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On H-closed topological semigroups and semilattices

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Abstract. In this paper, we show that if S is an H-closed topological semigroup and e is an idempotent of S, then eSe is an H-closed topological semigroup. We give sufficient conditions on a linearly ordered topological semilattice to be H-closed. Also we prove that any H-closed locally compact topological semilattice and any H-closed topological weakly U-semilattice contain minimal idempotents. An example of countably compact topological semilattice whose topological space is H-closed is constructed.

Introduction

In this paper, all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 2, 3, 4]. If A is a subset of a topological space X, then by $\mathrm{cl}_X(A)$ we denote the closure of the set A in X and by $\mathrm{Int}(A)$ the interior of A in X. By ω we denote the first infinite cardinal.

If S is a semigroup, then by $E(S)$ we denote the subset of idempotents of S. A topological space S that is algebraically a semigroup with a continuous semigroup operation is called a *topological semigroup*. A *topological inverse semigroup* is a topological semigroup S that is algebraically an inverse semigroup with the continuous inversion.

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A *semilattice* is a semigroup with a commutative idempotent semigroup operation. A *topological semilattice* is a topological semigroup which is algebraically a semilattice.

If E is a semilattice, then the semilattice operation on E determines the partial order $\leqslant$ on E:

$$e \leqslant f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element e of a semilattice E is called *minimal* (*maximal*) if $f \leqslant e$ ($e \leqslant f$) for $f \in E$ implies $f = e$. For elements e and f of a semilattice E we write $e < f$ if $e \leqslant f$ and $e \neq f$. A semilattice E is called *linearly ordered* if the semilattice operation admits a linear natural order on E.

Let S be a semilattice and $e, q \in S$. We denote ${\downarrow}e = \{f \in S \mid f \leqslant e\}$, ${\uparrow}e = \{f \in S \mid e \leqslant f\}$. Obviously, if S is a topological semilattice then ${\uparrow}e$ and ${\downarrow}e$ are closed subsets in S for any $e \in S$.

Let $\mathcal{S}$ be some class of topological semigroups. A semigroup $S \in \mathcal{S}$ is called *H-closed in* $\mathcal{S}$ if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*. The H-closed topological semigroups were introduced by J. W. Stepp in [9], where they were called *maximal semigroups*. A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class* $\mathcal{S}$, if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

An algebraic semigroup S is called *algebraically h-closed in* $\mathcal{S}$, if S with discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathcal{S}$ and $(S, \mathfrak{d}) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*. Absolutely H-closed topological semigroups and algebraically h-closed semigroups were introduced by J. W. Stepp in [10], where they were called *absolutely maximal* and *algebraic maximal*, respectively.

J. W. Stepp [9] showed that any locally compact topological semigroup is a dense subsemigroup of an H-closed topological semigroup. O. V. Gutik and K. P. Pavlyk [5, 6] proved that a topological inverse semigroup S is [absolutely] H-closed in the class of topological inverse semigroups if and only if any topological Brandt λ-extension of S is an [absolutely] H-closed semigroup in the class of topological inverse semigroups. The topological Brandt λ-extensions which preserve the H-closedness and the absolute H-closedness were constructed in [5, 8].

In [10] J. W. Stepp proved that a semilattice E is algebraically h-closed if and only if any maximal chain in E is finite and he posed therein the question: *Is any H-closed topological semilattice absolutely H-closed?* In [6] O. V. Gutik and K. P. Pavlyk remarked that a topological semilattice is [absolutely] H-closed if and only if it is [absolutely] H-closed in the class of topological semilattices. O. V. Gutik and D. Repovš [7] established properties of linearly ordered H-closed topological semilattices and showed that any linearly ordered H-closed topological semilattice is absolutely H-closed. Also they constructed therein an example of a linearly ordered H-closed locally compact topological semilattice which is not embedded into a compact topological semilattice.

In this paper, we show that if S is an H-closed topological semigroup and e is an idempotent of S, then eSe is an H-closed topological semigroup. We give sufficient conditions on a linearly ordered topological semilattice to be H-closed. Also we prove that any H-closed locally compact topological semilattice and any H-closed topological weakly U-semilattice contain minimal idempotents. An example of countably compact topological semilattice whose topological space is H-closed is constructed.

