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On classification of CM modules over hypersurface singularities

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Abstract. This article is devoted to a special case of classification problem for Cohen-Macaulay modules over hypersurface singularities.

1. Introduction

Recall that with respect to classification of Cohen-Macaulay modules they distinguish tree types of algebras:

1. Cohen-Macaulay finite: An algebra R is called *Cohen-Macaulay finite* if it has only finitely many non-isomorphic indecomposable Cohen-Macaulay modules.

2. Cohen-Macaulay tame: Let R be a ring. R is called *Cohen-Macaulay tame* if indecomposable modules of fixed rank form finitely many 1-parametric families.

3. Cohen-Macaulay wild: R is called *Cohen-Macaulay wild* if for every finitely generated algebra A there is an exact functor from the category of finite dimensional A-modules to the category of Cohen-Macaulay modules over this singularity, which maps non-isomorphic modules to non-isomorphic ones and indecomposable to indecomposable.

It happens that these notions are closely related to the deformation and modality properties of singularities. For the definitions and results concerning modalities and deformations we refer to [1]. Till now the following results have been obtained:

1. A hypersurface singularity is of finite Cohen-Macaulay type if and only if it is simple (i.e. has zero modality or, the same, is of type A_n, D_n, E_6, E_7, E_8 in Arnold's classification). (See [10, 3].)

2. A curve singularity is of finite Cohen-Macaulay type if and only if it dominates one of the simple plane curve singularities (i.e. A_n, D_n, E_6, E_7, E_8 in Arnold's classification) (see [9]). (Recall here that A' dominates A means that $A \subset A'$ and A'/A is an A-module of finite length).

3. A surface singularity is of finite Cohen-Macaulay type if and only if it a quotient singularity, i.e. so that $R \simeq k[|x, y]]^G$, the ring of invariants of finite subgroup $G \subset GL(2, k)$ (see [2, 8]).

4. A curve singularity is Cohen-Macaulay tame if and only if it dominates one of the unimodal singularities T_{pq}. (See [5].)

5. A minimally elliptic surface singularity (in the sense of [11]) is tame if and only if it is either a simple elliptic singularity or a cusp singularity. (See [6].)

6. Hypersurface singularities of type T_{pqr} are Cohen-Macalay tame (see [6]).

In [6] it was conjectured that all other hypersurface singularities are Cohen-Macaulay wild. In this article we will prove the following part of this conjecture:

Theorem 1.1. *Let $R = k[[x_1, x_2, \ldots, x_n]]/(f)$. If $n = 3$ and $f \in m^4$, then R is Cohen-Macaulay wild.*

2. Prerequisites

In this section, we will review some basic definitions and facts we will need then (see [12]). Through the article, we denote commutative Cohen-Macaulay algebras R over an algebraically closed field k. For the sake of simplicity, we also suppose that *char* $k = 0$, though some of results remain valid for positive characteristic too. We suppose that R is local, complete, noetherian and $R/m = k$, where m is the maximal ideal of R. All modules in the article considered to be finitely generated.

Proposition 2.1. *If $0 \neq f \in k[[t_0, \ldots, t_d]]$, then $R = k[[t_0, \ldots, t_d]]/(f)$ is a Cohen-Macaulay ring of Krull dimension d.*

Let S be a regular local ring, and R is a homomorphic image of the regular local ring S, that is $R = S/I$.

Definition 2.2. A pair of square matrices (φ, ψ) with entries in S satisfying the conditions

$$\varphi\psi = fI$$

$$\psi\varphi = fI$$

is called a matrix factorisation of f.

