ (\alpha_3 x + \alpha_7 x^3)\widehat{y} \in C_L(y)$ and $\beta = \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i + (\beta_3 x + \beta_7 x^3)\widehat{y} \in C_L(y)$ $\sum_{i=0}^{2}\alpha_i = 1$, $\sum_{i=0}^{2}\alpha_{i+3} = 0$, $\sum_{i=0}^{2}\beta_i = 1$, $\sum_{i=0}^{2}\beta_{i+3} = 0$ and $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Then

$$\alpha\beta = \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i$$
$$+ \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i(\beta_3 x + \beta_7 x^3)\widehat{y}$$
$$+ (\alpha_3 x + \alpha_7 x^3)\widehat{y}\sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i + (\alpha_3 x + \alpha_7 x^3)\widehat{y}(\beta_3 x + \beta_7 x^3)\widehat{y}$$
$$= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i$$
$$+ \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)\widehat{y}(\beta_3 x + \beta_7 x^3) + (\alpha_3 x + \alpha_7 x^3)\sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)\widehat{y}$$
$$= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i$$
$$+ \widehat{y}(\beta_3 x + \beta_7 x^3) + (\alpha_3 x + \alpha_7 x^3)\widehat{y}$$

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$$= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i$$
$$+ ((\alpha_3 + \beta_3)x + (\alpha_7 + \beta_7)x^3)\widehat{y}$$

$$\beta\alpha = \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i$$
$$+ \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i(\alpha_3 x + \alpha_7 x^3)\widehat{y}$$
$$+ (\beta_3 x + \beta_7 x^3)\widehat{y}\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (\beta_3 x + \beta_7 x^3)\widehat{y}(\alpha_3 x + \alpha_7 x^3)\widehat{y}$$
$$= \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i$$
$$+ \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)\widehat{y}(\alpha_3 x + \alpha_7 x^3)$$
$$+ (\beta_3 x + \beta_7 x^3)\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)\widehat{y}$$
$$= \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i$$
$$+ \widehat{y}(\alpha_3 x + \alpha_7 x^3) + (\beta_3 x + \beta_7 x^3)\widehat{y}$$
$$= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i$$
$$+ ((\alpha_3 + \beta_3)x + (\alpha_7 + \beta_7)x^3)\widehat{y}$$

Let $\gamma = \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i \in L$ and $\delta = \sum_{i=0}^{2}(\beta_i + \beta_{i+4}x^2)y^i \in L$. Clearly $\gamma, \delta \in V(\mathbb{F}_{3^k}C_6)$ since $\langle x^2, y\rangle \cong C_6$ and $\gamma\delta = \delta\gamma$. Therefore $C_L(y) \cong {C_3}^{6k}$. $\square$

Lemma 7. *Let T be the subgroup of L consisting of elements of the form $1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y}$ where $\check{y} = \sum_{i=0}^{2} iy^i$ and $r, r_1 \in \mathbb{F}_{3^k}$. Then $T \cong {C_3}^{4k}$.*

Proof. Let $t_1 = 1 + (a_1 + b_1x^2)\widehat{y} + (rx + r_1x^3)\check{y} \in T$ and $t_2 = 1 + (a_2 +$

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$b_2x^2)\widehat{y} + (sx + s_1x^3)\check{y} \in T$ where $a_i, b_j, r, r_1, s, s_1 \in \mathbb{F}_{3^k}$. Then

$$\begin{aligned}
t_1t_2 &= 1 + (a_2 + b_2x^2)\widehat{y} + (sx + s_1x^3)\check{y} + (a_1 + b_1x^2)\widehat{y} \\
&+ (a_1 + b_1x^2)\widehat{y}(a_2 + b_2x^2)\widehat{y} + (a_1 + b_1x^2)\widehat{y}(sx + s_1x^3)\check{y} \\
&+ (rx + r_1x^3)\check{y} + (rx + r_1x^3)\check{y}(a_2 + b_2x^2)\widehat{y} \\
&+ (rx + r_1x^3)\check{y}(sx + s_1x^3)\check{y} \\
&= 1 + ((a_1 + a_2) + (b_1 + b_2)x^2)\widehat{y} + ((r + s)x + (r_1 + s_1)x^3)\check{y} \\
&+ (rx + r_1x^3)\check{y}(sx + s_1x^3)\check{y} \\
&= 1 + ((a_1 + a_2 + \delta_1) + (b_1 + b_2 + \delta_2)x^2)\widehat{y} \\
&+ ((r + s)x + (r_1 + s_1)x^3)\check{y} \in T
\end{aligned}$$

where $\delta_1 = 2(rs_1 + sr_1)$ and $\delta_2 = 2(rs + r_1s_2)$. It can easily be shown that T is abelian. Therefore $T \cong {C_3}^{4k}$. □

Lemma 8. $L \cong {C_3}^{6k} \rtimes {C_3}^{2k}$.