1. H-closed and absolutely H-closed topological semigroups

Lemma 1.1. *Let S be a dense subsemigroup of a topological semigroup T and let e be a left (right) unity of S. Then e is a left (right) unity of T.*

Proof. Suppose, on the contrary, that e is not a left unity of the topological semigroup T. Then there exists $t \in T$ such that $e \cdot t \neq t$. We put $a = e \cdot t$. Let $W(a)$ and $W(t)$ be open neighbourhoods of the points a and t, respectively, such that $W(a) \cap W(t) = \varnothing$. Since T is a topological semigroup, there exist open neighbourhoods $V(e)$ and $V(t)$ of the points e and t, respectively, such that $V(t) \subseteq W(t)$ and $V(e) \cdot V(t) \subseteq W(a)$. Since S is a dense subsemigroup of T, there exists $s \in S$ such that $s \in V(t)$, and hence $e \cdot s = s \in V(t) \subseteq W(t)$, a contradiction. Therefore e is a left unity of T.

The proof in the case if e is a right unity of S is similar. □

Theorem 1.1. *Let S be an H-closed topological semigroup and let e be an idempotent of S. Then $eSe = eS \cap Se$ is an H-closed topological semigroup.*

Proof. Suppose the contrary, i.e., that $T = eSe$ is not an H-closed topological semigroup. Then e is the unity of T and there exists a topological semigroup G which contains T as a non-closed subsemigroup. Without loss of generality we can assume that $\mathrm{cl}_G(T) = G$. Then $G \setminus T \neq \varnothing$ and by Lemma 1.1 e is the unity of G.

We define $A = S \cup G$ and extend the semigroup operation from S and G onto A as follows:

$$x \cdot_A y = \begin{cases} x \cdot y, & \text{if } x, y \in S; \\ x \cdot y, & \text{if } x, y \in G; \\ x \cdot e \cdot y, & \text{if } x \in S \text{ and } y \in G; \\ x \cdot e \cdot y, & \text{if } x \in G \text{ and } y \in S. \end{cases}$$

Let τ_S be the topology on S and τ_G be the topology on G. We define a topology τ_A on A as follows: $U \in \tau_A$ if and only if $U \cap S \in \tau_S$ and $U \cap G \in \tau_G$. Obviously, (A, τ_A) is a Hausdorff topological space and the semigroup operation "$\cdot_A$" on A is continuous.

Therefore S is a dense subsemigroup of the topological semigroup A, a contradiction. The obtained contradiction implies the statement of the theorem. □

Corollary 1.1. *Let S be an H-closed topological semigroup and let e be an idempotent of S such that $ex = xe$ for all $x \in eS \cup Se$. Then $eS = Se$ is an H-closed topological semigroup.*

Theorem 1.2. *Let S be an H-closed topological semigroup and let x be a regular element of S. If y is an inverse element to x then $xSy = xS \cap Sy$ is an H-closed topological semigroup.*

Proof. By Lemma 1.13 [2], $xSy = eS \cap Se$ for an idempotent $e = xy$ of the semigroup S. Then Theorem 1.1 implies the statement of the theorem. □

Corollary 1.2. *Let S be an H-closed regular topological semigroup and let x and y be inverse elements of S, i.e. $xyx = x$ and $yxy = y$. Then $xSy = xS \cap Sy$ is an H-closed topological semigroup.*

Since the band of a Clifford inverse semigroup S lies in the center of S, Corollary 1.2 implies Corollaries 1.3 and 1.4 below.

