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RESEARCH ARTICLE

On division rings with general involution

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Abstract. In this work we consider division rings with general involution. Properties of such division rings are investigated. General valuation of such division rings is introduced. We extend to this general notion some result on the extension of valuations.

1. Introduction

In this paper, we study a generalization of the notion of an involution, which we will refer to as an ε-involution. An ε-involution on a division ring D is an anti-automorphism $*$ whose square is a conjugation by an element $\varepsilon \in D$ such that $\varepsilon\varepsilon^* = \varepsilon^*\varepsilon = 1$. In Section 2, we establish elementary properties and basic facts about the considered division rings. We show for instance that, if all the symmetric elements in a division ring D with ε-involution are central, then D is a commutative field.

Valuations of division rings with ε-involution are introduced in Section 3. In [1], a necessary and sufficient condition is given for extending an abelian valuation from a division ring to an over division ring. We solve here a $*$-version of this problem, where valuations are replaced with ε-valuations.

2. Properties of division rings with ε-involution

Let D be a division ring with centre $Z(D)$.

Definition 2.1. *An anti-automorphism $x \mapsto x^*$ of the division ring D is called an ε-involution if there is $\varepsilon \in D$ such that for each $x \in D$, $(x^*)^* = \varepsilon^{-1}x\varepsilon$; where $\varepsilon\varepsilon^* = \varepsilon^*\varepsilon = 1$.*

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The general idea is to consider an anti-automorphism $* : D \mapsto D$ on a division ring D such that $(x^*)^* = \varepsilon^{-1}x\varepsilon$ for some ε in D (i.e. up to an inner automorphism it has order 2). Indeed, the map sending x to εx^* is a map on D of order 2 (although in general not an anti-automorphism). As a first observation, when $\varepsilon = 1$, then $*$ is an involution in the usual sense. More generally, if ε is in the centre $Z(D)$, again $*$ is an involution.

Definition 2.2. *Let D be a division ring with ε-involution $*$. An element $x \in D$ is called successively* ***a symmetric element*** *if $\varepsilon x^* = x$,* ***a skew-symmetric element*** *if $\varepsilon x^* = -x$.*

Let S, K denote the additive subgroups of the division ring D formed by the symmetric elements and the skew-symmetric elements respectively. In symbols:

$$S = \{x \in D | \varepsilon x^* = x\}; \quad K = \{x \in D | \varepsilon x^* = -x\}.$$

Let T denote the subgroup of traces: $T = \{x + \varepsilon x^* | x \in D\}$.

Examples 2.3.

1. For $D = C$, the complex field, complex conjugation is an ε-involution for every complex number ε of norm 1.

2. Let $H = \{i, j | i^2 = -1, j^2 = -1, ij = -ji\}$ stand for the classical real quateraionic algebra $\{a_0 + a_1 i + a_2 j + a_3 k | k = ij, a_i \in R\}$. Take for $*$ the mapping defined by

$$(a_0 + a_1 i + a_2 j + a_3 k)^* = a_0 - a_1 i - a_2 j + a_3 k \quad (a_i \in R).$$

It is easy to check that $*$ is an anti-autmorphism of H such that $x^{**} = \varepsilon^{-1}x\varepsilon$ for every $x \in H$, where $\varepsilon = i$. Thus $*$ is an ε-involution. Here the additive subgroup of symmetric elements is the 1-dimensional subspace $R(1+i)$, the additive subgroup of skew symmetric elements is $R(i-1) + R(j-k) + R(k-j)$.

3. Let H stand for the classical real quateraionic algebra given in 2. Take for $*$ the conjugation mapping defined by

$$(a_0 + a_1 i + a_2 j + a_3 k)^* = a_0 - a_1 i - a_2 j - a_3 k \quad (a_i \in R).$$

Then, $*$ is an ε-involution with $\varepsilon = 1$. In fact, $*$ is the unique 1-involution of H such that $S \supset R$, $T \subset R$, and $xx^* \in R$ for all $x \in R$. We note that H has many 1-involutions with $S \supset R$, in fact any map $x \mapsto ux^*u^{-1}$, u a unit of H with $u^* = \pm u$, is an involution. Conversely, any involution $\sharp$ of H with $S = R$ can be expressed as $x^\sharp = ux^*u^{-1}$, for $u^* = \pm u$.