Definition 2.3. A morphism between matrix factorisations (φ, ψ) and (φ', ψ') is a pair of matrices (S, T) with $S\varphi = \varphi' T$ and $T\psi = \psi' S$:

$$\begin{array}{ccccc} S^{(n_1)} & \xrightarrow{\psi} & S^{(n_1)} & \xrightarrow{\varphi} & S^{(n_1)} \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle S} \\ S^{(n_2)} & \xrightarrow{\psi'} & S^{(n_2)} & \xrightarrow{\varphi'} & S^{(n_2)} \end{array} \tag{1}$$

Note, that the commutativity of the right square in **(1)** implies the commutativity of the left. In fact, multiplying $S\varphi = \varphi' T$ by ψ, ψ', we will have $f\psi' S = \psi' S\phi\psi = \psi'\phi' T\psi = fT\psi$, hence $\psi' S = T\psi$. Nevertheless, during the computations in the chapter **3** it will be convenient to use both two equalities:

$$\begin{array}{c} S\varphi = \varphi' T \\ \psi' S = T\psi \end{array} \tag{2}$$

Definition 2.4. Two matrix factorisations (φ, ψ) and (φ', ψ') are called equivalent if and only if there exists a morphism(S, T) between (φ, ψ) and (φ', ψ'), such that (S, T) is an isomorphism, i.e. $\det(S) \neq 0$ and $\det(T) \neq 0$. We will denote this fact using the notation $(\varphi, \psi) \sim (\varphi', \psi')$.

Therefore, we obtain the category $MF(f)$ of matrix factorizations of (f). And now we will formulate an important result about the connection between CM modules and matrix factorisations:

Theorem 2.5 (Eisenbud [7])**.** *Let F_1 and F_2 be two functors*

$$F_1 : MF(f) \longrightarrow CM(R),$$

and

$$F_2 : CM(R) \longrightarrow MF(f)$$

defined as follows. If we have matrix factorisation (φ, ψ) we get a CM R-module $M = S^n/\varphi(S^n)$ (S^n is a free S-module and φ is considered as a homomorphism $\varphi : S^n \longrightarrow S^n$). This defines the functor F_1. Conversely, we have for a CM R-module M a free resolution over S:

$$0 \longrightarrow S^{(n)} \xrightarrow{\varphi} S^{(n)} \longrightarrow M \longrightarrow 0$$

And there is another homomorphism $\psi : S^n \longrightarrow S^n$*, such that* $\varphi\psi = \psi\varphi = fI$*. This defines the functor* F_2*.*

Then the functors F_1 *and* F_2 *establish an equivalence between the category of CM modules over* R *and the category of matrix factorisations of* (f)*.*

Recall, thet an exact functor $F : A - mod \to B - Mod$ is called a representation embedding if and only if the following conditions hold:

1. $F(M)$ is indecomposable if and only if M is indecomposable.

2. $F(M) \simeq F(M')$ if and only if $M \simeq M'$.

By definition, a Cohen-Macaulay algebra R is called Cohen-Macaulay wild (CM wild) if and only if for every finitely generated k-algebra A there exists a representation embedding $F : A - mod \longrightarrow CM(R)$. We will show now, that we need to check it only for a "special algebra" A, i.e. $A = k\langle x, y\rangle$ (free non-commutative algebra), or $A = K[x, y]$, or $A = K[[x, y]]$. First notice the following obvious result.

Lemma 2.6. *Suppose that a* k*-algebra* A_0 *is wild in the sense that for every finitely generated* k*-algebra* A *there is a representation embedding* $A - mod \to A_0 - mod$*. Then a CM algebra* R *is CM wild if and only if there is a representation embedding* $CM(R) \to A_0 - mod$*.*

Lemma 2.7. *The algebras* $k\langle x, y\rangle$*,* $k[x, y]$ *and* $k[[x, y]]$ *are wild.*