Corollary 1.3. *Let S be an H-closed Clifford inverse topological semigroup (in the class of inverse topological semigroups) and $x \in S$. Then xS is an H-closed inverse topological semigroup (in the class of inverse topological semigroups).*

Corollary 1.4. *Let S be an H-closed Clifford topological inverse semigroup (in the class of topological inverse semigroups) and $x \in S$. Then xS is an H-closed topological inverse semigroup (in the class of topological inverse semigroups).*

Theorem 1.3. *Let S be an absolutely H-closed topological semigroup and e be an idempotent of S such that $ex = xe$ for all $a \in S$. Then eS is an absolutely H-closed topological semigroup.*

Proof. Suppose, on the contrary, that eS is not an absolutely H-closed topological semigroup. Then there exists a topological semigroup T and a continuous homomorphism $h\colon eS \to T$ such that $h(eS)$ is not a closed subsemigroup of T. Without loss of generality we can assume that $h(eS)$ is a dense subsemigroup of the topological semigroup T and $T \backslash h(eS) \neq \varnothing$. We define the map $g\colon S \to T$ as follows:

$$g(x) = h(ex) \quad \text{for all} \quad x \in S.$$

Then

$$g(s\cdot t) = h(e \cdot s\cdot t) = h(e\cdot e \cdot s\cdot t) = h(e\cdot s \cdot e\cdot t) = h(e\cdot s) \cdot h(e\cdot t) = g(s) \cdot g(t)$$

for $s, t \in S$ and hence $g\colon S \to T$ is a homomorphism. Moreover, $g(x) = h(x)$ for $x \in eS$ and $g(S) = h(eS)$. Therefore $g(S)$ is a dense subsemigroup of the topological semigroup T and $T \setminus g(S) \neq \varnothing$, a contradiction. The obtained contradiction implies the statement of the theorem. □

Corollary 1.5. *Let S be an absolutely H-closed Clifford inverse topological semigroup (in the class of inverse topological semigroups) and $x \in S$. Then xS is an absolutely H-closed inverse topological semigroup (in the class of inverse topological semigroups).*

Proof. Since S is a Clifford inverse semigroup, $xS = Sx$ for all $x \in S$ and there exists an idempotent e in S such that $xS = eS$. Then we apply Theorem 1.3. □

Similarly we get

Corollary 1.6. *Let S be an absolutely H-closed Clifford topological inverse semigroup (in the class of topological inverse semigroups) and $x \in S$. Then xS is an absolutely H-closed topological inverse semigroup (in the class of topological inverse semigroups).*

2. H-closed topological semilattices

Proposition 2.1. *Let (S, τ_S) be an H-closed topological subsemilattice of a linearly ordered topological semilattice (T, τ_T) and $x \in T$. Then the set $\uparrow x \cap S$ contains a minimal idempotent.*

Proof. Suppose the contrary, i.e., that the set $A = {\uparrow}x \cap S$ does not contain a minimal idempotent.

Since the topological semilattice is H-closed, for any idempotent $x \in T \setminus S$ there exists an open neighbourhood $U(x)$ of x such that $U(x) \cap S = \varnothing$. We define

$$A^{-}(x) = \{e \in T \setminus S \mid e < y \quad \text{for any} \quad y \in A\}.$$

Therefore $A^{-}(x)$ is an open subset in T.

Let $e_0 \notin T$. On the set $T^* = T \cup \{e_0\}$ we define the semigroup operation as follows

$$t \cdot e_0 = e_0 \cdot t = \begin{cases} e_0, & \text{if} \quad t = e_0; \\ e_0, & \text{if} \quad t \in {\uparrow}A; \\ t, & \text{if} \quad t \in {\downarrow}(A^{-}(x)). \end{cases}$$

It is obvious that T^* with so defined semigroup operation is a linearly ordered semilattice.

We define a topology τ^* on T^* as follows:

1) the bases of the topologies τ^* and τ_T at the point $e \in T = T^* \setminus \{e_0\}$ coincide;

2) the family
$$\mathcal{B}(e_0) = \{U_f(e_0) = [e_0; f) \mid f \in A\}$$
is a base of the topology τ^* at the point $e_0 \in T^*$.

Obviously, the conditions (BP1)–(BP3) of [3] hold for the family $\mathcal{B}(e_0)$ and hence $\mathcal{B}(e_0)$ is a base of a topology τ^* at the point $e_0 \in T^*$.