Remark 2.4. *If $char(D) \neq 2$, then*

(1) For each $x \in D$, $x = s + k$ where s is a symmetric element and k is a skew-symmetric element.

(2) $T = S$.

(3) S and K are preserved under the 2-sided translation $a \mapsto xax^*$, $x \in D$. If $char(D) = 2$, we still may have $T = S$, e.g. if there exists an element z in the center of D with $z^* \neq z$.

Proof. (1) Since $char(D) \neq 2$ it follows that $x = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*) = s + k$ as desired.

(2) Clearly $T \subset S$. Since each symmetric element can be written in the form $s = (\frac{1}{2})s + \varepsilon((\frac{1}{2})s)^*$, it follows that $T = S$.

(3) If a is a symmetric element, then $\varepsilon(xax^*)^* = \varepsilon\varepsilon^{-1}x\varepsilon.a^*.x^* = xax^*$, so that xax^* is a symmetric element. Similarly one can prove the result for a skew-symmetric element a. $\square$

We note that any ε-involution $*$ with $\varepsilon \neq 1$ is such that ε is not symmetric. In fact, if ε is symmetric, then $\varepsilon\varepsilon^* = \varepsilon$, so that $\varepsilon = 1$, a contradiction. We also note that if k is a skew symmetric element and s is a symmetric element of D, then the following are symmetric elements: $k(1+\varepsilon)k^* = kk^* - k^2$; $s(1+\varepsilon)s^* = ss^* + s^2$. For $a, b \in D$, put $(a, b) = ab - ba$.

Lemma 2.5. *Let D be a division ring with ε-involution $*$ such that $char(D) \neq 2$, and suppose that $sk - ks = 0$ for all symmetric s and skew symmetric k in D. Then for all $s \in S$ and, $x \in D$, follows $(s, (s, x)) = 0$.*

Proof. Firstly we claim that $\varepsilon \in Z$. Since $1 + \varepsilon$ is a symmetric element, it follows that $1 + \varepsilon$ commutes with each skew symmetric element in D, as does ε. Also, since $1 - \varepsilon$ is a skew symmetric element of D, it follows that $1 - \varepsilon$ commutes with each symmetric element, as does ε. Thus ε commutes with all symmetrics and skew symmetrics in D. In view of Remark 2.4, ε is in the centre of D.

Now, let s and s' be two arbitrary symmetric elements of D. It is easy to check that $(1 + \varepsilon^*)(s, s')$ is a skew symmetric element; and consequently, this element commutes with s. Since $\varepsilon \in Z(D)$, it follows that $\varepsilon^* \in Z(D)$ so that $1 + \varepsilon^* \in Z(D)$. Hence $(s, (s, s')) = 0$. Since we also have $(s, (s, k)) = 0$ for all skew symmetric elements $k \in D$, it follows that $(s, (s, x)) = 0$ for all $x \in D$. $\square$

Lemma 2.6 ([2]). *Let D be a division ring with $char(D) \neq 2$. If $(a, (a, x)) = 0$ for all $x \in D$, then a is in the centre of D.*

The following corollary is an immediate consequence of the above two lemmas.

