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Absolutely ubiquitous groups

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Abstract. We characterise absolutely ubiquitous groups in the class of model complete ω-categorical groups.

1. Introduction

Let M be a countable structure over a finite language. The *age* of M, denoted by $\mathcal{J}(M)$, is the set of all isomorphism types of finite substructures of M. The structure M is *absolutely ubiquitous* if it is uniformly locally finite and for any countable locally finite structure N with $\mathcal{J}(M) = \mathcal{J}(N)$ we have $M \cong N$. This notion has been introduced by P. Cameron (see [3]). It covers the case of ω-categorical universally axiomatisable structures already studied by E. A. Palyutin in [8].

In fact in [8] model-theoretic investigations of absolutely ubiquitous structures have been initiated. It turns out that some basic problems studied there are still open. It is still unknown if absolutely ubiquitous stuctures are ω-stable (or even stable). On the other hand H. D. Macpherson has noticed in [7] that absolutely ubiquitious structures are ω-categorical, model complete and do not have the strict order property. Together with I. Hodkinson he has described all absolutely ubiquitous structures in relational languages. Recently Gabor Sagi has extended this (and [5]) to structures with linear growth of substructures [9].

In [7] Dugald Macpherson has begun investigations of absolutely ubiquitous groups. He has shown that such a group has a characteristic

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subgroup of finite index which is a direct product of elementary abelian groups of infinite rank. Using this it can be shown that an absolutely ubiquitous group is ω-stable.

In our paper we continue this theme in order to obtain an algebraic characterisation of absolutely ubiquitous groups. This question was also discussed in [7] and some partial results were given there.

The idea of our method is that we study absolutely ubiquitous groups in the language extended by automorphisms $L' = (\cdot, \alpha_1, ..., \alpha_n)$ and we show that some stronger version of the theorem of Macpherson cited above holds in this situation: an absolutely ubiquitous $(G, \alpha_1, ..., \alpha_n)$ has a characteristic abelian subgroup $A < G$ of finite index with the following properties. Let $\gamma_1, ..., \gamma_m$ be the automorphisms of A which are the conjugations induced by a transversal of G/A. Then $(A, \alpha_1, ..., \alpha_n, \gamma_1, ..., \gamma_m)$ is absolutely ubiquitous. As a module over $R = \mathbf{Z}[\alpha_1, ..., \alpha_n, \gamma_1, ..., \gamma_m]$ the group A is a direct sum of finitely many R-modules of the form B^ω, where B is irreducible, i.e. does not have non-trivial submodules (this follows just from absolute ubiquity of A). Then we apply these results to some characterisation of absolutely ubiquitous groups in terms of their submodules of finite index.

We now give some basic definitions and facts. We fix a language L and study absolutely ubiquitous L-structures. Note that if M_0, M_1, M_2 are L-structures where M_0 is absolutely ubiquitous, $M_2 \models Th(M_0)$ and $M_0 \subseteq M_1 \subseteq M_2$, then $\mathcal{J}(M_0) = \mathcal{J}(M_1) = \mathcal{J}(M_2)$ and $M_1 \models Th(M_0)$ (see [7]).

We do not use any involved model-theoretical material. In particular we do not assume that the reader knows what a stable (or ω-stable) theory is. On the other hand to follow the logical structure of the paper it is worth knowing that ω-stable theories are stable and stable structures do not have the strict order property. The latter is defined as follows:

> there is a natural number k and a $2k$-ary formula $\phi(\bar{x}, \bar{y})$ (possibly with parameters) which defines a partial order on M^k with an infinite chain.

In fact our model-theoretic arguments are restricted to some applications of the compactness theorem and some basic properties of ω-categorical and model complete theories. We remind the reader that a theory T is model complete if and only if any formula ϕ of the language L is equivalent with respect to T to some existential formula ψ and to some universal formula θ. When $\phi(\bar{x}, \bar{y})$ is a formula, M is a structure and $\bar{b}$ is a tuple from M, then we denote by $\phi(M, \bar{b})$ the set $\{\bar{a} \in M : M \models \phi(\bar{a}, \bar{b})\}$.

We now give some preliminaries concerning ω-categorical groups. In this paper we study groups $(G, \cdot, \alpha_1, ..., \alpha_m)$ in the language extended by automorphisms $\alpha_1, ..., \alpha_m$. By $\bar{\alpha}$ we denote $\alpha_1, ..., \alpha_m$. Our basic algebraic notation is standard (the same as in [2] and [7]). For example $[x, y]$ is an abbreviation for $x^{-1}y^{-1}xy$.

Since all $Aut(G)$-invariant relations of an ω-categorical group are definable (i.e. of the form $\phi(M, \bar{b})$), all characteristic subgroups of G are definable. It is also worth noting that for any natural k such a group has only finitely many $Aut(G)$-invariant k-ary relations. We will often use Lemma 5.1 from [7] that, if F is a finite subgroup of an ω-categorical nilpotent group H, then the centralizer $C_H(F)$ is infinite. The following theorem of Macpherson from [7] will play an important role in our arguments.

> **Theorem A.** If G is an ω-categorical group and $Th(G)$ does not have the strict order property, then G has a characteristic nilpotent subgroup of finite index.

This theorem generalises one of the main results of [2].

Let $(G, p_1, ..., p_m, P_1, ..., P_k)$ be a group G in the language extended by functions p_i and relations $P_j \subseteq G^{l_j}$. For $A, B \subset G$ we denote by $\langle A\rangle^{p_{i_1},...,p_{i_j},B}$ the smallest subgroup of G, containing A and closed under the functions $p_{i_1}, ..., p_{i_j}$ [1] and conjugation by elements of B. Generalising Lemma 3.2 from [7] we obtain the following lemma.

Lemma 1.1. *If $(G, \cdot, p_1, ..., p_m, P_1, ..., P_k)$ is an ω-categorical structure, then there is a formula $\psi(x, y)$ such that $(G, \bar{p}, \bar{P}) \models \psi(g, h)$ if and only if $h \in \langle g\rangle^{\bar{p},G}$.*

Proof. Any element of $\langle g\rangle^{\bar{p},G}$ can be expressed by a term built from $\cdot$, all p_i and x^y (with $y \in G$). By ω-categoricity the depth of such terms can be bounded by a natural number. This allows us to find ψ as in the formulation. □

2. Submodules of finite index

In this section we develop Theorem 1.3 from [7] by some inspection of Macpherson's proof.

