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Semi-commutativity criteria and self-coincidence of elements expressed by vectors properties of n-ary groups

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Abstract. In this paper new criteria of semi-commutativity and results on self-coincidence of an arbitrary point P in the terms of properties of vectors of n-ary groups are obtained.

It is well known that the most important tool for investigation of n-ary groups and for development of their applications is the concept of semi-commutativity. In this connection see for example [1, 2, 3, 4, 5, 6, 7, 8].

In the paper [9] P.A. Alexandrov introduced the concept of self-coincidence for geometric figures. He used this concept to construct different types of groups.

The results by S.A. Rusakov [5] and P.S. Alexandrov [9] allowed to introduce the concept of self-coincidence of points (of elements) of an n-ary group G.

Finding of new semi-commutativity criteria of n-ary groups as well as the study of self-coincidence of some elements of geometric figures constructed on the basis of an n-ary group is a very topical problem in our opinion.

The results presented in the paper are connected with the above-mentioned field of investigation. It should be noted that vector equalities which are presented in our theorems not only describe semi-commutativity criteria of an n-ary group $G = \langle X, (\ ), ^{[-2]} \rangle$ but establish the fact of self-coincidence of an arbitrary point $p \in X$ as well.

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Recall that an n-ary group G is said to be semi-abelian if the equality

$$(x_1 x_2^{n-1} x_n) = (x_n x_2^{n-1} x_1)$$

holds for any sequence $x_1^n \in X^n$. Further for the elements of an n-ary group $G = \langle X, (\ ), ^{[-2]} \rangle$ we use the term a *point*.

A point

$$S_a(b) = (a b^{[-2]} \overset{2n-4}{b} a)$$

is called a point that is symmetric with a point b relatively a point a. The sequence of k elements of X is called a k-gon of G. A tetragon $\langle a, b, c, d\rangle$ of an n-ary group G is called a parallelogram of G if

$$(a b^{[-2]} \overset{2n-4}{b} c) = d.$$

Let's say that a point $p \in X$ self-coincides if there is a sequence of symmetries of this point relatively other points of X, in the result of which this point maps into itself.

An ordered pair $\langle a, b\rangle$ of points $a, b \in X$ is called a directed segment of an n-ary group G and it is denoted by $\overline{ab}$.

If $a, b, c, d \in X$, then the directed segments $\overline{ab}$ and $\overline{cd}$ are called to be equal and they write $\overline{ab} = \overline{cd}$ if the tetragon $\langle a, c, d, b\rangle$ is a parallelogram of G.

Let $\overline{V}$ be the set of all directed segments of an n-ary group G. According to Proposition 1 in the paper [5] the binary relation $=$ on the set $\overline{V}$ is a relation of equivalence and partitions the set V into disjoint classes. The class generated by the directed segment $\overline{ab}$ has the following form:

$$K(\overline{ab}) = \{\overline{uv} \mid \overline{uv} \in \overline{V},\ \overline{uv} = \overline{ab}\}.$$

A vector $\overrightarrow{ab}$ of an n-ary group G is a class $K(\overline{ab})$, i.e. $\overrightarrow{ab} = K(\overline{ab})$.

Other notations, definitions and results used in the paper can be found in the following papers [4, 5, 6, 7, 8].

Now let us introduce the obtained results.

Theorem 1. *Let a, b, c, p be arbitrary points of X and $d \in X$ be a point such that the tetragon $\langle a, b, c, d\rangle$ is a parallelogram of G. An n-ary group G is semi-abelian if and only if the following equality holds:*

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a p))c} + \overrightarrow{S_c(S_b(S_a(p)))d} = \overrightarrow{0}. \tag{1}$$

Proof. 1. Let G be a semi-abelian n-ary group. Let's establish the validity of (1).