Corollary 2.7. *If D is a division ring with ε-involution $*$ such that $sk - ks = 0$ for all symmetrics s and skew symmetrics k in D, then all the symmetric elements of D are central.*

Theorem 2.8. *Let D be a division ring with ε-involution $*$, where $\varepsilon \neq 1$. Suppose that all the symmetric elements in D are central. Then D is a commutative field.*

Proof. Since $1 + \varepsilon \in S \subset Z(D)$, it follows that $\varepsilon \in Z(D)$. Two cases to be considered:
(i) If $\varepsilon \neq -1$: Let $k_0 = \frac{1-\varepsilon}{1+\varepsilon}$. Then $k_0 \in Z(D)$ and $\varepsilon = \frac{1-k_0}{1+k_0}$. Also, $(k_0)^* = -k_0$. Now, we claim that each skew symmetric element σ is central. Because, $\varepsilon(k_0\sigma)* = \varepsilon\sigma^*(k_0)^* = \sigma k_0 = k_0\sigma$ (where $k_0 \in Z(D)$) , it follows that $k_0\sigma$ is a symmetric element. Then $k_0\sigma \in Z(D)$. Since $k_0 \in Z(D)$, it follows that $\sigma \in Z(D)$.
(ii) If $\varepsilon = -1$: In this case s is a symmetric element if and only if $s^* = -s$, and k is a skew symmetric element if and only if $k^* = k$. Then each $s = -s^*$ is central. For a symmetric element s and a skew symmetric element k in D one has $(sk)^* = k^*s^* = -ks = -(sk)$. Thus $sk \in Z(D)$ giving $k \in Z(D)$. □

Remark 2.9. *Let D be a division ring with ε-involution $*$. If p is any non-zero element in D, then the formula $x^\sharp = p^{-1}xp$ defines an ε_1-involution $\sharp$ of D, where $\varepsilon_1 \in D$ can be found.*

Proof. Clearly $\sharp$ is an antiautomorphism. Put $\varepsilon_1 = \varepsilon(p^{-1})^*p$. It is to be shown that $\sharp$ is an ε_1-invo1ution. For,

$$\begin{aligned} x^{\sharp\sharp} &= p^{-1}(p^{-1}x^*p)^*p \\ &= p^{-1}p^*\varepsilon^{-1}x(p^{-1})^*p \\ &= \varepsilon_1^{-1}x\varepsilon_1. \end{aligned}$$

One also has $\varepsilon_1\varepsilon_1^\sharp = \varepsilon(p^{-1})^*pp^{-1}p^*\varepsilon^{-1} = 1$. Thus $\sharp$ is an ε_1-involution. □

By analogy with the usual notion of co-gradient involution, refer to $\sharp$ appearing above to be a co-gradient ε-invo1ution (ε need not be the same). It is appropriate to observe that while in the classical case one requires p to be a symmetric element, here we will dispense with this requirement. To find a co-gradient involution $\sharp$, all that is needed is that $\varepsilon_1 = \varepsilon(p^{-1})^*p = 1$, meaning that p^{-1} is a symmetric element relative to the initial ε-invo1ution.

Theorem 2.10. *If D is a division ring, then D admits ε-involution if and only if it admits a usual involution.*

Proof. If $x^* = x$ for each $x \in D$, then evidently D is commutative, in which case there is nothing more to prove. If, on the other hand, there is $x \neq x^*$, we contend that D contains a symmetric element a. For, put $a = x + \varepsilon x^*$. If $a \neq 0$, then we are done. If, on the other hand, $a = 0$ then $\varepsilon x^* = -x$, so that $b = xx^* - x^*$ is a non-zero symmetric element. Now, take p to be a^{-1}. Define $\sharp$ by:

$$a^\sharp = p^{-1}a^*p(a \in D).$$

Thus $\sharp$ is an involution. Note that if all the symmetric elements are central, then by Theorem 2.8, D is commutative.

For the converse, take $\sharp$ to be a usual involution of D. Define $*$ of D by:

$$x^* = b^{-1}x^\sharp b \text{ (for some } b \in D, b \neq \pm b^\sharp).$$

Clearly $*$ is an anti-automorphism of D, and $x^{**} = \varepsilon^{-1}x\varepsilon$ when $\varepsilon = (b^{-1})^\sharp b$. Also $\varepsilon\varepsilon^* = (b^{-1})^\sharp b b^{-1}\varepsilon^\sharp b = (b^{-1})^\sharp b^\sharp b^{-1} b = 1$, as desired. $\square$