Theorem 2.1. *Let $(G, \bar{\alpha}, P_1, ..., P_k)$ be an absolutely ubiquitous group in a language extended by automorphisms and relations. Then G has a characteristic subgroup of finite index which is a direct product of elementary abelian groups of infinite rank.*

[1] $\bar{p}$-closed

The situation with this theorem is slightly unusual. The point is that there is no reason why a reduct of an absolutely ubiquitous structure is absolutely ubiquitous. Thus the original Macpherson's theorem (proved for the pure group language) cannot be applied here. We have found that the proof given in [7] works in our case with obvious changes: basically we add everywhere the condition of $\bar{\alpha}$-closedness of subgroups. On the other hand the proof from [7] is quite long and involved, and we are not sure that the reader can easily trust this recipe of the corresponding conversion. Thus we have decided to include our version of Macpherson's proof into the paper.

Our starting point is the same as in [7]. This is the observation that if $(G, \bar{\alpha}, P_1, ..., P_k)$ is absolutely ubiquitous then G has a nilpotent characteristic subgroup N of finite index. This follows from Theorem A and the fact that G is ω-categorical and does not have the strict order property. Now let N be a nilpotent characteristic subgroup of G of finite index. We want to prove that N has an abelian characteristic subgroup of finite index. The key result is as follows.

Lemma 2.2 (Main Lemma)**.** *Let M be a characteristic subgroup of N such that N/M is of nilpotency class two. Then N/M has an abelian characteristic subgroup of finite index.*

Proof. By inspection of the proof of Theorem 1.3 from [7]. The presentation below is divided into Lemmas 2.3 - 2.7 and the concluding part of the proof.

Let $\Theta = \{t_1, .., t_p\}$ be a transversal of G/N. We shall suppose for a contradiction that $|N/M : Z(N/M)|$ is infinite. By ω-categoricity there are formulas $\phi_M(x)$, $\phi_N(x)$ and $\phi_Z(x)$ in the language of groups, such the following holds:

$G \models \phi_M(x)$ if and only if $x \in M$
$G \models \phi_N(x)$ if and only if $x \in N$
$G \models \phi_Z(x)$ if and only if $xM \in Z(N/M)$.

Let Ω be the set of all non-trivial term functions of the language $\langle \alpha_1, ..., \alpha_m \rangle$ in G. By ω-categoricity Ω is finite. Modifying the corresponding place from [7] we say that $g \in N$ is *good* if

(i) $\langle g \rangle^{\bar{\alpha}, G} M/M$ is finite ;
(ii) there is $H \leq G$ with $M \leq H$ such that $g \notin H$, $G = \langle g \rangle^{\bar{\alpha}, G} H$, the groups H and G are isomorphic in the language $\langle \cdot, \alpha_1, ..., \alpha_m, P_1, ..., P_k \rangle$, and $G \models \phi_M([(\gamma(g))^b, h])$ for all $h \in H \cap N$, $b \in G$ and $\gamma \in \Omega$.

In this case we say that the pair (H, g) is good in G. Since the set of good elements is invariant under $Aut(G, \bar{\alpha}, \bar{P})$, there is a formula $\psi_{gd}(x)$ such that $(G, \bar{\alpha}, \bar{P}) \models \psi_{gd}(h)$ if and only if h is good. This formula enables us to talk about good elements in isomorphic copies of $(G, \bar{\alpha}, \bar{P})$.

Lemma 2.3. *There are ω-sequences $(G_i : i < \omega)$ of $\bar{\alpha}$-closed subgroups of G containing M and $(g_i : i < \omega)$ of good elements of G such that*
(i) $G_0 = G$ and $G_{i+1} \leq G_i$ for all $i < \omega$
(ii) $\bigcap(G_i : i < \omega)$ contains no good elements of G
(iii) for any $i < \omega$ the pair (G_{i+1}, g_i) is good in G_i
(iv) for any $i < \omega$, g_i has infinitely many translates under $Aut(G, \bar{\alpha}, P_1, ..., P_k)$ among $(g_j : j < \omega)$.

Proof. We show first that there is a pair (H, g) which is good in $(G, \bar{\alpha}, \bar{P})$. Extend the language L to the language $L^* = L \cup \{b, a_i : i < \omega\}$, where the a_i are interpreted bijectively by the elements of G. Let T^* denote the complete theory of $(G, \bar{\alpha}, \bar{P})$ over $L^* \setminus \{b\}$. For each $i < \omega$, let σ_i be the sentence

$$\bigwedge\{\neg\phi_M((\gamma(b))^t a_i^{-1}) \wedge \phi_N((\gamma(b))^t) \wedge [\phi_N(a_i) \rightarrow \\ \rightarrow \phi_M([(\gamma(b))^t, a_i])] : \gamma \in \Omega, t \in \Theta\}.$$

Claim 1. $T^* \cup \{\sigma_i : i < \omega\}$ is consistent.

It suffices to show that for any $n < \omega$, $T^* \cup \{\sigma_i : i \leq n\}$ is consistent. Let $a_1, ..., a_n \in G$ and $\{a_1, ..., a_n\} \cap N = \{a_1, ..., a_l\}$. Then $F_1 = \langle a_1, ..., a_n\rangle^{\Omega,\Theta} M/M$ and $F_2 = \langle a_1, ..., a_l\rangle^{\Omega,\Theta} M/M$ are finite. From Lemma 5.1 of [7] [2] $C_{N/M}(F_2)$ is infinite. Hence, for each $n < \omega$, $T^* \cup \{\sigma_i : i \leq n\}$ has an interpretation for b from $C_{N/M}(F_2) \setminus F_1$. □

By Claim 1 $T^* \cup \{\sigma_i : i < \omega\}$ has a countable model $\tilde{G}$, which contains G. Let $G^* = \langle G, b\rangle^{\bar{\alpha}}$. Then $\langle b\rangle^{\bar{\alpha},G}$ is finite and $(G, \bar{\alpha}, \bar{P})$ is an elementary substructure of $(G^*, \bar{\alpha}, \bar{P})$.

Claim 2. The pair $(G \cdot \phi_M(G^*), b)$ is good in G^*.

To see condition (i) of the definition present an element g of G^* in the form $g = g'b'tg''$, where $g' \in \phi_N(G)$, $g'' \in \phi_M(G^*)$, $t \in \langle\Theta\rangle$ and $b' \in \langle b\rangle^{\Omega,G}$. Then we have modulo $\phi_M(G^*)$: $b^g = t^{-1}b'^{-1}bb't \in \langle t_1, ..., t_p\rangle \cdot \langle b\rangle^{\Omega,G}$ and the latter subgroup is finite. Thus $\langle b\rangle^{\Omega,G^*}\phi_M(G^*)/\phi_M(G^*)$ is finite.