Proof. First prove this result for the algebra $k\langle x, y\rangle$. Indeed, let $A = k\langle a_1, a_2, \dots, a_m\rangle$ be any finitely generated k-algebra. Any d-dimensional representation M of the algebra A is given by m matrices of size $d \times d$ $A_1, A_2, \dots, A_m$. Define the representation $F(M)$ of the free algebra $k\langle x, y\rangle$ such that it maps

$$x \mapsto M_x = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad y \mapsto M_y = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_m \end{pmatrix},$$

where I denotes the identity matrix of size $d \times d$. It gives us a functor $F : A - mod \to k\langle x, y\rangle - mod$. Suppose that another A-module N is given by the matrices $B_1, B_2, \dots, B_m$ and $\Phi : F(M) \to F(N)$ is a homomorphism. Then Φ is a $md \times md$-matrix such that $\Phi M_x = N_x\Phi$

and $\Phi M_y = N_y \Phi$. The first of these equalities implies that

$$\Phi = \begin{pmatrix} T & * & * & \dots & * \\ 0 & T & * & \dots & * \\ 0 & 0 & T & \dots & * \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & T \end{pmatrix},$$

where T is a $d \times d$-matrix. The second equality implies now that $TA_i = B_i T$ for all $i = 1, 2, \dots, m$, therefore T is a homomorphism $M \to N$. Moreover, Φ is invertible (i.e. isomorphism) if and only if so is T, and if Φ is an idempotent, so is T too. It means that the functor F is indeed a representation embedding.

Now we construct, following [4], a representation embedding $k\langle x, y\rangle - mod \to k[[x, y]] - mod$ (it will imply that $k[[x, y]]$ and all the more $k[x, y]$ are wild). Namely, let a representation M of $k\langle x, y\rangle$ is given by two matrices X, Y (the images of x and y). Define the representation $F(M)$ of $k[[x, y]]$ such that

$$x \mapsto M_x = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto M_y = \begin{pmatrix} Y_1 & 0 & Y_2 \\ 0 & 0 & Y_3 \\ 0 & 0 & Y_1 \end{pmatrix},$$

where

$$Y_1 = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & B & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & C & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} c_1 I & 0 & 0 & 0 & 0 \\ 0 & c_2 I & 0 & 0 & 0 \\ 0 & 0 & c_3 I & 0 & 0 \\ 0 & 0 & 0 & c_4 I & 0 \\ 0 & 0 & 0 & 0 & c_5 I \end{pmatrix}, \quad C = \begin{pmatrix} I & 0 & I & I & I \\ 0 & I & I & X & Y \end{pmatrix}.$$

with c_1, c_2, c_3, c_4, c_5 are pairwise distinct elements.

Just as above, one can easily check that any homomorphism $\Phi : F(M) \to F(N)$ induces a homomorphism $T : M \to N$; moreover, Φ is an isomorphism (an idempotent) if and only if so is T. Thus F is also a representation embedding. □

Therefore, to prove that an algebra R is CM wild, it is enough to construct a representation embedding $k[x, y] - mod \to CM(R)$ or $k[[x, y]] - mod \to CM(R)$. In what follows, we will construct such functors.

To prove these facts we will use the following lemma.

Lemma 2.8. *Let f be a polynomial that has a presentation as $f = f_1g_1 + f_2g_2 + f_3g_3$, where f_i, g_i are polynomials of order at least 1 for all $i = 1, 2, 3$. Then there exists a matrix factorization of f: $fI_4 = \varphi\psi = \psi\varphi$, where et φ and ψ are the following matrices:*

$$\begin{aligned} \varphi &= \begin{pmatrix} f_1 & g_2 & f_3 & 0 \\ f_2 & -g_1 & 0 & f_3 \\ g_3 & g_3 & -g_1 - f_2 & f_1 - g_2 \\ -g_3 & 0 & g_1 & g_2 \end{pmatrix}, \\ \psi &= \begin{pmatrix} g_1 & g_2 & 0 & -f_3 \\ f_2 & -f_1 & f_3 & f_3 \\ g_3 & 0 & -g_2 & -g_2 + f_1 \\ 0 & g_3 & g_1 & f_2 + g_1 \end{pmatrix}. \end{aligned} \tag{3}$$

Proof. The proof is an easy straightforward calculations. □

The idea of the proof of theorem 1.1 is the following.