Let $p \in {\uparrow}e_0 \setminus \{e_0\}$. Since the set A does not contain a minimal idempotent there exists an idempotent $f \in A$ such that $e_0 < f < p$ and for an open neighbourhood $V_f(p) = T^* \setminus {\downarrow}f$ of the point p in T^* we have

$$V_f(p) \cdot U_f(e_0) \subseteq U_f(e_0).$$

Also for any idempotent $f \in A$ we have

$$U_f(e_0) \cdot U_f(e_0) \subseteq U_f(e_0).$$

Let $q \in P = {\downarrow}e_0 \setminus \{e_0\} \subseteq T^*$. Then $P = T^* \setminus {\uparrow}e_0$ and P is an open subset in T^*. Hence for any open neighbourhood $W(q) \subseteq P$ of q and for any $f \in A$ we have

$$W(q) \cdot U_f(e_0) \subseteq W(q).$$

Therefore (T^*, τ^*) is a topological semilattice and obviously (S, τ_S) is not a closed a subsemilattice of (T^*, τ^*), which contradicts the H-closedness of the semilattice (S, τ_S). The obtained contradiction implies the statement of the proposition. □

The proof of Proposition 2.2 is similar to Proposition 2.1.

Proposition 2.2. *Let* (S, τ_S) *be an* H*-closed topological subsemilattice of a linearly ordered topological semilattice* (T, τ_T) *and* $x \in T$*. Then the set* ${\downarrow}x \cap S$ *contains a maximal idempotent.*

Propositions 2.1 and 2.2 and Propositions 4 and 5 of [7] imply

Corollary 2.1. *Let* (S, τ_S) *be an* H*-closed topological subsemilattice of a linearly ordered topological semilattice* (T, τ_T)*. Then for any* $x \in T$ *the subsets* ${\uparrow}x \cap S$ *and* ${\downarrow}x \cap S$ *of* T *with induced semilattice operation are* H*-closed topological semilattices.*

Let C be a maximal chain of a topological semilattice E. Then $C = \bigcap_{e \in C}({\downarrow}e \cup {\uparrow}e)$, and hence C is a closed subsemilattice of E. Therefore we get

Lemma 2.1. *Let* L *be a linearly ordered subsemilattice of a topological semilattice* E*. Then* $\mathrm{cl}_E(L)$ *is a linearly ordered subsemilattice of* E*.*

A subsemilattice L of a linearly ordered semilattice S is called a L-*chain* in S if ${\uparrow}e \cap {\downarrow}f \subseteq L$ for any $e, f \in L$, $e \leqslant f$.

Theorem 2.1. *Let* S *be a linearly ordered topological semilattice and let* L *be a subsemilattice of* S *such that* L *is an* H*-closed topological semilattice and any maximal* $S \setminus L$*-chain in* S *is an* H*-closed semilattice. Then* S *is an* H*-closed semilattice.*

Proof. Suppose, on the contrary, that the topological semilattice S is not H-closed. Then there exists a topological semilattice T which contains S as a non-closed subsemilattice. By Lemma 2.1, $\mathrm{cl}_T(S)$ is a linearly ordered topological subsemilattice of T. Therefore without loss of generality we can assume that S is a dense subsemilattice of a linearly ordered topological semilattice T.

Let $x \in T \setminus S$. The conditions of the theorem imply that the set $S \setminus L$ is a disjunctive union of maximal $S \setminus L$-chains K_α, $\alpha \in \mathcal{A}$, which are H-closed semilattices. Therefore any open neighbourhood of the point x intersects infinitely many sets K_α, $\alpha \in \mathcal{A}$.

Since any maximal $S \setminus L$-chain in S is an H-closed topological semilattice, one of the following conditions hold:

$$ {\uparrow}x \cap L \neq \varnothing \quad \text{or} \quad {\downarrow}x \cap L \neq \varnothing. $$

We consider the case when the sets ${\uparrow}x \cap L$ and ${\downarrow}x \cap L$ are not empty. The proofs in the other cases are similar.