To see condition (ii) note that $G \leq G \cdot \phi_M(G^*) \leq G^* \leq \tilde{G}$, $G \models T$, $\tilde{G} \models T$ and $G^* \cong G \cdot \phi_M(G^*)$. The rest is easy. □

[2] see also Introduction

We now repeat the arguments from the corresponding part of the proof of Lemma 5.3 from [7]. By Claim 2 there is a pair (H, g) which is good in G. Since $H \cong G$, we may apply this remark to H and to the corresponding subgroups of H. It follows that G contains infinitely many good elements; let these be listed as $\{h_i : i < \omega\}$. Let $O_1, ..., O_l$ be the set orbits of $Aut(G, \bar{\alpha}, \bar{P})$ on G which intersect $\{h_i : i < \omega\}$ non-trivially. Clearly each such intersection is infinite.

Suppose that after r steps we have constructed a finite chain $G_r < G_{r-1} < ... < G_0 = G$ together with elements $g_{r-1}, ..., g_0$ satisfying the conditions of the lemma. Since $(G_r, \bar{\alpha}, \bar{P})$ is an elementary submodel of $(G, \bar{\alpha}, \bar{P})$, the group G_r contains good elements and these elements are precisely the good elements of G which lie in G_r. Let $r \equiv s \bmod l$ where $s \in \{1, ..., l\}$. Choose the least g_r among the elements of the set $\{h_i : i < \omega\}$ which lie in $G_r \bigcap O_s$, and choose $G_{r+1} \leq G_r$ such that the pair (G_{r+1}, g_r) is good in G_r. This construction yields the lemma. □

For the set of g_i from Lemma 2.3 put $V = \langle \langle g_i \rangle^{\Omega, G} : i < \omega \rangle M$.

Lemma 2.4. *The group V/M is abelian, and lies in $Z(N/M)$.*

Proof. As in the proof of the previous lemma any $g \in G$ is of the form $g'_0 g'_1 ... g'_i g' t$, where $g' \in \phi_N(G_{i+1})$, $g'_j \in \langle g_j \rangle^{\Omega, G_j}$ for $j = 1, ..., i$ and t is an element of a transversal of $G_{i+1}/\phi_N(G_{n+1})$ (it is clear that if $g \in \phi_N(G)$ then $t = 1$). Thus $\langle g_i \rangle^{\Omega, G} M/M = \langle g_i \rangle^{\Omega, G_i} M/M$ and V/M is a commuting product of finite groups $\langle g_i \rangle^{\Omega, G} M/M$. If we can show that each group $\langle g_i \rangle^{\Omega, G} M/M$ is abelian, we see $\langle g_i \rangle^{\Omega, G} M/M \leq Z(N/M)$.

Suppose that $\langle g_{i_0} \rangle^{\Omega, G} M/M$ is non-abelian, and let $\rho(x)$ be a formula such that $G \models \rho(g)$ iff g is good and $\langle g \rangle^{\Omega, G} M/M$ is non-abelian. By the choice of g_i, the formula $\rho(x)$ is satisfied by infinitely many elements, say $\{g_{i_k} : i_k < \omega\}$ from $\{g_i : i < \omega\}$. Write $h_k := g_{i_k}$ for all $k < \omega$.

Claim 1. Let B be the least 0-definable subgroup of G with $M \leq B \leq N$ such that whenever g is good in G, $\gamma \in \Omega$ and $h \in G$ we have $[g, (\gamma(g))^h] \in B$. Then $|B : M|$ is finite.

The proof of this claim is a slight modification of the proof of Claim 1 from Lemma 5.4 from [7]. Assuming the contrary we find $\gamma \in \Omega$ and an infinite sequence of elements of N/M of the form $[h_i, \gamma(h_i^{k_i})]M$, $k_i \in G$. Assuming for simplicity that this sequence depends on all $i \in \omega$ consider the elements $h_0 M, h_0 h_1 M, h_0 h_1 h_2 M, ...$ Conjugating them by products of elements $\gamma(h_i^{k_i})M$ we see that they lie in finite conjugacy classes of arbitrary large size (contradicting ω-categoricity). □

Claim 2. Put $h = h_0 ... h_k$ for $k \in \omega$. Then $G \models \rho(h)$.

It suffices to repeat the corresponding part of [7]. Find $g \in G$ and $\gamma \in \Omega$ such that $[(\gamma(h_0))^g, h_0]M \neq M$. Then $[(\gamma(h_0))^g, h]M \neq M$, so $hM \notin Z(N/M)$. Assume $h_k = g_r$. Then $N/M = ((G_{r+1} \cap N)/M) \cdot$

$\langle\langle g_j\rangle^{\Omega,G}M/M : 0 \leq j \leq r\rangle$ and $h \notin G_{r+1}$. Since the pair (h, G_{r+1}) is good in $\langle h, G_{r+1}\rangle^{\Omega}$, the element h is good in G. If $\langle h\rangle^{\Omega,G}M/M$ were abelian, then by arguments above, h would satisfy $\phi_Z(x)$ in $\langle h, G_{r+1}\rangle^{\Omega}$. Since the latter is an elementary submodel of G, h would satisfy $\phi_Z(x)$ in G. This is a contradiction. □

As in [7] by an application of Ramsey's Theorem we may suppose that all pairs (h_iM, h_jM) for $i < j < \omega$, lie in the same orbit of $Aut(G, \bar{\alpha}, \bar{P})$. This assumption and the finiteness of G/N and B/M imply that

> for any $g \in G$ and $\gamma \in \Omega$ there is $b_{g\gamma} \in B$ such that $[(\gamma(h_i))^g, h_i] \in b_{g\gamma}M$ for all $i < \omega$.

Now let $s = |B : M|$. For any $g \in G$, and $\gamma \in \Omega$

$$[(\gamma(h_0 ... h_{s-1}))^g, (h_0 ... h_{s-1})]M =$$
$$= \Pi_{i=0}^{s}[(\gamma(h_i))^g, h_i]M = b_{g\gamma}^s M = (b_{g\gamma}M)^s = M.$$

This contradicts Claim 2. □

Now put $\hat{Q} = \bigcap(G_i/M : i < \omega)$ and $Q = \bigcap(G_i : i < \omega)$. Clearly Q is Ω-closed and contains no good elements of G. We want to show that every finite subgroup of N/M has finite normal closure in N/M. The argument of Lemma 5.5 [7] shows, that

Lemma 2.5. $[N/M, N/M] \leq \hat{Q}$.