- Firstly, we find a presentation of f as $f_1g_1 + f_2g_2 + f_3g_3$, where f_i depend on some parameters $\lambda, \mu, \dots$ and g_i have a big enough order. Thus we obtain a matrix factorization of the form (3) depending on parameters $\lambda, \mu, \dots$.

- Secondly, we "blow" the matrix factorization (3). Namely, we replace each constant $a \in k$ by the scalar matrix aI_m for some m and the parameters $\lambda, \mu, \dots$ by commuting $m \times m$ matrices $\Lambda, \mathrm{M}, \dots$. It is obvious that in this way we obtain a marix factorization $fI_{4m} = \Phi\Psi = \Psi\Phi$.

- Then we prove that these new factorizations are equivalent if and only if the corresponding matrices $\Lambda, \mathrm{M}, \dots$ are conjugate. Thus we obtain a representation embedding $k[[x, y]] - mod \to CM(R)$, therefore R is CM wild.

Certainly, this is only a general outline of the proof; for the details see below.

3. Proof of the Theorem 1.1

Lemma 3.1. *Let $f \in k[[x, y, z]]$ be a polynomial of order 4. Then f has a presentation $f = f_1g_1 + f_2g_2 + f_3g_3$, where $f_1 = x + \lambda z, f_2 = y + \mu z, f_3 = z^2$, $ord(g_1) \geq 3$, $ord(g_2) \geq 3$, $ord(g_3) \geq 2$ and g_3 is divisible by z.*

Proof. Let $\mathfrak{l} = (x+\lambda z, y+\mu z)$, where λ, μ are parameters. Firstly we check that $\mathfrak{l}\mathfrak{m}^2 + (z^3) = \mathfrak{m}^3$, where $\mathfrak{m} = (x, y, z)$. Indeed, multiplying $x + \lambda z$ and $y + \mu z$ by momomials of degree 2, we obtain polynomials

$$x^3+\lambda x^2z,\ x^2y+\lambda xyz,\ x^2z+\lambda xz^2,\ xy^2+\lambda y^2z,\ xyz+\lambda yz^2,\ xz^2+\lambda z^3,$$
$$x^2y+\mu x^2z,\ xy^2+\mu xyz,\ xyz+\mu xz^2,\ y^3+\mu y^2z,\ y^2z+\mu yz^2,\ yz^2+\mu z^3.$$

Together with z^3 they generate $(x, y, z)^3$.

Since $\mathfrak{m}^4 = \mathfrak{m}^3\mathfrak{m}$, we can present any polynomial $f \in \mathfrak{m}^4$ in the form $f = (x+\lambda z)g_1 + (y+\mu z)g_2 + z^3 g$, where $ord(g_1) \geq 3$, $ord(g_2) \geq 3$, $ord(g) \geq 1$, so it is enough to denote $g_3 = zg$. □

According to lemma 2.8, we obtain a matrix factorization $fI_4 = \varphi\psi = \psi\varphi$, where

$$\varphi = \begin{pmatrix} x+\lambda z & g_2 & z^2 & 0 \\ y+\mu z & -g_1 & 0 & z^2 \\ g_3 & g_3 & -y-\mu z-g_1 & x+\lambda z-g_2 \\ -g_3 & 0 & -g_1 & g_2 \end{pmatrix}.$$

$$\psi = \begin{pmatrix} g_1 & g_2 & 0 & -z^2 \\ y+\mu z & -x-\lambda z & z^2 & z^2 \\ g_3 & 0 & -g_2 & x+\lambda z-g_2 \\ 0 & g_3 & g_1 & y+\mu z+g_1 \end{pmatrix}.$$

