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On the Amitsur property of radicals

N. V. Loi and R. Wiegandt

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Abstract. The Amitsur property of a radical says that the radical of a polynomial ring is again a polynomial ring. A hereditary radical γ has the Amitsur property if and only if its semisimple class is polynomially extensible and satisfies: $f(x) \in \gamma(A[x])$ implies $f(0) \in \gamma(A[x])$. Applying this criterion, it is proved that the generalized nil radical has the Amitsur property. In this way the Amitsur property of a not necessarily hereditary normal radical can be checked.

1. Introduction

All rings considered are associative, not necessarily with unity element. Radicals are meant in the sense of Kurosh and Amitsur. A radical γ is *hereditary*, if $I \triangleleft A \in \gamma$ implies $I \in \gamma$. For details of radical theory the readers are referred to [3].

Many classical radicals, for instance, the Baer, Levitzki, Köthe, Jacobson, and Brown–McCoy radicals, enjoy an important property concerning polynomial rings, called the Amitsur property: the radical of a polynomial ring is again a polynomial ring.

In several cases it is not so easy to decide that a given radical has the Amitsur property. So it seems to be desirable to have equivalent conditions (as Krempa's condition [5]) for testing the Amitsur property of radicals. We are going to prove such a criterion for hereditary radicals in Theorem 2.4.

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A radical γ has the *Amitsur property*, if for every polynomial ring $A[x]$ it holds

$$\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]. \tag{A}$$

The Amitsur property of a radical states that the radical of a polynomial ring is again a polynomial ring. It seems to be folklore that also the converse is true.

Proposition 1.1. *A radical γ has the Amitsur property if and only if $\gamma(A[x])$ is a polynomial ring in x.*

Proof. If $\gamma(A[x])$ is a polynomial ring $B[x]$, then the constant polynomials on both sides are equal. Hence $\gamma(A[x]) \cap A = B$, and so γ has the Amitsur property. □

A useful criterion for the Amitsur property of a radical was given by Krempa [5].

Proposition 1.2. *For a radical γ to have the Amitsur property a necessary and sufficient condition is*

$$\gamma(A[x]) \cap A = 0 \quad \textit{implies} \quad \gamma(A[x]) = 0 \tag{K}$$

for all rings A.

Let $Z(A^1)$ denote the center of the Dorroh extension A^1 of a ring A. We say that a radical γ is closed under *linear substitutions*, if $f(x) \in \gamma(A[x])$ implies $f(ax+b) \in \gamma(A[x])$ for all rings A and all $a, b \in Z(A^1)$.

Proposition 1.3. *If a radical γ has the Amitsur property, then γ is closed under linear substitutions. If a radical γ is closed under linear substitutions, then γ satisfies condition*

$$f(x) \in \gamma(A[x]) \quad \textit{implies} \quad f(0) \in \gamma(A[x]) \tag{T}$$

for all rings A.

Proof. Suppose that γ has the Amitsur property and let

$$f(x) = \sum_{i=0}^{n} c_i x^i \in \gamma(A[x]) = (\gamma(A[x]) \cap A)[x].$$

Then for any $a, b \in Z(A^1)$ we have

$$f(ax+b) = \sum_{i=0}^{n} c_i (ax+b)^i = g(x).$$

Since each $c_i \in \gamma(A[x]) \cap A$ and $a, b \in Z(A^1)$, all the coefficients of $g(x)$ are in $\gamma(A[x]) \cap A$. Hence

$$f(ax+b) = g(x) \in (\gamma(A[x]) \cap A)[x] = \gamma(A[x]).$$

If a radical γ is closed under linear substitutions then γ satisfies trivially condition (T). □

We say that the *semisimple class* $\mathcal{S}\gamma$ of a radical γ is *polynomially extensible* if $A \in \mathcal{S}\gamma$ implies $A[x] \in \mathcal{S}\gamma$. This notion was introduced and studied in connection with the Amitsur property in [9].

Proposition 1.4. *If a radical γ has the Amitsur property, then its semisimple class*
$\mathcal{S}\gamma$ is polynomially extensible.