Taking into account Theorem 8 in [8], Definition 4 in [5], Proposition 1 in [8], Equality 3.28 in [4], and the fact that for any $x \in X$ sequences

$\overset{2n-4}{x^{[-2]}\,x}$ and $\overset{2n-4}{x^{[-2]}\,x}x$ are neutral $2(n-1)$-sequences the following can be obtained:

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} = \overrightarrow{p(a(S_a(p))^{[-2]}\underbrace{S_a(p)\ldots}_{2n-4}b)} =$$

$$= \overrightarrow{p(a(a\overset{2n-4}{p^{[-2]}\,p}a)^{[-2]}\underbrace{(a\overset{2n-4}{p^{[-2]}\,p}a)\ldots}_{2n-4}b)} =$$

$$= \overrightarrow{p(a\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b)} = \overrightarrow{p(p\overset{2n-4}{a^{[-2]}\,a}b)}. \quad (2)$$

Taking into account (2) one can obtain

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} = \overrightarrow{p(p\overset{2n-4}{a^{[-2]}\,a}b)} + \overrightarrow{S_b(a\overset{2n-4}{p^{[-2]}\,p}a)c} =$$

$$= \overrightarrow{p((p\overset{2n-4}{a^{[-2]}\,a}b)(S_b(a\overset{2n-4}{p^{[-2]}\,p}a))^{[-2]}\underbrace{S_b(a\overset{2n-4}{p^{[-2]}\,p}a)\ldots}_{2n-4}c)} =$$

$$= \overline{p((\overset{2n-4}{a^{[-2]}\,a}b)(b(a\overset{2n-4}{p^{[-2]}\,p}a)^{[-2]}\underbrace{(a\overset{2n-4}{p^{[-2]}\,p}a)\ldots}_{2n-4}b)^{[-2]}}$$

$$\overrightarrow{\underbrace{(b(a\overset{2n-4}{p^{[-2]}\,p}a)^{[-2]}\underbrace{(a\overset{2n-4}{p^{[-2]}\,p}a)\ldots}_{2n-4}b)\ldots}_{2n-4}c)} =$$

$$= \overline{p((p\overset{2n-4}{a^{[-2]}\,a}b)(b\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b)^{[-2]}}$$

$$\overrightarrow{\underbrace{(b\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b)\ldots}_{2n-4}c)} =$$

$$= \overrightarrow{p(p\overset{2n-4}{a^{[-2]}\,a}b\overset{2n-4}{b^{[-2]}\,b}a\overset{2n-4}{p^{[-2]}\,p}a\overset{2n-4}{b^{[-2]}\,b}c)} = \overrightarrow{p(a\overset{2n-4}{b^{[-2]}\,b}c)}. \quad (3)$$

Now taking into account Definition 4 in [5], Equality 3.28 in [4], Proposition 1 in [8] we have

$$S_c(S_b(S_a(p))) = S_c(S_b(a\overset{2n-4}{p^{[-2]}\,p}a)) =$$

$$= S_c(b(a\overset{2n-4}{p^{[-2]}\,p}a)^{[-2]}\underbrace{(a\overset{2n-4}{p^{[-2]}\,p}a)\ldots}_{2n-4}b) =$$

$$= S_c(b\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b) =$$

$$= (c(b\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b)^{[-2]}\underbrace{(b\overset{2n-4}{a^{[-2]}\,a}p\overset{2n-4}{a^{[-2]}\,a}b)\ldots}_{2n-4}c) =$$

$$= (cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c). \quad (4)$$

Taking into consideration (3) and (4), the fact that the tetragon $\langle a, b, c, d\rangle$ is a parallelogram of G and that G is a semi-abelian group one can obtain

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} + \overrightarrow{S_c(S_b(S_a(p)))d} =$$
$$= \overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}\;c)} + \overrightarrow{(cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c)d} =$$
$$= \overline{p((ab^{[-2]}\overset{2n-4}{b}\;c)(cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c)^{[-2]}}$$
$$\overrightarrow{\underbrace{(cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c)\ldots}_{2n-4} d)} =$$
$$= \overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}\;cc^{[-2]}\overset{2n-4}{c}\;ba^{[-2]}\overset{2n-4}{a}\;pa^{[-2]}\overset{2n-4}{a}\;bc^{[-2]}\overset{2n-4}{c}\;d)} =$$
$$= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}\;bc^{[-2]}\overset{2n-4}{c}\;d)} = \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}\;bc^{[-2]}\overset{2n-4}{c}(ab^{[-2]}\overset{2n-4}{b}\;c))} =$$
$$= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}\;bc^{[-2]}\overset{2n-4}{c}(cb^{[-2]}\overset{2n-4}{b}\;a))} = \overrightarrow{pp} = \overrightarrow{0}.$$

Thus we proved the equality (1).