Now we use the blowing procedure. Namely, let Λ and M be two commuting $m \times m$ matrices. Then we can consider the matrix factorization

$$fI_{4m} = \Phi(\Lambda,\mathrm{M})\Psi(\Lambda,\mathrm{M}) = \Psi(\Lambda,\mathrm{M})\Phi(\Lambda,\mathrm{M}),$$

where $\Phi = \Phi(\Lambda,\mathrm{M})$ and $\Psi = \Psi(\Lambda,\mathrm{M})$ are given by the formulas:

$$\begin{aligned} \Phi &= \begin{pmatrix} xI+\Lambda z & G_2 & z^2I & 0 \\ yI+\mathrm{M}z & -G_1 & 0 & z^2I \\ G_3 & G_3 & -yI-\mathrm{M}z-G_1 & xI+\Lambda z-G_2 \\ -G_3 & 0 & -G_1 & G_2 \end{pmatrix}, \\ \Psi &= \begin{pmatrix} G_1 & G_2 & 0 & -z^2I \\ yI+\mathrm{M}z & -xI-\Lambda z & z^2I & z^2I \\ G_3 & 0 & -G_2 & xI+\Lambda z-G_2 \\ 0 & G_3 & G_1 & yI+\mathrm{M}z+G_1 \end{pmatrix}. \end{aligned} \quad (4)$$

Here $I = I_m$, and G_i denotes the matrix obtained from the polynomial g_i by replacing all coefficients $a \in k$ by the scalar matrix aI, parameter λ by the matrix Λ and parameter μ by the matrix M.

In the following computations, we denote by G^w the coefficient of the monomial w in the polynomial matrix G. Especially G^0 denote the constant term of this polynomial matrix.

Any pair (Λ, M) of commuting matrices defines a representation $L = L(\Lambda, \mathrm{M})$ of the polynomial algebra $k[x, y]$, namely

$$x \mapsto \Lambda, \quad y \mapsto \mathrm{M}.$$

If (Λ', M') is another such pair and $L' = L(\Lambda', \mathrm{M}')$, a homomorphism $L \to L'$ is a matrix A such that $A\Lambda = \Lambda' A$ and $A\mathrm{M} = \mathrm{M}'A$. In particular, L and L' are isomorphic if and only if the pairs (Λ, M) and (Λ', M') are *conjugate*, i.e. there is an invertible matrix A such that $\Lambda' = A\Lambda A^{-1}$ and $\mathrm{M}' = A\mathrm{M}A^{-1}$. Moreover, $L = L(\Lambda, \mathrm{M})$ is devomposable if and only if there is a non-trivial idempotent endomorphism $L \to L$, i.e. a matrix A such that $A\Lambda = \Lambda A$, $A\mathrm{M} = \mathrm{M}A$, $A^2 = A$ and $A \neq 0$, $A \neq I$.

On the other hand, a homomorphism of matrix factorizations $(\Phi, \Psi) \to (\Phi', \Psi')$ is given by a pair of matrices (S, T) such that $S\Phi = \Phi' T$ and $T\Psi = \Psi' S$. Therefore, all we need is to prove the following fact.

Lemma 3.2. *Let $\Phi = \Phi(\Lambda, \mathrm{M})$, $\Psi = \Psi(\Lambda, \mathrm{M})$, $\Phi' = \Phi(\Lambda', \mathrm{M}')$, $\Psi' = \Psi(\Lambda', \mathrm{M}')$ as defined by the formulas* (4)*. If a pair (S, T) defines a homomorphism of matrix factorizations $(\Phi, \Psi) \to (\Phi', \Psi')$, their constant terms S^0, T^0 are of the form:*

$$S^0 = \begin{pmatrix} A & 0 & 0 & \xi_1 \\ 0 & A & 0 & \xi_2 \\ \xi_3 & \xi_4 & A & \xi_5 \\ 0 & 0 & 0 & A \end{pmatrix}, \qquad T^0 = \begin{pmatrix} A & 0 & 0 & 0 \\ \theta_1 & A & \theta_2 & \theta_3 \\ \theta_4 & 0 & A & 0 \\ \theta_5 & 0 & 0 & A \end{pmatrix}, \tag{5}$$

where all matrices are with entries from the field k, such that $A\Lambda = \Lambda' A$ and $A\mathrm{M} = \mathrm{M}'A$.