Proof. Any two h and $h' \in N$ can be presented as $h = h_i z_i$ and $h' = h'_i z'_i$ with $h_i, h'_i \in G_i$ and $z_i, z'_i \in \langle g_0, ..., g_{i-1}\rangle^{\Omega,G}$. Then by Lemma 2.4 $[h, h']M = [h_i, h'_i]M \in G_i/M$. Since i was arbitrary, $[h, h']M \in \hat{Q}$. □

Lemma 2.6. *$\hat{Q} \cap Z(N/M)$ is finite.*

Proof. Suppose $\hat{Q} \cap Z(N/M)$ is infinite. Note that if $g \in Q$ then $G \models \neg\psi_{gd}(g)$. To obtain a contradiction we want to show that G contains a good element b with $G \models \neg\psi_{gd}(b)$. Find a formula $\psi(x)$ such that $G \models \psi(g)$ if and only if $\langle g\rangle^{\Omega,G}\phi_M(G)/\phi_M(G)$ is abelian. Extend the language L to the language $L^* = L \cup \{b, a_i : i < \omega\}$, where the a_i are interpreted bijectively by the elements of G. Let T^* denote the complete theory of $(G, \bar{\alpha}, \bar{P})$ over $L^* \setminus \{b\}$. For each $i < \omega$, let σ_i be the sentence

$$\bigwedge\{\neg\phi_M((\gamma(b))^t a_i^{-1}) \wedge \phi_N((\gamma(b))^t) \wedge \neg\psi_{gd}((\gamma(b))^t) \wedge \psi(b) \wedge$$
$$[\phi_N(a_i) \rightarrow \phi_M([(\gamma(b))^t, a_i])] : \gamma \in \Omega, t \in \Theta\}.$$

To see that $T^* \cup \{\sigma_i : i < \omega\}$ is consistent let us show that for any $n < \omega$, $T^* \cup \{\sigma_i : i \leq n\}$ is consistent. Let $a_1, ..., a_n \in G$. Then

$F = \langle a_1, ..., a_n \rangle^{\Omega,\Theta} M/M$ is finite. Hence, for each $n < \omega$, $T^* \cup \{\sigma_i : i \leq n\}$ has an interpretation for b from $(\hat{Q} \cap Z(N/M))M \setminus FM$ (to see $\langle b \rangle^{\Omega,G} M/M \subseteq Z(N/M)$ for such an interpretation, apply the fact that $bM \in Z(N/M)$).

Find a countable model $\tilde{G} \models T^* \cup \{\sigma_i : i < \omega\}$ which contains G. Let $G^* = \langle G, b \rangle^{\Omega}$. As in the proof of Lemma 2.3 the pair $(G \cdot \phi_M(G^*), b)$ is good in G^* and $G^* \models \neg\psi_{gd}(b)$, a contradiction. □

Lemma 2.7. *If F is a finite subgroup of N/M, then the group $F \cdot (\hat{Q} \cap Z(N/M))$ is a finite normal subgroup of N/M.*

Proof of this statement is the same as in [7]. The finiteness follows from Lemma 2.6. By Lemma 2.5 for any $f \in F$ and any $h \in N/M$ we have $h^{-1}fhf^{-1} \in \hat{Q}$. By the assumption that N/M has nilpotency class two, $h^{-1}fhf^{-1} \in Z(N/M)$. Thus there is $z \in \hat{Q} \cap Z(N/M)$ with $h^{-1}fh = zf$. □

Proof of Lemma 2.2. We shall show that G contains a good element b with $bM \notin Z(N/M)$. This will contradict Lemma 2.4. Extend the language L to the language $L^* = L \cup \{b, a_i : i < \omega\}$, where the a_i are interpreted bijectively by the elements of G. Let T^* denote the complete theory of $(G, \bar{\alpha}, \bar{P})$ over $L^* \setminus \{b\}$. For each $i < \omega$, let σ_i be the sentence

$$\bigwedge\{\neg\phi_M((\gamma(b))^t a_i^{-1}) \wedge \phi_N((\gamma(b))^t) \wedge$$

$$[\phi_N(a_i) \rightarrow \phi_M([(\gamma(b))^t, a_i])] \wedge \neg\phi_Z(b) : \gamma \in \Omega, t \in \Theta\}.$$

It suffices to show that for any $n < \omega$, $T^* \cup \{\sigma_i : i \leq n\}$ is consistent. Let $a_1, .., a_n \in G$, where $\{a_1, .., a_n\} \cap N = \{a_1, ..., a_l\}$. Then $F_1 = \langle a_1, ..., a_n \rangle^{\Omega,\Theta} M/M$ and $F_2 = \langle a_1, ..., a_l \rangle^{\Omega,\Theta} M/M$ are finite. By Lemma 2.7, $C_{N/M}(F_2 \cdot (\hat{Q} \cap Z(N/M)))$ has finite index in N/M and by asumptions $Z(N/M)$ has infinite index in N/M. Thus we can find an interpretation for b in $C_{N/M}(F_2 \cdot (\hat{Q} \cap Z(N/M)))M \setminus (Z(N/M)M \cup F_1 M)$. The rest is as above. □

Proof of Theorem 2.1. It follows from Lemma 2.2 that if G has a nilpotent characteristic subgroup of finite index and nilpotency class l, then G also has a characteristic subgroup of finite index and nilpotency class $l - 1$. By induction we obtain that if $(G, \bar{\alpha}, \bar{P})$ is an absolutely ubiquitous group, then G has a characteristic abelian subgroup A of finite index. Let Θ be a set of representatives of G/A. Let p_i $(1 \leq i \leq q)$ be the primes dividing the orders of elements of A, and let $S_1, ..., S_q$ be the corresponding Sylow p_i-subgroup of A; then $A = S_1 \times ... \times S_q$. For $i = 1, ..., q$ let V_i be the subgroup of S_i consisting of elements of order p_i. To complete the proof of the theorem we must show that for any i, $|S_i/V_i|$

is finite. If $|S_i/V_i|$ is infinite, put $p := p_i$, $S := S_i$ and $V := V_i$. Let $\phi(x)$ be the formula $x \in A \wedge x^p = 1 \wedge x \neq 1 \wedge (\exists y \in A)(y^p = x)$. Extend L to the language $L^* \cup \{b, c_i : i < \omega\}$, where c_i are interpreted bijectively by the elements of G. Let T^* be the theory of $(G, \bar{\alpha}, \bar{P})$ over $L^* \setminus \{b\}$. Since S/V is infinite, for any finite subset of $\Sigma^* = T^* \cup \{b \neq c_i\} \cup \{\phi(b)\}$, an interpretation for b can be found in V.

Let G^* be a model of Σ^* containing G, and put $\tilde{G} = \langle G, b\rangle^{\bar{\alpha}}$. Then $\tilde{G} \models \phi(b)$.