Proof. The statement is a special case of [9, Proposition 3.4]. □

Let us observe that *the Amitsur property of a radical γ is independent from the polynomial extensibility of γ* (that is $A \in \gamma$ implies $A[x] \in \gamma$), as proved in [9, Corollary 3.8 (iii)].

2. Hereditary radicals and the Amitsur property

We shall denote by $(f(x))_{A[x]}$ the principal ideal of the polynomial ring $A[x]$ generated by the polynomial $f(x) \in A[x]$.

Proposition 2.1. *For a hereditary radical γ condition* (T) *is equivalent to*

$$(f(x))_{A[x]} \in \gamma \quad \textit{implies} \quad (f(0))_{A[x]} \in \gamma. \tag{S}$$

Proof. Straightforward. □

The following Lemma may be useful also in other contexts.

Lemma 2.2. *Let γ be a hereditary radical. If $A \in \gamma$ and $\gamma(A[x]) \subseteq xA[x]$, then $\gamma(A[x]) = 0$.*

Proof. Let us consider the set

$$K = \{f \in xA[x] \mid xf \in \gamma(A[x])\}.$$

Clearly $\gamma(A[x]) \subseteq K \lhd A[x]$.

For arbitrary polynomials $f, g \in K$ we have $xfg \in \gamma(A[x])$ and $g = xh$ with a suitable polynomial $h \in A[x]$. Hence $fh \in K$, so $fg = xfh \in \gamma(A[x])$. Thus $K^2 \subseteq \gamma(A[x])$, that is, $(K/\gamma(A[x]))^2 = 0$.

We define a mapping $\varphi : K \to \gamma(A[x])/x\gamma(A[x])$ by the rule

$$\varphi(f) = xf + x\gamma(A[x]) \qquad \forall f \in K.$$

Obviously this mapping preserves addition. Further,

$$\ker\varphi = \{f \in K \mid xf \in x\gamma(A[x])\},$$

so to each $f \in \ker\varphi$ there exists a $g \in \gamma(A[x])$ such that $xf = xg$, that is, $x(f-g) = 0$. Since x is an indeterminate, $f = g$ follows. Hence $\ker\varphi \subseteq \gamma(A[x])$. The inclusion $\gamma(A[x]) \subseteq \ker\varphi$ is obvious, therefore $\ker\varphi = \gamma(A[x])$. Taking into account that

$$\operatorname{im}\varphi \cong K/\ker\varphi = K/\gamma(A[x]),$$

by $(K/\gamma(A[x]))^2 = 0$ we conclude that φ is a ring homomorphism. Since γ *is hereditary*, from

$$K/\gamma(A[x]) \cong \operatorname{im}\varphi \triangleleft \gamma(A[x])/x\gamma(A[x]) \in \gamma$$

it follows that $K/\gamma(A[x]) \in \gamma$. We have also

$$K/\gamma(A[x]) \triangleleft A[x]/\gamma(A[x]) \in \mathcal{S}\gamma,$$

and therefore $K/\gamma(A[x]) \in \gamma \cap \mathcal{S}\gamma = 0$. Thus $K = \gamma(A[x])$.

Let us define the ideal

$$M = \{f \in A[x] \mid xf \in \gamma(A[x])\}$$

of $A[x]$. Obviously $M \cap xA[x] = K = \gamma(A[x])$. Then $M/\gamma(A[x]) \triangleleft A[x]/\gamma(A[x]) \in \mathcal{S}\gamma$ implies $M/\gamma(A[x]) \in \mathcal{S}\gamma$. Further,

$$M/\gamma(A[x]) = M/(M \cap xA[x]) \cong (M + xA[x])/xA[x] \triangleleft A[x]/xA[x] \cong A.$$

Since $A \in \gamma$, by the *hereditariness of* γ we have

$$(M + xA[x])/xA[x] \in \gamma \cap \mathcal{S}\gamma = 0,$$

and so $M \subseteq xA[x]$. This implies

$$\gamma(A[x]) = M \cap xA[x] = M.$$

Suppose that $\gamma(A[x]) \neq 0$ and $p = \sum_{i=1}^{t} a_i x^i \in \gamma(A[x])$ is a polynomial of minimal degree. Taking into consideration that $\gamma(A[x]) \subseteq xA[x]$, we have $a_0 = 0$ and $p = xq$ with an appropriate polynomial $q \in A[x]$. By the definition of M we have that $q \in M = \gamma(A[x])$. But the degree of q is $t - 1 < t$, a contradiction. This proves $\gamma(A[x]) = 0$. □

The next statement is crucial in proving Theorem 2.4.