Proof. Recall that all entries of the polynomial matrices G_1, G_2 are of order at least 3, and those of G_3 are of order at least 2; moreover, all entries of G_3 are divisible by z. We write S and T as block matrices with the blocks of size $m \times m$, namely, $S = (\xi_{i,j})$, $T = (\theta_{i,j})$ $(i, j = 1, 2, 3, 4)$. First, we will consider the equality $S\Phi = \Phi' T$ and compare all (i, j)-components.

$(1, 1)$:

$\xi_{1,1}(xI+\Lambda z)+\xi_{1,2}(yI+\mathrm{M}z)+\xi_{1,3}G_3-\xi_{1,4}G_3=(xI+\Lambda' z)\theta_{1,1}+G_2\theta_{2,1}+z^2\theta_{3,1}$;

$x:\ \xi^0_{1,1}=\theta^0_{1,1}$,

$y:\ \xi^0_{1,2}=0$,

$z:\ \xi^0_{1,1}\Lambda+\xi^0_{1,2}\mathrm{M}=\Lambda'\theta^0_{1,1}$. Since $\xi^0_{1,2}=0$,

$y^2:\ \xi^y_{1,2}=0$ (recall that G_3 is divisible by z), we have:

$$\xi^0_{1,1}\Lambda=\Lambda'\theta^0_{1,1} \tag{6}$$

$(1,2)$

$\xi_{1,1}G_2-\xi_{1,2}G_1+\xi_{1,3}G_3=(xI+\Lambda' z)\theta_{1,2}+G_2\theta_{2,2}+z^2\theta_{3,2}$

$x:\ \theta^0_{1,2}=0$

$(2,1)$

$\xi_{2,1}(xI+\Lambda z)-\xi_{2,2}(yI+\mathrm{M}z)+\xi_{2,3}G_3-\xi_{2,4}G_3=(yI+\mathrm{M}'z)\theta_{1,1}-G_1\theta_{2,1}+z^2\theta_{4,1}$

$x:\xi^0_{2,1}=0$

$y:\xi^0_{2,2}=\theta^0_{1,1}$

$z:\xi^0_{2,1}\Lambda+\xi^0_{2,2}\mathrm{M}=\mathrm{M}'\theta^0_{1,1}$. Since $\xi^0_{2,1}=0$ and $\theta^0_{4,1}=0$ we have:

$$\xi^0_{2,2}\mathrm{M}=\mathrm{M}'\theta^0_{1,1} \tag{7}$$

$(4,1)$

$\xi_{4,1}(xI+\Lambda z)+\xi_{4,2}(yI+\mathrm{M}z)+\xi_{4,3}G_3-\xi_{4,4}G_3=-G_3\theta_{1,1}-G_1\theta_{3,1}+G_2\theta_{4,1}$

$x:\xi^0_{4,1}=0$

$y:\xi^0_{4,2}=0$

$(4,3)$

$\xi_{4,1}z^2I+\xi_{4,3}(-yI-\mathrm{M}z-G_1)-\xi_{4,4}G_1=-G_3\theta_{1,3}-G_1\theta_{3,3}+G_2\theta_{4,3}$

$y:\xi^0_{4,3}=0$

$(3,2)$

$\xi_{3,1}G_2-\xi_{3,2}G_1+\xi_{3,3}G_3=G_3\theta_{1,2}-G_3\theta_{2,2}+(-yI-\mathrm{M}'z-G_1)\theta_{3,2}+(x+\Lambda' z-G_2)\theta_{4,2}$

$x:\theta^0_{3,2}=0$

$y:\theta^0_{4,2}=0$

$(3,1)$

$\xi_{3,1}(xI+\Lambda z)-\xi_{3,2}(yI+\mathrm{M}z)+\xi_{3,3}G_3-\xi_{3,4}G_3=G_3\theta_{1,1}-G_3\theta_{2,1}+(-yI-\mathrm{M}'z-G_1)\theta_{3,1}+(x+\Lambda' z-G_2)\theta_{4,1}$