2. Now we suppose that (1) is true. We shall prove that G is semi-abelian.

From (1) we have

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} = -\overrightarrow{S_c(S_b(S_a(p)))d}$$

and so

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} = \overrightarrow{dS_c(S_b(S_a(p)))}.$$

Therefore from (3) and (4) we have

$$\overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}\;c)} = \overrightarrow{d(cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c)}. \quad (5)$$

From (5) on the basis of Definition 2 in [5] we conclude that the tetragon

$$\langle p, d, (cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c), (ab^{[-2]}\overset{2n-4}{b}\;c)\rangle$$

is a parallelogram of G, so the equality

$$(pd^{[-2]}\overset{2n-4}{d}(cb^{[-2]}\overset{2n-4}{b}\;ap^{[-2]}\overset{2n-4}{p}\;ab^{[-2]}\overset{2n-4}{b}\;c)) = (ab^{[-2]}\overset{2n-4}{b}\;c). \quad (6)$$

holds.

Since by the hypothesis the tetragon $\langle a,b,c,d\rangle$ is a parallelogram of G the equality

$$(ab^{[-2]}\overset{2n-4}{b}c)=d. \tag{7}$$

is valid.

In view of (7) we obtain from (6) that

$$(p(ab^{[-2]}\overset{2n-4}{b}c)^{[-2]}\underbrace{(ab^{[-2]}\overset{2n-4}{b}c)\ldots}_{2n-4}(cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c))=$$
$$=(ab^{[-2]}\overset{2n-4}{b}c)$$

and hence

$$(pc^{[-2]}\overset{2n-4}{c}ba^{[-2]}\overset{2n-4}{a}(cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c))=(ab^{[-2]}\overset{2n-4}{b}c).$$

Therefore

$$(cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c)=(ab^{[-2]}\overset{2n-4}{b}cp^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c),$$

so

$$(cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}cc^{[-2]}\overset{2n-4}{c}ba^{[-2]}\overset{2n-4}{a}p)=(ab^{[-2]}\overset{2n-4}{b}c).$$

Then

$$(cb^{[-2]}\overset{2n-4}{b}a)=(ab^{[-2]}\overset{2n-4}{b}c). \tag{8}$$

Since a,b,c are arbitrary points of X then on the basis of Proposition 4 in [7] and (8) we conclude that G is a semi-abelian n-ary group.

The proof is complete.

Theorem 2. *Let a,b,c,d,p be arbitrary points of X. An n-ary group G is semi-abelian if and only if the following equality holds:*

$$\overrightarrow{pa}+\overrightarrow{S_a(p)b}+\overrightarrow{S_b(S_a(p))c}+\overrightarrow{S_c(S_b(S_a(p)))d}+$$
$$+\overrightarrow{S_d(S_c(S_b(S_a(p))))(dc^{[-2]}\overset{2n-4}{c}b)}+$$
$$+\overrightarrow{S_{(dc^{[-2]}\overset{2n-4}{c}b)}(S_d(S_c(S_b(S_a(p)))))}=\overrightarrow{0}. \tag{9}$$

Proof. 1. Let G be a semi-abelian n-ary group. We shall show that Equality (9) is true. In order to prove this we sequentially summarize vectors mentioned in (9) taking into account Theorem 8 in [8], Definition 4 in [5], Equality 3.28 in [4], Proposition 1 in [8], and the fact that for any $x \in X$ the sequences $x^{[-2]}\overset{2n-4}{x}\,x$ and $xx^{[-2]}\overset{2n-4}{x}$ are neutral $2(n-1)$-sequences.