Proof. Let $t = 1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y} \in T$ and $c = \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y} \in C_L(y)$ where $\sum_{i=0}^{2}\alpha_i = 1$, $\sum_{i=0}^{2}\alpha_{i+3} = 0$ and $\alpha_i, a, b, r, r_1 \in \mathbb{F}_{3^k}$. Then

$$\begin{aligned}
c^t &= (1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y})^2 \\
&\times \left(\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y}\right) \\
&\times (1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y}) \\
&= (1 - (a + bx^2)\widehat{y} - (rx + r_1x^3)\check{y} - (rx + r_1x^3)^2\widehat{y}) \\
&\times \left(\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + (a + bx^2)\widehat{y}\right. \\
&+ \left.\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i(rx + r_1x^3)\check{y} + (\alpha_3x + \alpha_7x^3)\widehat{y}\right) \\
&= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i(rx + r_1x^3)\check{y} \\
&+ (\alpha_3x + \alpha_7x^3)\widehat{y} - (rx + r_1x^3)\check{y}\sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i
\end{aligned}$$

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$$= \sum_{i=0}^{2}(\alpha_i + \alpha_{i+4}x^2)y^i + ((\alpha_3 + \delta_1)x + (\alpha_7 + \delta_2)x^3)\widehat{y} \in C_L(y)$$

where $\delta_1 = r(\alpha_2 - \alpha_1) + r_1(\alpha_6 - \alpha_5)$, $\delta_2 = r(\alpha_6 - \alpha_5) + r_1(\alpha_2 - \alpha_1)$.

Clearly $c^t \in C_L(y)$ and T normalizes $C_L(y)$. Let

$$U = C_L(y) \cap T = \left\{1 + (a + bx^2)\widehat{y} \mid a, b \in \mathbb{F}_{3^k}\right\}.$$

Clearly $|C_L(y) \cap T| = 3^{2k}$ and by the second Isomorphism Theorem $|C_L(y)T| = 3^{8k}$. Therefore $L = C_L(y)T$. T is an elementary abelian 3-group ($Y \cong U \times V$). Clearly $V \cap C_L(y) = \{1\}$ and V normalizes $C_L(y)$. Therefore $L = C_L(y)V \cong {C_3}^{6k} \rtimes {C_3}^{2k}$. □

Recall that $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}C_4)$. Now

$$\mathbb{F}_{3^k}C_4 \cong \begin{cases} {\mathbb{F}_{3^k}}^4 & \text{if } 4 \mid (3^k - 1) \\ {\mathbb{F}_{3^k}}^2 \oplus \mathbb{F}_{3^{2k}} & \text{if } 4 \nmid (3^k - 1). \end{cases}$$

$\mathbb{F}_{3^k}C_4 \cong {\mathbb{F}_{3^k}}^2 \oplus \mathbb{F}_{3^{2k}}$. Thus $\mathcal{U}(\mathbb{F}_{3^k}C_4) \cong {C_{3^k-1}}^4$ if $4 \mid (3^k - 1)$ or $\mathcal{U}(\mathbb{F}_{3^k}C_4) \cong {C_{3^k-1}}^2 \times C_{3^{2k}-1}$ if $4 \nmid (3^k - 1)$. Therefore

$$\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \cong$$
$$\cong \begin{cases} [({C_3}^{6k} \rtimes {C_3}^{2k}) \rtimes C_{3^k-1}^3] \times \mathcal{U}(\mathbb{F}_{3^k}) & \text{if } 4 \mid (3^k - 1) \\ [({C_3}^{6k} \rtimes {C_3}^{2k}) \rtimes (C_{3^k-1} \times C_{3^{2k}-1})] \times \mathcal{U}(\mathbb{F}_{3^k}) & \text{if } 4 \nmid (3^k - 1). \end{cases}$$

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