$x:\xi^0_{3,1}=\theta^0_{4,1}$

$y:\xi^0_{3,2}=\theta^0_{3,1}$

$(1,3)$

$\xi_{1,1}z^2-\xi_{1,3}(-yI-\mathrm{M}z-G_1)+\xi_{1,4}G_1=(xI+\Lambda' z)\theta_{1,3}-G_2\theta_{2,3}+z^2\theta_{3,3}$

$x:\ \theta^0_{1,3}=0$

$x^2: \ \theta^x_{1,3} = 0$
$y: \ \xi^0_{1,3} = 0$
$y^2: \ \xi^y_{1,3} = 0$

$$\begin{aligned} z^2 &: \xi^0_{1,1} + \xi^z_{1,3}\mathrm{M} = \Lambda'\theta^z_{1,3} + \theta^0_{3,3}, \\ xz &: \theta^z_{1,3} = -\xi^x_{1,3}\mathrm{M}, \\ yz &: \xi^z_{1,3} = -\Lambda'\theta^y_{1,3}. \end{aligned} \tag{8}$$

$(2,3)$
$\xi_{2,1}z^2 - \xi_{2,3}(-yI - \mathrm{M}z - G_1) + \xi_{2,4}G_3 = (yI + \mathrm{M}'z)\theta_{1,3} - G_1\theta_{2,3} + z^2\theta_{4,3}$
$y: \ \xi^0_{2,3} = \theta^0_{1,3} = 0$, from $(1,3)x$

$$y^2: \ \xi^y_{2,3} = -\theta^y_{1,3} \tag{9}$$

$(1,4)$
$\xi_{1,2}z^2 + \xi_{1,3}(xI + \Lambda z - G_2) + \xi_{1,4}G_2 = (xI + \Lambda z)\theta_{1,4} - G_2\theta_{2,4} + z^2\theta_{3,4}$
$x: \theta^0_{1,4} = \xi^0_{1,3} = 0$, since $(1,1)y$
$(3,3)$
$\xi_{3,1}z^2 - \xi_{3,3}(-yI - \mathrm{M}z - G_1) - \xi_{2,4}G_1 = G_3\theta_{1,3} - G_3\theta_{2,3} + (y + \mathrm{M}'z - G_1)\theta_{3,3} + (x + \Lambda z - G_2)\theta_{4,3}$
$x: \theta^0_{4,3} = 0$
$y: \xi^0_{3,3} = \theta^0_{3,3}$.
$(3,4)$
$\xi_{3,2}z^2 - \xi_{3,3}(xI - \Lambda z - G_2) - \xi_{3,4}G_2 = G_3\theta_{1,4} - G_3\theta_{2,4} + (y + \mathrm{M}'z - G_1)\theta_{3,3} + (x + \Lambda z - G_2)\theta_{4,4}$
$x: \xi^0_{3,3} = \theta^0_{4,4}$
$y: \theta^0_{3,4} = 0$.
So, we have

$$S^0 = \begin{pmatrix} | & 0 & 0 & \xi^0_{1,4} \\ 0 & | & 0 & \xi^0_{2,4} \\ \xi^0_{3,1} & \xi^0_{3,2} & || & \xi^0_{3,4} \\ 0 & 0 & 0 & \xi^0_{4,4} \end{pmatrix},$$

$$T^0 = \begin{pmatrix} | & 0 & 0 & 0 \\ \theta^0_{2,1} & \theta^0_{2,2} & \theta^0_{2,3} & \theta^0_{2,4} \\ \xi^0_{3,2} & 0 & || & 0 \\ \xi^0_{3,1} & 0 & 0 & || \end{pmatrix}.$$

And also we have two important equalities (6) and (7).