So we have

$$\begin{aligned}
\overrightarrow{pa} + \overrightarrow{S_a(p)b} &= \overrightarrow{p(a(S_a(p))^{[-2]}\underbrace{S_a(p)\dots}_{2n-4})} = \\
&= \overrightarrow{p(a(ap^{[-2]}\overset{2n-4}{p}a)^{[-2]}\underbrace{(ap^{[-2]}\overset{2n-4}{p}a))\dots}_{2n-4}b)} = \\
&= \overrightarrow{p(aa^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)} = \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}b)};
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} &= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}b)} + \overrightarrow{S_b(S_a(p))c} = \\
&= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}b)} + \overrightarrow{(b(S_a(p))^{[-2]}\underbrace{S_a(p)\dots}_{2n-4}b)c} = \\
&= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}b)} + \overrightarrow{(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)c} = \\
&= \overline{p((pa^{[-2]}\overset{2n-4}{a}b)(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)^{[-2]}} \\
&\qquad \overrightarrow{\underbrace{(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)\dots}_{2n-4}c)} = \\
&= \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}bb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c)} = \overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}c)}. \quad (10)
\end{aligned}$$

Taking into account Definition 4 in [5], Equality 3.28 in [4], and Proposition 1 in [8] we have

$$\begin{aligned}
S_c(S_b(S_a(p))) &= S_c(S_b(ap^{[-2]}\overset{2n-4}{p}a)) = \\
&= S_c(b(ap^{[-2]}\overset{2n-4}{p}a)^{[-2]}\underbrace{(ap^{[-2]}\overset{2n-4}{p}a)\dots}_{2n-4}b) = \\
&= S_c(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b) = \\
&= (c(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)^{[-2]}\underbrace{(ba^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}b)\dots}_{2n-4}c) = \\
&= (cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}c). \quad (11)
\end{aligned}$$

Hence in view of (10) and (11) we obtain

$$\begin{aligned}
&\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} + \overrightarrow{S_c(S_b(S_a(p)))d} = \\
&\quad = \overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}\,c)} + \overrightarrow{(cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}\,ab^{[-2]}\overset{2n-4}{b}\,c)d} = \\
&\quad \overline{p((ab^{[-2]}\overset{2n-4}{b}\,c)(cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}\,ab^{[-2]}\overset{2n-4}{b}\,c)^{[-2]}} \\
&\quad \overrightarrow{\underbrace{(cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}\,ab^{[-2]}\overset{2n-4}{b}\,c)\dots}_{2n-4} d)} = \\
&\quad = \overrightarrow{p(ab^{[-2]}\overset{2n-4}{b}\,cc^{[-2]}\overset{2n-4}{c}\,ba^{[-2]}\overset{2n-4}{a}\,pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)} = \\
&\quad = \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)}. \qquad (12)
\end{aligned}$$

Taking into consideration (11) and the previous arguments we have

$$\begin{aligned}
&S_d(S_c(S_b(S_a(p)))) = \\
&\quad = (d(cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}\,ab^{[-2]}\overset{2n-4}{b}\,c)^{[-2]} \\
&\quad \underbrace{(cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}\,ab^{[-2]}\overset{2n-4}{b}\,c)\dots}_{2n-4} d) = \\
&\quad = (dc^{[-2]}\overset{2n-4}{c}\,ba^{[-2]}\overset{2n-4}{a}\,pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d). \qquad (13)
\end{aligned}$$