On the other hand the group $\langle A, b\rangle^{\bar{\alpha},\Theta}$ is of index $|G : A|$ in $\tilde{G}$; thus is defined by the formula for A in G. Now it is easily seen that this formula cannot be realised by an element a in $\tilde{G}$ with $a^r = b$. This is a contradiction. □

Remark. Note that the relations P_i from the formulation do not play any special role in our arguments. Moreover in the next section we will only consider groups expanded by automorphisms. On the one hand we have included relations into our language in order to keep statements as general as possible. On the other hand we believe that some form of the results below concerning absolutely ubiquitous modules hold for abelian structures in general. This suggests that the results of the paper can be extended to expansions of groups by relations defining subgroups in appropriate G^l.

3. Absolutely ubiquitous modules

In this section let $(G, \bar{\alpha})$ be an expansion of a group G by a tuple of automorphisms. Consider the following situation. Assume that G has a characteristic abelian subgroup H of finite index, $|G : H| = m$, and H is a direct product of elementary abelian groups of infinite rank. Let $g_1, ..., g_m$ represent all cosets of H in G and let $f_i : H \to H$, $i = 1, 2, ..., m$, be the group automorphism defined by $f_i(h) = g_i^{-1} h g_i$.

It is clear that for any $\beta_1, ..., \beta_k \in Aut(H)$, the structure $(H, \beta_1, ..., \beta_k)$ can be considered as a $\mathbf{Z}[\beta_1, ..., \beta_k]/I$-module on H, where I is the annulator of $\mathbf{Z}[\beta_1, ..., \beta_k]$ on H. Moreover in the ω-categorical case H becomes a module over a finite ring. This explains why we start this section with a description of absolutely ubiquitous modules.

Any ω-categorical module over a finite ring R can be presented as follows

$$M = \bigoplus_{i=1}^{s} B_i^{\omega} \oplus \bigoplus_{i=1}^{t} C_i^{k_i},$$

where $s, t, k_i \in \omega$ and all B_i, C_i are indecomposable finite submodules. This presentation of M is unique (see [1], [10]). The case $M = B^{\omega}$

with irreducible B (does not have proper non-trivial submodules) will be principal in further arguments. Taking a quotient of R if necessary we may assume that B is an exact R-module. It is well-known that in this case R is simple and is isomorphic to the ring of all linear transformations of the vector space B over the centraliser of B (see Sections 1-3 in [6]).

Proposition 3.1. *Let R be any finite ring and M be an R-module $\bigoplus_1^s B_i^\omega \oplus \bigoplus_1^t C_i^{k_i}$, where B_i,C_i are indecomposable submodules. Then M is absolutely ubiquitous if and only if all modules B_i are irreducible.*

Proof. ($\Rightarrow$) Suppose that there is a number i such that B_i has a non-trivial proper submodule A. Let components B_j with $q < j \leq s$, be all B_k properly imbeddable into B_i. In the case $q \neq s$ consider the R-module

$$M' = \bigoplus_{i=1}^{q} B_i^\omega \oplus \bigoplus_{i=1}^{t} C_i^{k_i}.$$

In the case $q = s$ let M' be the R-module

$$M' = \bigoplus_{i=1}^{s} B_i^\omega \oplus \bigoplus_{i=1}^{t} C_i^{k_i} \oplus A.$$

It is easy to see that $\mathcal{J}(M) = \mathcal{J}(M')$ and $M \not\cong M'$ (by ω-categoricity). This contradicts absolute ubiquity.

($\Leftarrow$) Assume that all B_i in the presentation of M above are irreducible. Let π be the natural projection from M to its completely reducible part $\bigoplus_{i=1}^{s} B_i^\omega$.

Claim. For any extension $E \subseteq F$ with $F \in \mathcal{J}(M)$ and $\bigoplus_{i=1}^{t} C_i^{k_i} \leq E$ there is D such that $F = D \oplus E$.

Consider the extension $\pi(E) \leq \pi(F)$. By Section 4.1 of [6] any submodule of a completely irreducible module A is a direct summand in A. Thus there is a submodule $D < \pi(F)$ such that $\pi(F) = D \oplus \pi(E)$. It is easy to see that $F = D \oplus E$. □

The claim easily implies that any element of $\mathcal{J}(M)$ containing $\bigoplus_{i=1}^{t} C_i^{k_i}$ is of the form $\bigoplus_{i=1}^{s} B_i^{l_i} \oplus \bigoplus_{i=1}^{t} C_i^{k_i}$.

Now assume that N is a module such that $\mathcal{J}(M) = \mathcal{J}(N)$. We have to construct an isomorphism $F : M \to N$. Find an embedding

$\tau_0 : \bigoplus_{i=1}^{t} C_i^{k_i} \to N$ and let $\bigoplus_{i=1}^{t} (C')_i^{k_i}$ be the image of τ_0. By a back-and-forth argument we extend τ_0 to a sequence of partial isomorphisms $\tau_i : M_i \to N_i$, $i \in \omega$, from substructures of M to N.

Let $M = \{a_i : i < \omega\}$ and $N = \{d_i : i < \omega\}$. At Step $2l$ find the first $a_j \in M \setminus M_{2l-1}$. By the claim above any proper extension of N_{2l-1} in N is of the form $D \oplus N_{2l-1}$ where N_{2l-1} is of the form $D' \oplus \bigoplus_{i=1}^{t} (C')_i^{k_i}$. Then $D \oplus D'$ is a completely reducible module and is of the form $\bigoplus_{i=1}^{s} B_i^{l_i}$. For an appropriate D (for example of the isomorphism type of $\bigoplus_{i=1}^{s} B_i^{l_i}$ with large l_i) there is an embedding

$$\tau_{2l} : \langle M_{2l-1}, a_j \rangle \to D \oplus N_{2l-1}$$

which extends τ_{2l-1}.