Lemma 2.3. *Let γ be a hereditary radical. If γ satisfies condition* (T) *and the semisimple class $\mathcal{S}\gamma$ is polynomially extensible, then γ satisfies Krempa's condition* (K)*.*

Proof. For proving the validity of Krempa's condition (K), we suppose that $\gamma(A[x]) \cap A = 0$, and have to show that $\gamma(A[x]) = 0$.

Let us consider an arbitrary polynomial $f(x) \in \gamma(A[x])$. By the assumption *condition* (T) *implies* that $f(0) \in \gamma(A[x]) \cap A = 0$. Hence we have got that $\gamma(A[x]) \subseteq xA[x]$.

If $A \in \gamma$ then an application of Lemma 2.2 yields that $\gamma(A[x]) = 0$.

If $\gamma(A) = 0$, then *by the polynomial extensibility of $\mathcal{S}\gamma$* it follows that $\gamma(A[x]) = 0$, and Krempa's condition is trivially fulfilled.

Hence we may confine ourselves to the case $0 \neq \gamma(A) \neq A$. We have to prove that $\gamma(A[x]) = 0$. Since *the semisimple class $\mathcal{S}\gamma$ is polynomially extensible*, $A/\gamma(A) \in \mathcal{S}\gamma$ implies that

$$A[x]/\gamma(A)[x] \cong (A/\gamma(A))[x] \in \mathcal{S}\gamma.$$

Hence $\gamma(A[x]) \subseteq \gamma(A)[x]$. For the radical $B = \gamma(A)$ of A, the hereditariness of γ yields

$$\gamma(B[x]) = \gamma(A[x]) \cap B[x] \subseteq \gamma(A[x]),$$

and so

$$\gamma(B[x]) \cap B \subseteq \gamma(A[x]) \cap A = 0$$

follows. Hence applying Lemma 2.2 to $B = \gamma(A) \in \gamma$, we get that $\gamma(B[x]) = 0$. Thus, we arrive at

$$\gamma(A[x]) = \gamma(\gamma(A[x])) \subseteq \gamma(\gamma(A)[x]) = \gamma(B[x]) = 0.$$

□

From Propositions 1.2, 1.3, 1.4, Lemmas 2.2 and 2.3 we get immediately

Theorem 2.4. *A hereditary radical γ has the Amitsur property if and only if γ satisfies condition* (T) *and its semisimple class $\mathcal{S}\gamma$ is polynomially extensible.* □

3. Strict and special radicals

In this section we shall look at the Amitsur property of strict special radicals.

A radical γ is *strict* if $S \subseteq A$ and $S \in \gamma$ imply $S \subseteq \gamma(A)$ for every subring S of every ring A.

Proposition 3.1. *If γ is a strict radical, then γ satisfies condition* (T).

Proof. The mapping $\varphi : A[x] \to A$ defined by $\varphi(f(x)) = f(0)$ for all $f(x) \in A[x]$, is obviously a homomorphism onto A. Since γ is strict, we have

$$\varphi(\gamma(A[x])) \subseteq \gamma(A) \subseteq \gamma(A[x]).$$

Hence $f(x) \in \gamma(A[x])$ implies that $f(0) \in \gamma(A[x])$. □

An ideal I of A is said to be *essential* in A if $I \cap K \neq 0$ for every nonzero ideal K of A, and we denote this fact by $I \triangleleft \cdot A$. A hereditary class ϱ of prime rings is called a *special class* if $I \triangleleft \cdot A$ and $I \in \varrho$ imply $A \in \varrho$. The upper radical

$$\gamma = \mathcal{U}\varrho = \{A \mid A \longrightarrow f(A) \in \varrho \Rightarrow f(A) = 0\}$$

is called a special radical. As is well known, every special radical is hereditary and every γ-semisimple ring $A \in \mathcal{S}\gamma$ is a subdirect sum of rings in ϱ, that is, $\mathcal{S}\gamma$ is the subdirect closure $\overline{\varrho}$ of the class ϱ (see, for instance [3, Theorem 3.7.12 and Corollary 3.8.5]).