Taking into account (12) and (13) we have

$$\begin{aligned}
&\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} + \overrightarrow{S_c(S_b(S_a(p)))d} + \\
&\quad + \overrightarrow{S_d(S_c(S_b(S_a(p))))(dc^{[-2]}\overset{2n-4}{c}\,b)} = \overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)} + \\
&\quad + \overrightarrow{(dc^{[-2]}\overset{2n-4}{c}\,ba^{[-2]}\overset{2n-4}{a}\,pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)(dc^{[-2]}\overset{2n-4}{c}\,b)} = \\
&= \overline{p((pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)(dc^{[-2]}\overset{2n-4}{c}\,ba^{[-2]}\overset{2n-4}{a}\,pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)^{[-2]}} \\
&\quad \overrightarrow{\underbrace{(dc^{[-2]}\overset{2n-4}{c}\,ba^{[-2]}\overset{2n-4}{a}\,pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,d)\dots}_{2n-4}(dc^{[-2]}\overset{2n-4}{c}\,b)} = \\
&\quad = \overline{p(pa^{[-2]}\overset{2n-4}{a}\,bc^{[-2]}\overset{2n-4}{c}\,dd^{[-2]}\overset{2n-4}{d}\,cb^{[-2]}\overset{2n-4}{b}\,ap^{[-2]}\overset{2n-4}{p}} \\
&\quad \overrightarrow{ab^{[-2]}\overset{2n-4}{b}\,cd^{[-2]}\overset{2n-4}{d}\,dc^{[-2]}\overset{2n-4}{c}\,b)} = \overrightarrow{pa}. \qquad (14)
\end{aligned}$$

But G is semi-ablelian and so in view of (13) we have

$$S_{(dc^{[-2]}\overset{2n-4}{c}\,b)}(S_d(S_c(S_b(S_a(p))))) =$$

$$= ((dc^{[-2]^{2n-4}}_{c}b)(dc^{[-2]^{2n-4}}_{c}ba^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}bc^{[-2]^{2n-4}}_{c}d)^{[-2]}$$
$$\underbrace{(dc^{[-2]^{2n-4}}_{c}ba^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}bc^{[-2]^{2n-4}}_{c}d)\dots}_{2n-4}(dc^{[-2]^{2n-4}}_{c}b)) =$$
$$= ((bc^{[-2]^{2n-4}}_{c}d)d^{[-2]^{2n-4}}_{d}cb^{[-2]^{2n-4}}_{b}ap^{[-2]^{2n-4}}_{p}ab^{[-2]^{2n-4}}_{b}cd^{[-2]^{2n-4}}_{d}$$
$$dc^{[-2]^{2n-4}}_{c}b) = (ap^{[-2]^{2n-4}}_{p}a). \quad (15)$$

Finally using (14) and (15) we obtain

$$\overrightarrow{pa} + \overrightarrow{S_a(p)b} + \overrightarrow{S_b(S_a(p))c} +$$
$$+ \overrightarrow{S_c(S_b(S_a(p)))d} + \overrightarrow{S_d(S_c(S_b(S_a(p))))(dc^{[-2]^{2n-4}}_{c}b)} +$$
$$+ \overrightarrow{S_{(dc^{[-2]^{2n-4}}_{c}b)}(S_d(S_c(S_b(S_a(p)))))a} = \overrightarrow{pa} + \overrightarrow{(ap^{[-2]^{2n-4}}_{p}a)a} =$$
$$= \overrightarrow{p(a(ap^{[-2]^{2n-4}}_{p}a)^{[-2]}\underbrace{(ap^{[-2]^{2n-4}}_{p}a)\dots}_{2n-4}a)} =$$
$$= \overrightarrow{p(aa^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}a)} = \overrightarrow{pp} = \overrightarrow{0}.$$

Consequently we have proved that (9) holds.

2. Suppose that (9) is true. We shall prove that G is a semi-abelian group.

Since in the previous arguments the property of semi-commutativity of G was used only in (15) we can conclude that (14) holds.