By Proposition 2.1 the set ${\uparrow}x \cap L$ contains a minimal idempotent e_m and by Proposition 2.2 the set ${\downarrow}x \cap L$ contains a maximal idempotent e_M. Then the sets ${\uparrow}e_m$ and ${\downarrow}e_M$ are closed in T and, obviously, $L \subset {\downarrow}e_M \cup {\uparrow}e_m$. Let $U(x)$ be an open neighbourhood of the point x in T. We define

$$U_0(x) = U(x) \setminus ({\downarrow}e_M \cup {\uparrow}e_m)\,.$$

Then $U_0(x)$ is an open neighbourhood of the point x in T which intersects at most one maximal $S \setminus L$-chain K_α, a contradiction.

Therefore S is an H-closed semilattice. □

Corollary 2.2. *Let S be a linearly ordered topological semilattice and let L be a subsemilattice of S such that L is a compact topological semilattice and any maximal $S \setminus L$-chain in S is a compact semilattice. Then S is an H-closed semilattice.*

Proposition 2.3. *Every H-closed locally compact topological semilattice contains a minimal idempotent.*

Proof. Suppose the contrary, i.e., that there exists an H-closed locally compact topological semilattice (E, τ_E) which does not contain a minimal idempotent. Let $a \notin S$. We put $E^* = E \cup \{a\}$ and define the semilattice operation on T as follows:

$$x \cdot y = \begin{cases} xy, & \text{if } \ x, y \in S; \\ a, & \text{if } \ \{x, y\} \ni a. \end{cases}$$

The topology τ^* on E^* is defined as follows. Let $\mathcal{B}(x)$ be a base of the topology τ_E at the point $x \in E$. Then for any $x \in E$ we put $\mathcal{B}^*(x) = \mathcal{B}(x)$ to be the base of the topology τ^* at $x \in E^* \setminus \{a\}$.

Let $x \in E$. We define

$$\mathcal{B}_C(x) = \{U \in \mathcal{B}(x) \mid \mathrm{cl}_E(U) \text{ is a compact subset of } E\}.$$

Then by Proposition VI-1.13(iii) [4], ${\uparrow}U$ is an open subset in E for every $U \in \mathcal{B}(x)$ and by Proposition VI-1.6(ii) [4], ${\uparrow}\,\mathrm{cl}_E(V)$ is a closed subset in E for any $V \in \mathcal{B}_C(x)$.

We put

$$\mathcal{B}^*(a) = \{V^*(a) = \{a\} \cup \big(E \setminus {\uparrow}\,\mathrm{cl}_E(V)\big) \mid V \in \mathcal{B}_C(x), x \in E\}.$$

Obviously, the conditions (BP1)–(BP3) of [3] hold for the family $\mathcal{B}^*(a)$ and hence $\mathcal{B}^*(a)$ is a base of a topology τ^* at the point $a \in E^*$. Since for any $x \in E$ there exists $V \in \mathcal{B}_C(x)$ such that $V \cap V^*(a) = \varnothing$, the topological space (E^*, τ^*) is Hausdorff.

For any $x \in E$ and $V \in \mathcal{B}_C(x)$ we have $V^*(a) \cdot V^*(a) \subseteq V^*(a)$ and $V \cdot V^*(a) \subseteq V^*(a)$, and hence (E^*, τ^*) is a topological semilattice which contains E as a dense subsemilattice. This is a contradiction to the H-closedness of E. The obtained contradiction implies the statement of the proposition. □

A topological semilattice L is called the *U-semilattice* if for every idempotent $e \in L$ and for any open neighbourhood $U(e)$ of e there exists an idempotent $y_e \in U(e)$ such that $e \in \operatorname{Int}(\uparrow y_e)$ [1].

A topological semilattice L is called the *weak U-semilattice* if for every idempotent $e \in L$ there exists an idempotent $y_e \in L$ such that $e \in \operatorname{Int}(\uparrow y_e)$. Obviously, every topological U-semilattice is a weak U-semilattice. Proposition 2.3 implies that any locally compact H-closed topological semilattice is a weak U-semilattice.