Example 2.11. *Let $D = H$ the classical real quatrenionic algebra. The usual involution of D is defined by*

$$(a_0 + a_1 i + a_2 j + a_3 k)^\sharp = a_0 - a_1 i - a_2 j - a_3 k.$$

With reference to the same ε-involution $*$ considered in Example 2.3(2), $(a_0 + a_1 i + a_2 j + a_3 k)^* = a_0 - a_1 i - a_2 j + a_3 k$, one can say that $*$ is a co-gradient ε-involution. For, one can check that $x^* = p^{-1}x^\sharp p(x \in D)$, where $p = 1 + i$. Changing $p = 1 + i$ to $p = 1 + j$ or $p = 1 + k$, this gives two additional ε-involutions of D.

In [3], W. Scharlau, making use of Hilbert's Theorem, established that if D is a finite dimensional central simple algebra with an anti-automorphism θ such that $\theta^2(x) = b^{-1}xb(x \in D)$, where b is such that $\theta(b)b = b\theta(b) = a\theta(a)$ for some central element a, then D possesses an involution. From Scharlau's arguments we may restrict our attention to the case where D is a division ring. In this case the given anti-automorphism θ is, in fact, an ε-involution. For, if $\varepsilon = a^{-1}b$, then

$$\begin{aligned} \varepsilon.\theta(\varepsilon) &= (a^{-1}b)\theta(a^{-1}) \\ &= a^{-1}b\theta(b)\theta(a^{-1}) = 1. \end{aligned}$$

Also, $\varepsilon^{-1}x\varepsilon = a(b^{-1}xb)a^{-1} = a(\theta^2(x))a^{-1} = \theta^2(x)$.

Therefore, Scharlau's criterion is simply Theorem 2.10. We note that in Theorem 2.10, we do not use Hilbert's Theorem.

The last topic of this section is about the symmetrics and the skew-symmetrics of $(D,*)$ and $(D,\sharp)$, where $\sharp$ is a co-gradient ε_1-involution of $D(\varepsilon_1 = \varepsilon(p^{-1})^* p$ as in Remark 2.9 above).

Theorem 2.12. *Let S_1 and K_1 be the additive subgroups of $(D,\sharp)$ of symmetric elements and skew symmetric elements respectively. Then $S_1 = Sp$; and $K_1 = Kp$.*

Proof. Let $s_1 \in S_1$. Then $\varepsilon_1 s_1^\sharp = s_1$ so that

$$\begin{aligned}\varepsilon(p^*)^{-1}bb^{-1}s_1^*b &= s_1,\\ \varepsilon(p^*)^{-1}s_1^*b &= s_1.\end{aligned}$$

Hence $\varepsilon(s_1b^{-1})^* = (s_1b^{-1})$, and so, s_1b^{-1} is a symmetric element with respect to $*$. Thus, $S_1b^{-1} \subseteq S$ that is $S_1 \subseteq Sb$.

Conversely, if $\varepsilon s^* = s$, then

$$\begin{aligned}\varepsilon_1(sb)^\sharp &= \varepsilon_1 b^\sharp s^\sharp\\ &= \varepsilon s^* b\\ &= sb.\end{aligned}$$

Hence, $sb \in S_1$, so that $Sb \subseteq S_1$. Similarly, one can show that $K_1 = Kb$. $\square$

3. ε-Valuations of division rings with ε-involution

Let D be a division ring with ε-involution $*$ and let $D^\bullet$ be the multiplicative group of nonzero elements of D . By ε-valuation we mean a mapping $\omega : D \to G$, where G is a linearly ordered additive group with positive infinity adjoined, such that

1. $\omega(xy) = \omega(x) + \omega(y)(x,y \in D)$;
2. $\omega(x+y) \geq min\{\omega(x),\omega(y)\}(x+y \neq 0)$;
3. ω maps D onto G; and
4. $\omega(\varepsilon x^*) = \omega(x)$.