The same argument works for Step $2l + 1$ (by replacing the pair M_{2l-1}, N_{2l-1} by the pair N_{2l}, M_{2l}). □

Lemma 3.2. *Let M be an absolutely ubiquitous module $\bigoplus_{1}^{s} B_i^{\omega} \oplus \bigoplus_{1}^{t} C_i^{k_i}$, where B_i,C_i are indecomposable submodules and B_i are irreducible. Then the expansion $Th(M, \{c : c \in \bigoplus_{1}^{t} C_i^{k_i}\})$ admits elimination of quantifiers.*

Proof. It suffices to show that every isomorphism over $\bigoplus_{1}^{t} C_i^{k_i}$ between finite substructures of M extends to an automorphism of M. Such an automorphism is constructed as in the proof of Proposition 3.1. □

We now return to the situation of groups $H \leq G$ described in the beginning of the section. The following theorem develops Proposition 3.1

Theorem 3.3. *If $(G, \bar{\alpha})$ is absolutely ubiquitous then $(H, \bar{\alpha}, \bar{f})$ is absolutely ubiquitous too.*

Proof. For $i, j \in \{1, .., m\}$ let $h_{ij} \in H$ be defined by the condition $g_i \cdot g_j = g_k \cdot h_{ij}$ for $k \in \{1, ..., m\}$. In this case let g_{ij} denote g_k. Suppose that $(H, \bar{\alpha}, \bar{f})$ is not absolutely ubiquitous. Then by ω-categoricity and Lemma 3.1 H has a decomposition into indecomposable submodules $\bigoplus_{i=1}^{s} B_i^{\omega} \oplus \bigoplus_{i=1}^{t} C_i^{k_i}$ over $\mathbf{Z}[\bar{\alpha}, f_1, ..., f_m]$ where some B_i (which will be denoted by B) has a nontrivial submodule A.

Fix a natural number $l > \sum_{i=1}^{t} k_i$. We can extend the multiplication $\cdot$ to the product $\{g_1, .., g_m\} \times H \times A^l$ as follows

$$(g_i, h, a_1, .., a_l)\cdot(g_j, h', a'_1, .., a'_l) = (g_{ij}, h_{ij}\cdot f_j(h)\cdot h', f_j(a_1)\cdot a'_1, .., f_j(a_l)\cdot a'_l).$$

As a result we obtain a group extension $0 \to (H \oplus A^l) \to \hat{G} \to (G/H) \to 0$. Any α_i extends to an automorphism β_i of $\hat{G}$ as follows:

$$\beta_i((g_j, h, a_1, .., a_l)) = (\alpha_i(g_j), \alpha_i(h), \alpha_i(a_1), .., \alpha_i(a_l))$$

(since each f_j is a conjugation it is easy to see that such β_i is an automorphism). It is clear that $(\{g_i\}_{i\le m} \times H, \bar{\beta}) \cong (G, \bar{\alpha})$.

In the case when A is isomorphic to some B_j, let H_1 be a $\mathbf{Z}[\bar{\alpha}, \bar{f}]$-submodule of H obtained from H by omitting all B_j^{ω} with B_j properly embeddable into B, and then adding all h_{ij}. If A is not isomorphic to any B_j then let $H_1 = H \oplus A^l$. Let $\hat{G}_1$ be the subgroup of $\hat{G}$ generated by H_1 and $\{g_1, ..., g_m\}$.

Claim. $\mathcal{J}(\hat{G}_1, \bar{\beta}) = \mathcal{J}(G, \bar{\alpha})$.

Since $(G, \bar{\alpha})$ is embeddable into $(\hat{G}, \bar{\beta})$ we have $\mathcal{J}(G, \bar{\alpha}) \subseteq \mathcal{J}(\hat{G}, \bar{\beta})$. By the choice of H_1 we have $\mathcal{J}(\hat{G}, \bar{\alpha}) \subseteq \mathcal{J}(\hat{G}_1, \bar{\beta})$.

It is clear that any structure from $\mathcal{J}(\hat{G}_1, \bar{\beta})$ can be embedded into a structure of the form $(\{g_i\}_{i\le m} \times H_0 \times A^l, \beta) \in \mathcal{J}(\hat{G}_1, \bar{\beta})$, where H_0 is a finite $\bar{\alpha}\bar{f}$-closed subgroup of H. Since $(H_0 \times A^l, \bar{\beta}, \bar{f})$ naturally embeds into $(H, \bar{\beta}, \bar{f})$ we see $(\{g_i\}_{i\le m} \times H_0 \times A^l, \bar{\beta}) \in \mathcal{J}(G, \bar{\alpha})$. □

To see $(\hat{G}_1, \bar{\beta}) \not\cong (G, \bar{\alpha})$ note that the intersection of H_1 with any $\bar{\alpha}\bar{f}$-closed subgroup of index m in $\hat{G}_1$ is an $\bar{\alpha}\bar{f}$-closed subgroup of index $\le m$ in H_1. In the case when $H_1 = H \oplus A^l$ for sufficiently large l, such a subgroup has suficiently many copies of A. Thus it cannot be a subgroup of index $\le m$ of a group isomorphic to H. In the remaining case this statement is obvious. As a result we have that $(G, \bar{\alpha})$ is not absolutely ubiquitous. □

4. Absolutely ubiquitous groups

As in the previous sections let $(G, \bar{\alpha})$ be an expansion of a group G by automorphisms. Assume that G has a characteristic abelian subgroup H of finite index, $|G : H| = m$. Let $g_1, ..., g_m$ represent all cosets of H in G and let $f_i : H \to H$, $i = 1, 2, .., m$, be the group automorphism defined by $f_i(h) = g_i^{-1} h g_i$.

From now on we concentrate on the question which consequences of absolute ubiquity of $(G, \bar{\alpha})$ become sufficient conditions for absolute ubiquity of $(G, \bar{\alpha})$ under the assumption that $(H, \bar{\alpha}, \bar{f})$ is absolutely ubiquitous. We know that if $(G, \bar{\alpha})$ is absolutely ubiquitous then it is model complete. Thus H is defined by an existential formula $\phi_H(x) = \exists \bar{y} \phi_1(x, \bar{y})$ and the latter is equivalent in $(G, \bar{\alpha})$ to some universal formula $\forall \bar{z} \phi_2(x, \bar{z})$. Let $m_H := max\{|\bar{y}|, |\bar{z}|\}$.

On the other hand if $(G, \bar{\alpha})$ is absolutely ubiquitous then for all natural numbers r and s, any finite substructure C of $(G, \bar{\alpha})$ extends to a finite substructure $(D, \bar{\alpha}) \leq (G, \bar{\alpha})$ with the property that for any $A \subseteq D$ with $|A| \leq r$ and any set of formulas $t(\bar{x})$ depending on an s-tuple of variables $\bar{x}$ and with parameters from A, if $t(\bar{x})$ has a realisation in $(G, \bar{\alpha})$ then $t(\bar{x})$ has a realisation in D (in this case we say that $(D, \bar{\alpha})$ is (r, s)-*saturated*). This follows from so called Palyutin's lemma presented in [5]. In fact the statement in [5] is much stronger: for an absolutely ubiquitous M there is a function $\rho_M(n)$ such that if a finite substructure D of M contains copies of all $\rho(r+s)$-generated substructures of M then D is (r, s)-saturated. We can now formulate the main result of the paper.