Proposition 3.2. *For a special class ϱ and special radical $\gamma = \mathcal{U}\varrho$ the following conditions are equivalent.*

(i) *$A \in \varrho$ implies $A[x] \in \overline{\varrho}$,*

(ii) *the semisimple class $\overline{\varrho} = \mathcal{S}\gamma$ is polynomially extensible.*

Proof. The implication (ii)⇒(i) is trivial.

Assume the validity of (i), and let $A \in \overline{\varrho}$. Then A is a subdirect sum of rings $A/I_\lambda \in \varrho$, $\lambda \in \Lambda$ and $\cap I_\lambda = 0$. By condition (i) we have $(A/I_\lambda)[x] \in \overline{\varrho}$ for every $\lambda \in \Lambda$. Since

$$A[x]/I_\lambda[x] \cong (A/I_\lambda)[x]$$

and $\cap I_\lambda[x] = 0$, the ring $A[x]$ is a subdirect sum of $(A/I_\lambda)[x] \in \overline{\varrho}$. Hence $A[x] \in \overline{\varrho}$ holds. □

Example 3.3. The *generalized nil radical* $\mathcal{N}_g$ is the upper radical of all domains, that is, of all rings without zero-divisors. It is well known that $\mathcal{N}_g$ is a strict special radical and the semisimple class $\mathcal{S}\mathcal{N}_g$ is the class of all reduced rings, that is, of all rings which do not possess nonzero nilpotent elements (see, for instance, [3, Theorem 4.11.11 and Proposition 4.11.12]). Hence by Proposition 3.1 the radical $\mathcal{N}_g$ satisfies condition (T) and a moment's reflection shows – without making use of Proposition 3.2 – that the semisimple class $\mathcal{SN}_g$ is polynomially extensible. Thus by Theorem 2.4 *the generalized nil radical* $\mathcal{N}_g$ *has the Amitsur property.*

Let us mention that by Puczyłowski [6] the generalized nil radical $\mathcal{N}_g$ is the smallest strict special radical.

4. Subidempotent, normal and A-radicals

A hereditary radical γ is called *subidempotent*, if the radical class γ consists of idempotent rings, or equivalently, the semisimple class $\mathcal{S}\gamma$ contains all nilpotent rings.

Proposition 4.1. $\gamma(A[x]) = 0$ *for every subidempotent radical* γ *and every ring* A*, and every subidempotent radical* γ *has the Amitsur property.*

Proof. If $\gamma(A[x]) \neq 0$ for a ring A, then by the hereditariness of γ we have that $x\gamma(A[x]) \in \gamma$. Hence

$$0 \neq x\gamma(A[x])/(x\gamma(A[x]))^2 \in \gamma,$$

and so the subidempotent radical γ contains a non-zero ring with zero multiplication, a contradiction. Thus $\gamma(A[x]) = 0$ follows. This means that Krempa's condition (K) in Proposition 1.2 is trivially fulfilled, and therefore γ has the Amitsur property. $\square$

A radical γ is said to be an *A-radical*, if the radicality depends only on the additive group of the ring; this may be defined as follows: $A \in \gamma$ if and only if the zero-ring $A^0 \in \gamma$.

