The structure of automorphism groups of semigroup inflations

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ABSTRACT.   In this paper we prove that the automorphism
group of a semigroup being an inflation of its proper subsemigroup
decomposes into a semidirect product of two groups one of which
is a direct sum of full symmetric groups.

1.   Introduction

In the study of a specific semigroup the description of all its automor-
phisms is one of the most important questions. The automorphism groups
of many important specific semigroups are described (see, for example, [7]
and references therein). It can be observed that for two types of semi-
groups of rather diverse nature the automorphism groups have similarities
in their structure: each of them decomposes into a semidirect product of
two groups one of which being a direct sum of full symmetric groups.
The mentioned two types of semigroups are variants of some semigroups
of mappings (see [4], [6]) and maximal nilpotent subsemigroups of some
transformation semigroups (see [2], [5]).

In the present paper we establish a general result, which, in particu-
lar, unifies all the above-mentioned examples. We deal with an abstract
semigroup being an inflation of its proper subsemigroup, constructed as
follows. Let S be a semigroup. First introduce an equivalence relation
h on S via $(a, b) \in h$ if and only if $ax = bx$ and $xa = xb$ for all $x \in S$,
that is a and b are not distinguished by multiplication from either side.
We would like to remark that the history of the study of h stems back at

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least to [6], where it was considered for the variants of certain semigroups of mappings. Set further

$$\psi = (h \cap ((S \setminus S^2) \times (S \setminus S^2))) \cup \{(a, a) : a \in S^2\}.$$

Denote by T any transversal of ψ. Then T is a subsemigroup of S, and S is an inflation of T. An automorphism τ of T will be called *extendable* provided that τ coincides with the restriction to T of a certain automorphism of S. Clearly, all extendable automorphisms of T constitute a subgroup, H, of the group $\mathrm{Aut}T$ of all automorphisms of T.

We state our result as the following theorem.

Theorem 1. *The group $\mathrm{Aut}S$ is isomorphic to a semidirect product of two groups one of which (the one which is normal) is the direct sum of the full symmetric groups on the ψ-classes and the other one is the group H consisting of all extendable automorphisms of T.*

All the papers [2], [4], [5], [6] deal precisely with the "semigroup inflation construction" introduced in this paper (though this construction is not mentioned explicitly in any of these papers), and their main results concerning automorphism groups can be deduced from our present general result — Theorem 1. Recently Theorem 1 was applied to describe automorphism groups of certain partition semigroups (see [3], Section 11 for details).

2. Construction

A semigroup S is called an *inflation* of its subsemigroup (see [1], Section 3.2) T provided that there is an onto map $\theta : S \to T$ such that:

- $\theta^2 = \theta$;

- $a\theta b\theta = ab$ for all $a, b \in S$.

In the described situation S is often referred to as an *inflation of T with an associated map θ* (or just *with a map θ*).

It is immediate that if S is an inflation of T then T is a retract of S (that is the image under an idempotent homomorphism) and that $S^2 \subset T$.

Lemma 2. *Suppose that S is an inflation of T with the map θ. Then $\ker\theta \subset h$.*

Proof. Let $(a, b) \in \ker\theta$ and $s \in S$. Then

$$as = a\theta s\theta = b\theta s\theta = bs; \quad sa = s\theta a\theta = s\theta b\theta = sb,$$

which implies that $(a, b) \in h$. $\qquad\square$

Lemma 3. *The equivalence ψ, defined in the Introduction, is a congruence on S.*

Proof. Obviously, ψ is an equivalence relation. Prove that ψ is left and right compatible. Let $(a, b) \in \psi$ and $a \neq b$. Then $(a, b) \in (h \cap ((S \setminus S^2) \times (S \setminus S^2)))$. Let $c \in S$. Since $(a, b) \in h$ we have that $ac = bc$ and $ca = cb$ for each $c \in S$. It follows that $(ac, bc) \in \psi$ and $(ca, cb) \in \psi$ as ψ is reflexive. □

Fix an arbitrary transversal of ψ and denote it by T.

Lemma 4. *T is a subsemigroup of S, and S is an inflation of T.*

Proof. T is a subsemigroup of S as $T \supset S^2$. Let θ be the map $S \to T$ which sends any element x from S to the unique element of the ψ-class of x, belonging to T. The construction implies that S is an inflation of T with the map θ. □

Let $S = \cup_{a \in T} X_a$ be a decomposition of S into the union of ψ-classes, where X_a denotes the ψ-class of a. Set G_a to be the full symmetric group acting on X_a and $G = \oplus_{a \in T} G_a$.

Lemma 5. *π is an automorphism of S, for every $\pi \in G$.*

Proof. It is enough to show that $(xy)\pi = x\pi y\pi$ whenever $x, y \in S$. Suppose first that $x, y \in S \setminus S^2$. Since $xy \in S^2$ it follows that π stabilizes xy, so that $(xy)\pi = xy$. Now, the inclusions $(x, x\pi) \in h$ and $(y, y\pi) \in h$ imply $x\pi y\pi = x\pi y = xy$. This yields $xy\pi = x\pi y\pi$, and the proof is complete. □

The following proposition gives a characterization of extendable automorphisms of T.