Theorem 4.1. *Let $(G, \bar{\alpha})$ be a model complete and ω-categorical structure such that*

(i) there is a characteristic abelian subgroup $H \leq G$ of index m satisfying the assumptions of the beginning of the section such that the corresponding expansion $(H, \bar{\alpha}, \bar{f})$ is absolutely ubiquitous;

(ii) any finite substructure of $(G, \bar{\alpha})$ extends to a finite $(1, m_H)$-saturated substructure of $(G, \bar{\alpha})$.

Then $(G, \bar{\alpha})$ is absolutely ubiquitous.

We need some lemmas.

Lemma 4.2. *Consider a structure $(G', \bar{\alpha}')$ such that $\mathcal{J}(G', \bar{\alpha}') = \mathcal{J}(G, \bar{\alpha})$. Let $H' = \forall \bar{z} \phi_2(G', \bar{z})$, where ϕ_2 is as above. Let D be an $(1, m_H)$-saturated substructure of G and $D' \subseteq G'$ be a substructure of $(G', \bar{\alpha}')$ which is isomorphic to D under an isomorphism $f : D \to D'$. Then $f(D \cap H) = D' \cap H'$.*

Proof. We start with the following claims.

Claim 1. $(D, \bar{\alpha}) \models (\exists \bar{y} \phi_1(x, \bar{y}) \leftrightarrow \forall \bar{z} \phi_2(x, \bar{z}))$.

The implication $\exists \bar{y} \phi_1(x, \bar{y}) \to \forall \bar{z} \phi_2(x, \bar{z})$ follows from the fact that $(G, \bar{\alpha}) \models \exists \bar{y} \phi_1(x, \bar{y}) \to \forall \bar{z} \phi_2(x, \bar{z})$. For the contrary implication apply the fact that the structure $(D, \bar{\alpha})$ is $(1, m_H)$-saturated. Thus for any $a \in D$ we have that if $(G, \bar{\alpha}) \models \exists \bar{y} \phi_1(a, \bar{y})$, then $(D, \bar{\alpha}) \models \exists \bar{y} \phi_1(a, \bar{y})$, and if $(D, \bar{\alpha}) \models \forall \bar{z} \phi_2(a, \bar{z})$, then $(G, \bar{\alpha}) \models \forall \bar{z} \phi_2(a, \bar{z})$. This together with assumptions on $\exists \bar{y} \phi_1(x, \bar{y})$ and $\forall \bar{z} \phi_2(x, \bar{z})$ implies the claim. □

Claim 2. For any $a' \in D'$, $(D', \bar{\alpha}') \models \forall \bar{z} \phi_2(a', \bar{z})$ if and only if $(G', \bar{\alpha}') \models \forall \bar{z} \phi_2(a', \bar{z})$.

It is obvious that $(G', \bar{\alpha}') \models \forall \bar{z} \phi_2(a', \bar{z})$ implies $(D', \bar{\alpha}') \models \forall \bar{z} \phi_2(a', \bar{z})$. To see the converse suppose that there is $a' \in D'$ such that $(G', \bar{\alpha}') \models$

$\neg(\forall\bar{z}\phi_2(a',\bar{z}))$ and $(D',\bar{\alpha}')\models\forall\bar{z}\phi_2(a',\bar{z})$. By Claim 1 the latter implies $(G',\bar{\alpha}')\models\exists\bar{y}\phi_1(a',\bar{y})$. As a result there are $\bar{b}'$ and $\bar{c}'$ such that $(G',\bar{\alpha}')\models\neg\phi_2(a',\bar{b}')\wedge\phi_1(a',\bar{c}')$. Since the formula $\neg\phi_2(x,\bar{z})\wedge\phi_1(x,\bar{y})$ does not have solutions in G, we have a contradition with $\mathcal{J}(G',\bar{\alpha}')=\mathcal{J}(G,\bar{\alpha})$. □

To finish the proof of the lemma we show that $a\in H$ if and only if $f(a)\in H'$, where $a\in D$. If $a\in H$ then $(G,\bar{\alpha})\models\forall\bar{z}\phi_2(a,\bar{z})$, $(D,\bar{\alpha})\models\forall\bar{z}\phi_2(a,\bar{z})$ and $(D',\bar{\alpha}')\models\forall\bar{z}\phi_2(f(a),\bar{z})$. By Claim 2 $(G',\bar{\alpha}')\models\forall\bar{z}\phi_2(f(a),\bar{z})$ and so $f(a)\in H'$.

If $a\notin H$ then $(G,\bar{\alpha})\models\forall\bar{y}\neg\phi_1(a,\bar{y})$, $(D,\bar{\alpha})\models\forall\bar{y}\neg\phi_1(a,\bar{y})$ and by Claim 1, $(D,\bar{\alpha})\models\neg\forall\bar{z}\phi_2(f(a),\bar{z})$. Thus $(D',\bar{\alpha}')\models\neg\forall\bar{z}\phi_2(f(a),\bar{z})$ and we have $(G',\bar{\alpha}')\models\neg\forall\bar{z}\phi_2(f(a),\bar{z})$. □

Now note that if a finite $(1,m_H)$-saturated substructure D of $(G,\bar{\alpha})$ contains more than m elements, then for any $d_1,...,d_{m+1}\in D$ there are $i,j\leq m+1$, $i\neq j$, such that $(D,\bar{\alpha})\models\phi_H(d_id_j^{-1})$. Under the circumstances of Lemma 4.2 we easily have that $|G:H|=m=|G':H'|$. Fix $g'_1,g'_2,...,g'_m\in G'$ representing all cosets of G'/H'. Define $f'_i(h'):=(g'_i)^{-1}h'g'_i$ for all $h'\in H'$.