From (9) we obtain

$$\overrightarrow{pa} + \overrightarrow{S_{(dc^{[-2]^{2n-4}}_{c}b)}(S_d(S_c(S_b(S_a(p)))))a} = \overrightarrow{0}$$

so taking into account (13) we have

$$\overrightarrow{pa} + \overrightarrow{S_{(dc^{[-2]^{2n-4}}_{c}b)}(dc^{[-2]^{2n-4}}_{c}ba^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}bc^{[-2]^{2n-4}}_{c}d)a} = \overrightarrow{0},$$

so

$$\overrightarrow{pa} + \overline{((dc^{[-2]^{2n-4}}_{c}b)(dc^{[-2]^{2n-4}}_{c}ba^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}bc^{[-2]^{2n-4}}_{c}d)^{[-2]}}$$
$$\overrightarrow{\underbrace{(dc^{[-2]^{2n-4}}_{c}ba^{[-2]^{2n-4}}_{a}pa^{[-2]^{2n-4}}_{a}bc^{[-2]^{2n-4}}_{c}d)\dots}_{2n-4}(dc^{[-2]^{2n-4}}_{c}b))a} = \overrightarrow{0},$$

and hence

$$\overrightarrow{pa} + \overrightarrow{((dc^{[-2]}\overset{2n-4}{c}b)d^{[-2]}\overset{2n-4}{d}cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}ab^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}dc^{[-2]}\overset{2n-4}{c}b)a} = \overrightarrow{0},$$

or

$$\overrightarrow{pa} + \overrightarrow{((dc^{[-2]}\overset{2n-4}{c}b)d^{[-2]}\overset{2n-4}{d}cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}a)a} = \overrightarrow{0}.$$

Then taking into consideration Theorem 8 in [8] we have

$$\overrightarrow{p(a(dc^{[-2]}\overset{2n-4}{c}bd^{[-2]}\overset{2n-4}{d}cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}a)^{[-2]}\underbrace{(dc^{[-2]}\overset{2n-4}{c}bd^{[-2]}\overset{2n-4}{d}cb^{[-2]}\overset{2n-4}{b}ap^{[-2]}\overset{2n-4}{p}a)\ldots a}_{2n-4})} = \overrightarrow{0},$$

so

$$\overrightarrow{p(aa^{[-2]}\overset{2n-4}{a}pa^{[-2]}\overset{2n-4}{a}bc^{[-2]}\overset{2n-4}{c}db^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}a)} = \overrightarrow{0},$$

and hence

$$\overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}bc^{[-2]}\overset{2n-4}{c}db^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}a)} = \overrightarrow{0}.$$

Since $\overrightarrow{0} = \overrightarrow{pp}$ holds for any $p \in X$, then

$$\overrightarrow{p(pa^{[-2]}\overset{2n-4}{a}bc^{[-2]}\overset{2n-4}{c}db^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}a)} = \overrightarrow{pp}.$$

From this equality it can be concluded that

$$(pa^{[-2]}\overset{2n-4}{a}bc^{[-2]}\overset{2n-4}{c}db^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}a) = p.$$

Let's multiply both parts of this equality by the expression $ap^{[-2]}\overset{2n-4}{p}$ from the left, and by the expression $a^{[-2]}\overset{2n-4}{a}dc^{[-2]}\overset{2n-4}{c}b$ from the right. Then

$$(ap^{[-2]}\overset{2n-4}{p}pa^{[-2]}\overset{2n-4}{a}(bc^{[-2]}\overset{2n-4}{c}d)b^{[-2]}\overset{2n-4}{b}cd^{[-2]}\overset{2n-4}{d}aa^{[-2]}\overset{2n-4}{a}dc^{[-2]}\overset{2n-4}{c}b) = (ap^{[-2]}\overset{2n-4}{p}pa^{[-2]}\overset{2n-4}{a}dc^{[-2]}\overset{2n-4}{c}b).$$

Hence taking into account the neutrality of the sequences $x^{[-2]}\overset{2n-4}{x}x$ and $xx^{[-2]}\overset{2n-4}{x}$ for any $x \in X$ we obtain

$$(bc^{[-2]}\overset{2n-4}{c}d) = (dc^{[-2]}\overset{2n-4}{c}b). \tag{16}$$

Taking into consideration the arbitrariness of points $b, c, d \in X$ on the basis of Proposition 4 in [7] and (16) we conclude that G is a semi-abelian n-ary group.

The proof is complete.

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