We first remark that any ε-valuation $\omega : D \to G$ is abelian, that is, its value group G is an abelian group, because

$$\omega(x)+\omega(y) = \omega(xy) = \omega((xy)^*) = \omega(y^*x^*) = \omega(y^*)+\omega(x^*) = \omega(y)+\omega(x).$$

Let $R = \{x \in D | \omega(x) \geq 0\}$. Evidently R is a subring of D, R is **total** (i.e., contains x or x^{-1} for every non-zero element x in D), and R is **symmetric** (i.e., contains $\varepsilon x^* x^{-1}$ for every element x in D). For, $\omega(\varepsilon x^* x^{-1}) = \omega(\varepsilon x^*) + \omega(x^{-1}) = \omega(x) + \omega(x^{-1}) = 0$.

Definition 3.1. *Any subring R of D which is total and symmetric is called* ***ε-valuation ring****.*

Theorem 3.2. *Given any ε-valuation ring R of a division ring D with ε-involution, there exist a linearly ordered abelian group G , and an ε-valuation $\omega : D \to G$ such that R coincides with the valuation ring of ω.*

Proof. Let U denote the multiplicative group of invertible elements of R. Then $\varepsilon x^* x^{-1} \in U$ for every $x \in D$, because $(\varepsilon x^* x^{-1})^{-1} = x(x^*)^{-1}\varepsilon^{-1} = \varepsilon(\varepsilon x^*)^*(\varepsilon x^*)^{-1} \in R$. Observe that ε and ε^* are in U. Also, since U is multiplicative, it follows that $x^* x^{-1} = \varepsilon^* \varepsilon x^* x^{-1} \in U$, for every $x \in D$.

We now claim that $xyx^{-1}y^{-1} \in U$ for every $x, y \in D$. For, one has the following identity

$$xyx^{-1}y^{-1} = [\varepsilon(x^*)^*(x^*)^{-1}][(\varepsilon y^* x)^*(\varepsilon y^* x)^{-1}][\varepsilon y^* y^{-1}].$$

Hence, U contains the commutator subgroup $[D, D]$ of $D^\bullet$. Thus U is a normal subgroup of $D^\bullet$, and consequently, the factor group $G = D^\bullet/U$ is abelian. Define $\omega : D^\bullet \to G$ to be the canonical mapping, and G order by setting

$$a \in D^\bullet, aU \geq 1 \text{ if and only if } \omega(a) \geq 0.$$

Switching from the multiplicative linearly ordered group G to the same ordered system where the multiplication is written addition, one can quickly check axioms 1 to 4, so that, ω is an ε-valuation of D with value group precisely G and with valuation ring evidently R. □

Lemma 3.3. *Any symmetric subring R of a division ring D with ε-involution has the following properties:*

1. *R is closed under the ε-involution ($x \in R \Rightarrow x^* \in R$).*
2. *R contains ε, ε^*, and $x^* x^{-1}$, for every $x \in D$.*
3. *R contains the commutator subgroup $[D, D]$, and hence R is preserved under conjugation ($x^{-1}Rx = R$, for every $x \in D$);*
4. *Each ideal I of R is closed under the ε-involution, is two sided, and is such that $xy \in I$ implies $yx \in I$.*

Proof. 1. If $x \in R$, then $x^* = \varepsilon^*.\varepsilon x^* x^{-1}.x \in R$. Hence, $R^* = R$. The statements 2. and 3. follow from the proof of Theorem 3.2.

4. If I is a left ideal of R, and if $x \in I$, then $x^* = x^* x^{-1}.x \in I$, where $x^* x^{-1} \in R$. Also if $y \in R$, then $x^* \in I$ and $\varepsilon y^* \in R$ together imply that $\varepsilon y^* x^* \in I$, that is, $(xy\varepsilon^*)^* \in I$. Hence, $\varepsilon^{-1} xy = (xy\varepsilon^*)^{**} \in I$. However $\varepsilon \in R$ and I is a left ideal of R, so that $xy \in R$. Thus I is also a right ideal of R. Finally, if $xy \in I$, then $x^* y^* = (x^* x^{-1})(xy)(y^{-1} y^*) \in I$, where $x^* x^{-1}, y^{-1} y^* \in R$. Thus, $yx = \varepsilon(x^* y^*)^* \varepsilon^{-1} \in I$. □

The following lemma is evident.