Lemma 4.3. *Under the circumstances of Lemma 4.2 there exists a permutation δ of the set $\{1,...,m\}$ such that*

$$\mathcal{J}(H,\bar{\alpha},f_1,...,f_m)=\mathcal{J}(H',\bar{\alpha}',f'_{\delta(1)},...,f'_{\delta(m)}).$$

Proof. We present H and H' as $\bigcup\{A_i : i\in\omega\}$ and $\bigcup\{A'_i : i\in\omega\}$ where all A_i (resp. A'_i) define an increasing sequence of finite $\bar{\alpha}\bar{f}$-closed substructures of H (resp. finite $\bar{\alpha}'\bar{f}'$-closed substructures of H'). We build two chains of embeddings of finite $(1,m_H)$-saturated substructures D_i of G and $D'_i\leq G'$, $i\in\omega$, respectively (more precisely: substructures D'_i are isomorphic to $(1,m_H)$-saturated substructures of G) such that $\langle A_i,g_1,...,g_n\rangle^{\bar{\alpha}}$ is embeddable into D_i and $\langle A'_i,g'_1,...,g'_n\rangle^{\bar{\alpha}'}$ is embeddable into D'_i. Since $\mathcal{J}(G,\bar{\alpha})=\mathcal{J}(G',\bar{\alpha}')$ we can additionally arrange the existence of a sequence of embeddings

$$D_0\stackrel{e_0}{\to}D'_0\stackrel{e'_0}{\to}D_1\stackrel{e_1}{\to}D'_1\stackrel{e'_1}{\to}....$$

By Lemma 4.2 the embeddings e_i,e'_i induce a sequence of embeddings of $\mathbf{Z}[\bar{\alpha}]$- (resp. $\mathbf{Z}[\bar{\alpha}']$-)modules

$$\phi_H(D_0)\to\phi_H(D'_0)\to\phi_H(D_1)\to\phi_H(D'_1)\to...\to...$$

such that for every i the $e_ie'_{i-1}...e_0(\bar{g})$-conjugating on $\phi_H(D'_i)$ agrees with the $e'_ie_ie'_{i-1}...e_0(\bar{g})$-conjugating under the embedding e'_i and the corresponding property holds for e_i. Let σ'_i be a permutation of set $\{1,...,m\}$

such that $g'_{\sigma'_i(j)} \cdot (e_i e'_{i-1} ... e_0(g_j))^{-1} \in \phi_H(D'_i)$, where g'_i are taken by the corresponding embedding of $\langle A'_i, g'_1, ..., g'_n \rangle^{\bar{\alpha}'}$ into D'_i. By the same method define σ_i for $\phi_H(D_i)$. Then all e_i and e'_i form a sequence of embeddings of $\mathbf{Z}[\bar{\alpha}\sigma_i(\bar{f})]$-(and $\mathbf{Z}[\bar{\alpha}'\sigma'_i(\bar{f}')]$-)modules. Choosing σ and σ' which occur infinitely often as σ_i and σ'_i we obtain a sequence of $\mathbf{Z}[\bar{\alpha}\sigma(\bar{f})]$-(and $\mathbf{Z}[\bar{\alpha}'\sigma'(\bar{f}')]$-)modules. Now let $\delta = \sigma'\sigma^{-1}$. By the choice of A_i and A'_i we have

$$\mathcal{J}(H, \bar{\alpha}, f_{\sigma(1)}, ..., f_{\sigma(m)}) = \mathcal{J}(\lim_i \phi_H(D_i), \bar{\alpha}, f_{\sigma(1)}, ..., f_{\sigma(m)}) =$$

$$\mathcal{J}(\lim_i \phi_H(D'_i), \bar{\alpha}', f'_{\sigma'(1)}, ..., f'_{\sigma'(m)}) = \mathcal{J}(H', \bar{\alpha}', f'_{\sigma'(1)}, ..., f'_{\sigma'(m)}).$$

As a result we have the conclusion of the lemma. □

We can now finish the *proof of the theorem.* By Lemma 4.3 and the first assumption of the theorem we may assume that the structures $(H, \bar{\alpha}, \bar{f})$ and $(H', \bar{\alpha}', \bar{f}')$ are isomorphic and can be presented as in Lemma 3.2. Let $H = \bigoplus_1^s B_i^\omega \oplus \bigoplus_1^t C_i^{k_i}$, where B_i,C_i are indecomposable $\mathbf{Z}[\bar{\alpha}, \bar{f}]$-submodules and B_i are irreducible.

Let $T = \langle g_1, g_2, ..., g_m, \bigoplus_1^t C_i^{k_i} \rangle^{\bar{\alpha}}$. Let $T \subseteq \hat{T}$ be a finite $(1, m_H)$-saturated substructure of $(G, \bar{\alpha})$. Find $\hat{T}' \subseteq G'$ and an isomorphism $I : \hat{T} \to \hat{T}'$. Let $T' := I(T)$. By Lemma 4.2 we see that $I(T \cap H) = T' \cap H'$.

By Lemma 3.2 (that the expansion $Th(M, \{c : c \in \bigoplus_1^t C_i^{k_i}\})$ admits elimination of quantifiers), the map $I : T \cap H \to T' \cap H'$ can be extended to an isomorphism of $\mathbf{Z}[\bar{\alpha}, \bar{f}]-$(resp. $\mathbf{Z}[\bar{\alpha}', \bar{f}^I]$-)modules $I_0 : H \to H'$. Now the rule $g_i h \to I(g_i)I_0(h)$ with $1 \leq i \leq m$ and $h \in H$, defines an isomorphism $(G, \bar{\alpha}) \to (G', \bar{\alpha}')$. For example, the equality $g_i h \cdot g_j h' = g_i g_j f_j(h) h' = g_{ij} h_{ij} f_j(h) h'$ corresponds to $I(g_i)I_0(h) \cdot I(g_j)I_0(h') = I(g_i)I(g_j)f'_j(I_0(h))I_0(h') = I(g_{ij})I(h_{ij})f'_j(I_0(h))I_0(h') = I(g_{ij})I_0(h_{ij})f'_j(I_0(h))I_0(h')$. □

We now formulate a theorem which summarises the material of the paper. It follows from Theorem 2.1, Theorem 3.3 and Theorem 4.1.

Theorem 4.4. *A model complete, ω-categorical group G is absolutely ubiquitous if and only if G has a characteristic abelian subgroup H of finite index such that the following conditions are satisfied.*

(i) Let $g_1, ..., g_m$ represent all cosets of H in G and let $f_i : H \to H$, $i = 1, 2, ..., m$, be the group automorphism defined by $f_i(h) = g_i^{-1} h g_i$. Then the $\mathbf{Z}[\bar{f}]$-module H is absolutely ubiquitous.

(ii) Let H be defined by an existential formula $\phi_H(x) = \exists \bar{y} \phi_1(x, \bar{y})$ and the latter is equivalent in G to some universal formula $\forall \bar{z} \phi_2(x, \bar{z})$.

Let $m_H := max\{|\bar{y}|, |\bar{z}|\}$*. Then any finite substructure of* G *extends to a finite* $(1, m_H)$*-saturated substructure of* G*.*

We consider this theorem as some kind of algebraic description of absolutely ubiquitous groups in the class of model complete ω-categorical groups. Indeed condition (i) is algebraic by our description of absolutely ubiquitous modules in Proposition 3.1. Condition (ii) looks more complicated. Nevertheless it is a substantial reduction of the property arising in Palyutin's lemma from [5] (where there is no bound on m in the appropriate condition of m-saturation).

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