Proposition 2.4. *Every H-closed topological weak U-semilattice contains a minimal idempotent.*

Proof. Suppose, on the contrary, that there exists an H-closed topological weak U-semilattice (S, τ_S) which does not contain a minimal idempotent. Let $e \notin S$. We define $T = S \cup \{e\}$ and extend the semilattice operation from S onto T as follows

$$x \cdot e = e \cdot x = e \cdot e = e \qquad \text{for all} \quad x \in S.$$

Obviously, T with so defined binary operation is a semilattice and e is zero of T.

We define a topology τ_T on T such that $\tau_T|_S = \tau_S$ in the following way. For any $x \in S \subset T$ the bases of topologies τ_T and τ_S at the point x coincide.

Since (S, τ_S) is a weak U-semilattice, for any idempotent $x \in S$ there exists an idempotent $y_x \in S$ such that $x \in \operatorname{Int}(\uparrow y_x)$. We put

$$U_x(e) = S \setminus (\uparrow y_x)$$

and define

$$\mathcal{B}(e) = \{U_x(e) \mid x \in S\}.$$

Evidently, the conditions (BP1)–(BP3) of [3] hold for the family $\mathcal{B}(e)$ and hence $\mathcal{B}(e)$ is a base of a topology τ_T at the point $e \in T$. Obviously, $U_x(e) \cap S$ is open subset of S for every idempotent $x \in S$. Since for any open neighbourhood $U(x)$ of an arbitrary idempotent $x \in S$ we have

$$(U(x) \cap \operatorname{Int}(\uparrow y_x)) \cap U_x(e) = \varnothing,$$

(T, τ_T) is a Hausdorff topological space.

For every idempotent $x \in S$ and any its open neighbourhood $U(x)$ we have

$$(U(x) \cap \operatorname{Int}(\uparrow y_x)) \cdot U_x(e) \subseteq U_x(e) \qquad \text{and} \qquad U_x(e) \cdot U_x(e) \subseteq U_x(e)$$

and therefore $(T, \tau_T, \cdot)$ is a topological semilattice.

Since the topological semilattice (S, τ_S) does not contain a minimal idempotent, (S, τ_S) is a dense subsemilattice of $(T, \tau_T, \cdot)$. This contradicts the H-closedness of (S, τ_S). The obtained contradiction implies the statement of the proposition. $\square$

Theorem 1.1 implies

Corollary 2.3. *Let S be an H-closed topological semilattice and $e \in S$. Then eS is an H-closed topological semilattice.*

Theorem 1.3 implies

Corollary 2.4. *Let S be an absolutely H-closed topological semilattice and $e \in S$. Then eS is an absolutely H-closed topological semilattice.*

O. Gutik and D. Repovš in [7] constructed an example of a countable metrizable locally compact H-closed topological semilattice which is not embeddable into a compact topological semilattice.

Example 2.1 shows that there exists a countably compact topological semilattice, whose space is H-closed. Also this example shows that there exists a countably compact zero-dimensional scattered topological semilattice which is not embeddable into a locally compact topological semilattice.

Example 2.1. Let $X = [0, \omega_1)$ with the order topology and semilattice operation $x \cdot y = \max\{x, y\}$. On $Y = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with natural topology we define the semilattice operation as follows: $x \cdot y = \max\{x, y\}$ for all $x, y \in Y$. Let $S = X \times Y$ with the product topology τ_p and the product operation. We extend the semilattice operation onto $S^* = S \cup \{\alpha\}$, where $\alpha \notin S$, as follows: $\alpha \cdot \alpha = x \cdot \alpha = \alpha \cdot x = \alpha$ for all $x \in S$, and define a topology τ as follows. The bases of topologies τ and τ_p at the point $x \in S$ coincide and at the point $\alpha \in S^*$ the family $\mathcal{B}(\alpha) = \{U(\alpha) \mid \alpha \in \omega_1\}$ is the base of the topology τ, where

$$U(\alpha) = \{\alpha\} \cup \big([0, \omega_1) \setminus [0, \alpha]\big) \times \big(\{0\} \cup \{1/n \mid n \in \mathbb{N}\}\big).$$

It is obvious that (S^*, τ) is a topological semilattice. Moreover, Proposition 3.12.5 [3] implies that (S^*, τ) is an H-closed countably compact zero-dimensional scattered non-regular topological space.

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