Proposition 4.2. *Every A-radical* γ *has the Amitsur property.*

Proof. Gardner's [2, Proposition 1.5 (ii)] states that $\gamma(A[x]) = \gamma(A)[x]$. Hence by Proposition 1.1 the assertion follows. $\square$

Next, we shall focus our attention to *normal radicals* which are defined via Morita contexts and characterized as left strong and principally left hereditary radicals. A radical γ is said to be *left strong*, if $L \triangleleft_\ell A$ and $L \in \gamma$ imply $L \subseteq \gamma(A)$, and *principally left hereditary* if $A \in \gamma$ implies $Aa \in \gamma$ for every $a \in A$. Jaegermann and Sands [4] proved the following result. Let

$$\gamma^0 = \{A \mid A^0 \in \gamma\}$$

be the A-radical determined by a radical γ, β the Baer (prime) radical and $\mathcal{L}(\gamma \cup \beta)$ the lower radical generated by γ and β, that is, $\mathcal{L}(\gamma \cup \beta)$ is the union $\gamma \vee \beta$ in the lattice of all radicals.

Proposition 4.3. *Every normal radical γ is the intersection $\gamma = \gamma^0 \cap \mathcal{L}(\gamma \cup \beta)$.* □

Notice that the normal radical γ as well as the A-radical γ^0 need not be hereditary, but $\mathcal{L}(\gamma \cup \beta)$, *as a supernilpotent normal radical is hereditary* (cf. [3, Theorem 3.18.12]).

Puczyłowski [7] and Tumurbat [8] kindly informed us about

Proposition 4.4. *The radicals with Amitsur property form a sublattice in the lattice of all radicals.*

Proof. Let γ, δ be radicals with Amitsur property. The union $\gamma \vee \delta$ in the lattice of all radicals is the lower radical $\vartheta = \mathcal{L}(\gamma \cup \delta)$ generated by γ and δ. By Krempa's criterion (K) it suffices to show that $\vartheta(A[x]) \neq 0$ implies $\vartheta(A[x]) \cap A \neq 0$. If $\vartheta(A[x]) \neq 0$, then either $\gamma(A[x]) \neq 0$ or $\delta(A[x]) \neq 0$. Thus by (K) one of them has nonzero intersection with A. Since both of them are contained in $\vartheta(A[x])$, necessarily also $\vartheta(A[x]) \neq 0$.

The meet $\tau = \gamma \wedge \delta$ is just the intersection $\tau = \gamma \cap \delta$ of the radical classes. For a given ring A, let I be the smallest ideal of A such that $\tau(A[x]) \subseteq I([x])$. Such an ideal I exists, it is the intersection of all ideals containing $\tau(A[x])$. We have

$$\tau(A[x]) = \tau(I[x]) \subseteq \gamma(A[x]),$$

and by the Amitsur property of γ it holds $\gamma(A[x]) = J[x]$ with some ideal J of I. Moreover, $J[x] = \gamma(I[x]) \triangleleft A[x]$, therfore $J \triangleleft A$. Hence by the minimality of I we conclude that $I = J$. Thus $I[x] \in \gamma$. By the same token also $I[x] \in \delta$. Consequently, $\tau(A[x]) = I[x]$ which means by Proposition 1.1 that τ has the Amitsur property. □

The next result shows that for the Amitsur property of a normal radical γ it is enough to check the hereditary radical $\mathcal{L}(\gamma \cup \beta)$.

Proposition 4.5. *A normal radical γ has the Amitsur property if and only if the hereditary normal radical $\mathcal{L}(\gamma \cup \beta)$ has the Amitsur property.*

Proof. Suppose that γ has the Amitsur property. Since β has the Amitsur property, by Proposition 4.4 also $\mathcal{L}(\gamma \cup \beta)$ has it.

Assume that $\mathcal{L}(\gamma \cup \beta)$ has the Amitsur property. Then by Propositions 4.2, 4.3 and 4.4 also $\gamma = \gamma^0 \cap \mathcal{L}(\gamma \cup \beta)$ has the Amitsur property. □

Corollary 4.6. *A normal radical* γ *has the Amitsur property if and only if the hereditary radical* $\mathcal{L}(\gamma \cup \beta)$ *satisfies condition* (T) *and its semisimple class is polynomially extensible.*

Proof. Apply Theorem 2.4 and Proposition 4.5. □

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Contact information

N. V. Loi and R. Wiegandt
A. Rényi Institute of Mathematics
P. O. Box 127
H–1364 Budapest
Hungary
E-Mail: nvloi@hotmail.com,
wiegandt@renyi.hu

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