But It's not enough and nothing new can be obtained from others cells of matrices from the equality $S\Phi = \Phi T$. So, we will use another equality we have $\Psi S = T\Psi'$

(2, 1)

$(yI+\mathrm{M}z)\xi_{1,1}-(xI+\Lambda t)\xi_{2,1}+z^2\xi_{3,1}+z^2\xi_{4,1}=\theta_{2,1}G_1+\theta_{2,2}(y+\mathrm{M}t)+\theta_{2,3}G_3$

$y:\xi^0_{1,1}=\theta^0_{2,2}$.

(3, 4)

$G_3\xi_{4,1}-G_2\xi_{4,3}+(xI+\Lambda z-G_2)\xi_{4,4}=-\theta_{3,1}z^2+\theta_{3,2}z^2+\theta_{3,3}(xI+\Lambda'z-G_2)+\theta_{3,4}(yI+\mathrm{M}'z+G_1)$

$x:\xi^0_{4,4}=\theta^0_{3,3}$.

(2, 3)

$(y+\mathrm{M}z)\xi_{1,3}-(x+\Lambda z)\xi_{2,3}+z^2\xi_{3,3}+z^2\xi_{4,3}=\theta_{2,2}z^2-\theta_{2,3}G_2+\theta_{2,4}G_1$

$xy:\ \xi^x_{1,3}=\xi^y_{2,3}$.

Therefore, from (8) and (9), we get

$$\xi^0_{1,1}=\theta^0_{3,3}+\Lambda'\theta^z_{1,3}-\xi^z_{1,3}\mathrm{M}=\theta^0_{3,3}-\Lambda'\xi^x_{1,3}\mathrm{M}+\Lambda'\theta^y_{1,3}\mathrm{M}= \\ =\theta^0_{3,3}-\Lambda'\xi^x_{2,3}\mathrm{M}+\Lambda'\xi^y_{2,3}\mathrm{M}=\theta^0_{3,3}$$

Thus, the matrices S, T have the form as in (5). □

Corollary 3.3. *The functor $k[x,y]-mod \to MF(f)$, which maps a representation of $k[x,y]$ given by the commuting matrices Λ,M to the matrix factorization $\Phi(\Lambda,\mathrm{M}),\Psi(\Lambda,\mathrm{M})$ defined by the formulas (4), is a representation embedding.*

Proof. If a pair of matrices (S,T) defines an isomorphism of the matrix factorization $(\Phi(\Lambda,\mathrm{M}),\Psi(\Lambda,\mathrm{M}))$ to $(\Phi(\Lambda',\mathrm{M}'),\Psi(\Lambda',\mathrm{M}'))$, it is of the form (5). Since S is invertible, so is A, and since $A\Lambda=\Lambda'A$, $A\mathrm{M}=\mathrm{M}'A$, the pairs (Λ,M) and (Λ',M') are conjugate. It means that the corresponding representations of $k[x,y]$ are isomorphic.

Recall that a representation is decomposbale if and only if it has a non-trivial idempotent endomorphism. If a pair (S,T) defines such an endomorphism of the matrix factorization $\Phi(\Lambda,\mathrm{M}),\Psi(\Lambda,\mathrm{M})$, then the matrix A in the form (5) is also an idempotent endomorphism of the representation of $k[x,y]$ given by the matrices Λ,M. Moreover, as $S^2=S$, $T^2=T$, it is easy to check that if $A=0$, then $S=T=0$, and if $A=I$, then $S=T=I$. Hence, if $\Phi(\Lambda,\mathrm{M}),\Psi(\Lambda,\mathrm{M})$ is decomposable, so is the corresponding representation of $k[x,y]$. □

Since $MF(f)\simeq CM(R)$, we also have the following result, which proves the theorem **1.1** .

Corollary 3.4. *If $n=3$ and $f\in m^4$, the algebra $R=k[[x,y]]/(f)$ is CM wild.*

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