Lemma 3.4. *The set J of non-invertible elements in R is a proper ideal of R that contains every other ideal. One has*

$$J = \{x \in R | \omega(x) > 0\}.$$

Then one can form the residue division ring $\overline{D} = R/J$. The map $*$ of $\overline{D}$ defined by $(x + J)^* = x^* + J, x \in R$ is an ε-involution of $\overline{D}$.

In [1], a necessary and sufficient condition is given for extending an abelian valuation from a division ring D to the over division ring K. We will solve here a $*$-version of this problem, where valuation is replaced with ε-valuation. Along the lines of the solution to the problem given in [1], we shall consider pairs (V, J), where V is a $*$-closed subring of D such that $V \supset [D, D]$, and J is a $*$-closed ideal of V. We note that, since $[D, D]$ is a group, every element of $[D, D]$ is a unit in V. Therefore, $[D, D] \cap J = \phi$. We shall write $(V, J) \leq (V', J')$ and say (V', J') dominates (V, J) if $V \subseteq V'$ and $J \subseteq J'$. Clearly this is a partial ordering for the collection of all such pairs (V, J).

Lemma 3.5. *Let D be a division ring with ε-involution $*$, R is a $*$-closed subring containing $[D, D]$ and M a $*$-closed proper ideal of R. Then there exist a *-closed subring V containing $[D, D]$, and a $*$-closed ideal J such that (V, J) is maximal among pairs dominating (R, M). Further, any such maximal pair (V, J) is made up of the ε-valuation ring V in D with maximal 2-sided ideal precisely J.*

Proof. Clearly the pairs (V, J) dominating (R, M) form an inductive system and by Zorn's Lemma, one can find a maximal member (V, J). Since V is closed under the ε-involution and preserved under conjugation, it follows that $\varepsilon x^* x^{-1} \in V$ for all $x \in D$. From the maximality follows that J is a maximal ideal of V, and by the localization at J, we can enlarge V to a local ring. Then by maximality, V is in fact a local ring with maximal ideal J. To complete the proof we must show that V is a total subring and this can be done as in the case of a commutative valuation subring, see [4]. □

Theorem 3.6. *Let $D \subset K$ be any division ring extension with ε-involution. Given any ε-valuation ω on D with ε-valuation subring V and maximal ideal J, there is an extension ε-valuation ω' of ω to K if and only if $J[K,K]$ is a proper ideal of $V[K,K]$, that is, if and only if there is no equation of the form*

$$\sum_i a_i c_i = 1, a_i \in J, c_i \in [K,K]. \tag{1}$$

Proof. If there is an extension ω' of ω, and (1) holds, then $0 = \omega'(1) \geq min\{\omega'(a_i), \omega'(c_i)\}$. Since $a_i \in J$, $\omega(a_i) > 0$, and $\omega'(c_i) = 0$ (for, c_i is a product of commutators), it follows that the right hand side of the preceding inequality is strictly positive, which is a contradiction. Hence no equation as in (1) holds true. Then $J[K,K]$ is a proper ideal of $V[K,K]$.

To prove the converse, we first note that $J[K,K] = [K,K]J$ and $V[K,K] = [K,K]V$ (for, $r(xyx^{-1}y^{-1}) = (rxyrr^{-1}x^{-1}y^{-1}r^{-1})r$). Since both V and $[K,K]$ are closed under the ε-involution, it follows that $(V[K,K])^* = [K,K]^*V^* = [K,K]V = V[K,K]$, that is, $V[K,K]$ is $*$-closed. Similarly, $J[K,K]$ is closed under the ε-involution. Now, clearly $V[K,K] \supset [K,K]$ and, hence, by Lemma 3.5, there is a maximal pair (V',J') dominating (V,J). Thus, V' is an ε-valuation ring of K, which defines the desired extension (by Theorem 3.2). □

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