Proposition 6. *An automorphism τ of T is extendable if and only if the following condition holds:*

$$(\forall a, b \in T) \; a\tau = b \Rightarrow |X_a| = |X_b|. \tag{1}$$

Proof. Suppose $\tau \in \mathrm{Aut}T$ is extendable and $a \in T$. In the case when $a \in S \setminus S^2$ we have

$$X_a = \{b \in S \mid (a, b) \in h \text{ and } b \in S \setminus S^2\}.$$

Clearly, $(a, b) \in h \iff (a\tau, b\tau) \in h$ and $b \in S \setminus S^2 \iff b\tau \in S \setminus S^2$ for all $a, b \in S$. It follows that

$$X_{a\tau} = \{b\tau \mid b \in X_a\},$$

which implies (1). The inclusion $a \in S^2$ is equivalent to $a\tau \in S^2$. But then $|X_a| = |X_{a\tau}| = 1$, which also implies (1).

Suppose now that (1) holds for certain $\tau \in \operatorname{Aut} T$. Then one can extend τ to $\overline{\tau} \in \operatorname{Aut} S$ as follows.

Fix a collection of sets I_a, $a \in T$, and bijections $f_a : I_a \to X_a$, $a \in T$, satisfying the following conditions:

- $|I_a| = |X_a|$;

- $I_a = I_b$ whenever $|X_a| = |X_b|$;

- $I_a \cap I_b = \emptyset$ whenever $|X_a| \neq |X_b|$;

- if $a, b \in T$ and $|X_a| = |X_b|$ then $a f_a^{-1} = b f_b^{-1}$.

It is straightforward that such collections I_a, $a \in T$, and f_a, $a \in T$, exist.

Consider $x \in S \setminus T$. Since T is a transversal of ψ, there is $a \in T$ such that $x \in X_a$. By the hypothesis we have $|X_a| = |X_{a\tau}|$. Set $\overline{\tau}$ on X_a to be the map from X_a to $X_{a\tau}$ defined via $x \mapsto x f_a^{-1} f_{a\tau}$. In this way we define a permutation $\overline{\tau}$ of S such that $\overline{\tau}|_T = \tau$. It will be called an *extension* of τ to S. To complete the proof, we are left to show that $\overline{\tau}$ is a homomorphism. Let $x, y \in S$, $x \in X_a$, $y \in X_b$. Then

$$(xy)\overline{\tau} = (ab)\tau = a\tau b\tau = x\overline{\tau} y\overline{\tau},$$

as required.

$\square$

Let $\tau \in H$. Of course, $\overline{\tau}$, constructed in the proof of Proposition 6, depends not only on τ, but also on the sets I_a and the maps f_a, so that τ may have several extentions to S. Fix any extention $\overline{\tau}$ of τ.

Lemma 7. $\tau \mapsto \overline{\tau}$ *is an embedding of H into* $\operatorname{Aut} S$.

Proof. Proof is immediate by the construction of $\overline{\tau}$. $\square$

Denote by $\overline{H}$ the image of H under the embedding of H into $\operatorname{Aut} S$ from Lemma 7.

3. Proof of Theorem 1

Proposition 8. $\overline{H}$ *acts on G by automorphisms via* $\pi^\tau = \tau^{-1}\pi\tau$, $\tau \in \overline{H}, \pi \in G$.

Proof. Let $\pi \in G$ and $\tau \in \overline{H}$. Show first that $\pi^\tau \in G$. Take any $x \in S$. Let X_a be the block which contains x. We consequently have $x\tau^{-1} \in X_{a\tau^{-1}}$, $x\tau^{-1}\pi \in X_{a\tau^{-1}}$ and $x\tau^{-1}\pi\tau \in X_{a\tau^{-1}\tau} = X_a$. Hence $x\pi^\tau \in X_a$. It follows that $\pi^\tau \in G$.

That $\pi \mapsto \pi^\tau$ is one-to-one, onto and homomorphic immediately follows from its definition. We are left to show that the map sending $\tau \in \overline{H}$ to $\pi \mapsto \pi^\tau \in \operatorname{Aut}G$ is homomorphic. The latter follows from the equalities $\pi^{\tau_1\tau_2} = (\tau_1\tau_2)^{-1}\pi(\tau_1\tau_2) = (\pi^{\tau_1})^{\tau_2}$. The proof is complete. $\square$

In the following two lemmas we show that G and $\overline{H}$ intersect by the identity automorphism and generate $\operatorname{Aut}S$.

Lemma 9. $G \cap \overline{H} = \{id\}$, *where id is the identity automorphism of S.*

Proof. The proof follows from the observation that the decomposition $S = \cup_{a \in T} X_a$ is fixed by each element of G, while only by the identity element of $\overline{H}$. $\square$

Lemma 10. $\operatorname{Aut}S = \overline{H} \cdot G$.

Proof. Let $\varphi \in \operatorname{Aut}S$. The definition of ψ implies that φ maps each ψ-class onto some other ψ-class. Define a bijection $\tau : T \to T$ via $a\tau = b$ provided that $X_a\varphi = X_b$. Show that τ is an extendable automorphism of T. The definition of τ assures that (1) holds, and thereby τ is extendable in view of Proposition 6. Let $\overline{\tau} \in \operatorname{Aut}S$ be an extension of τ. The construction implies that $\varphi(\overline{\tau})^{-1} \in G$. $\square$

Now the proof of Theorem 1 follows from Proposition 8 and Lemmas 9 and 10.

4.   Acknowledgements

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