Twisted conjugacy classes of Automorphisms of Baumslag-Solitar groups

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Communicated by R. I. Grigorchuk

ABSTRACT.   Let $\phi : G \to G$ be a group endomorphism where
G is a finitely generated group of exponential growth, and denote
by $R(\phi)$ the number of twisted ϕ-conjugacy classes. Fel'shtyn and
Hill [7] conjectured that if ϕ is injective, then $R(\phi)$ is infinite. This
conjecture is true for automorphisms of non-elementary Gromov
hyperbolic groups, see [17] and [6]. It was showed in [12] that the
conjecture does not hold in general. Nevertheless in this paper,
we show that the conjecture holds for injective homomorphisms for
the family of the Baumslag-Solitar groups $B(m, n)$ where $m \neq n$
and either m or n is greater than 1, and for automorphisms for the
case $m = n > 1$. family of the Baumslag-Solitar groups $B(m, n)$
where $m \neq n$.

1. Introduction

The fixed point classes of a surface homeomorphism were introduced by
J. Nielsen in [20]. Subsequently, K. Reidemeister [21] laid for any map of
compact polyhedron the algebraic foundation for what is now known as
Nielsen fixed point theory. As result of Reidemeister's work, the twisted
conjugacy classes of a group homomorphism was introduced. It turns out

that the fixed point classes of a map can easily be identified with the conjugacy classes of liftings of this map to the universal covering of compact polyhedron, and these last ones with the twisted conjugacy classes of the homomorphism induced in the fundamental group of the polyhedron. Let G be a finitely generated group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be $\phi-conjugate$ if there exists $\gamma \in G$ with $\alpha' = \gamma\alpha\phi(\gamma)^{-1}$. The number of twisted ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G. Let X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Reidemeister number $R(f)$, which is simply the cardinality of the set of twisted ϕ-conjugacy classes where $\phi = f_\#$ is the induced homomorphism on the fundamental group, is relevant for the study of fixed points of f in the presence of the fundamental group. In fact the finiteness of Reidemeister number plays an important role. See for example [22], [13], [7], [9] and the introduction of [12]. It is proved in [8] for a wide class of groups including polycyclic and finitely generated polynomial growth groups that the Reidemeister number of an automorphism ϕ is equal to the number of finite-dimensional fixed points of $\widehat{\phi}$ on the unitary dual, if the Reidemeister number is finite. This theorem is a natural generalization to infinite groups of the classical Burnside theorem which says that the number of classes of irreducible representations of a finite group is equal to the number of conjugacy classes of elements of this group.

From another side for any automorphism $\phi : G \to G$ of a nonelementary Gromov hyperbolic group G the number of twisted ϕ-conjugacy classes is infinite see [17] and [6]. Furthermore, using co-Hofian property, it was showed in [6] that if in addition G is torsion-free and freely indecomposable then for every injective endomorphism ϕ, $R(\phi)$ is infinite. This result gives supportive evidence to a conjecture of [7] which states that $R(\phi) = \infty$ if G is a finitely generated torsion-free group with exponential growth.

This conjecture was showed to be false in general. In [12] was constructed automorphisms $\phi : G \to G$ on certain finitely generated torsion-free exponential growth groups G that are not Gromov hyperbolic with $R(\phi) < \infty$.

In the present paper we study this problem for a family of finitely generated groups which have exponential growth but are not Gromov hyperbolic. They are the Baumslag-Solitar groups. We recall its definition. They are indexed by pairs of natural numbers and have the following

presentation:

$$B(m, n) = \langle a, b : a^{-1}b^m a = b^n \rangle, m, n > 0.$$

It is easy to see that $B(m, n)$ is isomorphic to $B(n, m)$. This family has different features from the one given in [12], which is a family of groups which are metabelian having as kernel the group $\mathbb{Z}^n$. So they contain a subgroup isomorphic to $\mathbb{Z}+\mathbb{Z}$. In the case of Baumslag-Solitar groups this happens if and only if $m = n$. For $m = n = 1$ the group $B(1, 1) = \mathbb{Z} + \mathbb{Z}$ does not have exponential growth. So without loss of generality we will consider $m \leq n$, and for $m = n$ $n > 1$. For more about these groups see [1, 4].

Some results in this work could be obtained using the description of the Automorphism group of a Baumslag-Solitar group (for those, see [10] and [19]).

Our main result is:

Theorem For any injective homomorphism of $B(m, n)$ where $m \neq n$ we have that the Reidemeister number is infinite if either $m > 1$ or $n > 1$. Also the same holds for automorphisms of $B(m, m)$ if $m > 1$.

This result corresponds to the three results Theorem 3.4, Theorem 4.4 and Theorem 5.4 for the various values of m and n.

We have been informed by G. Levitt that our results can be generalized when looked at in the context of generalized Baumslag-Solitar groups (fundamental groups of graphs of groups with all edge and vertex groups infinite cyclic) where the basic techniques used are mostly due to Max Forester.

The paper is organized into four sections. In section 2, we develop some preliminaries about the Reidemeister classes of a pair of homomorphism between short exact sequences. In section 3 we study the case $B(1, n)$ for $n > 1$. The main result is Theorem 3.4. In section 4, we consider the cases $B(m, n)$ for $1 < m < n$. The main result is Theorem 4.4. In section 5, we consider the cases $B(n, n)$ for $n > 1$. The main result is Theorem 5.3.

2. Preliminaries

In this section we recall some facts about Reidemeister classes of a pair of homomorphism of a short exact sequences. The set of the Reidemeister classes will be denoted by $R[\ ,\ ]$ and the number of such classes by $R(\ ,\ )$. This will be used for the case where the two sequences are the same and one of the homomorphism is the identity. The main reference for this section is [11] and more details can be found there.

Let us consider a diagram of two short exact sequences of groups and maps between these two sequences:

$$
\begin{array}{ccccccccc}
1 & \to & H_1 & \overset{i_1}{\to} & G_1 & \overset{p_1}{\to} & Q_1 & \to & 1 \\
 & & f' \downarrow\downarrow g' & & f \downarrow\downarrow g & & \overline{f} \downarrow\downarrow \overline{g} & & \qquad (2.1) \\
1 & \to & H_2 & \overset{i_2}{\to} & G_2 & \overset{p_2}{\to} & Q_2 & \to & 1
\end{array}
$$

where $f' = f|_{H_1}$, $g' = g|_{H_1}$.

We recall that the Reidemeister classes $R[f_1, f_2]$ relative to homomorphisms $f_1, f_2 : K \to \pi$ are the equivalence classes of elements of π given by the following relation: $\alpha \sim f_2(\tau)\alpha f_1(\tau)^{-1}$ for $\alpha \in \pi$ and $\tau \in K$.

Also the diagram (2.1) above will provide maps between sets

$$
R[f', g'] \overset{\widehat{i_2}}{\to} R[f, g] \overset{\widehat{p_2}}{\to} R[\overline{f}, \overline{g}]
$$

where the last map is clearly surjective. As obvious consequence of this fact will be used to solve some of the cases that we are going to discuss, and will appear below as Corollary 2.2. For the remain cases we need further information about the above sequence and we will use the Corollary 2.4.

Proposition 1.2 in [11] says

Proposition 2.1 Given the diagram (2.1) we have a short sequence of sets

$$
R[f', g'] \overset{\widehat{i_2}}{\to} R[f, g] \overset{\widehat{p_2}}{\to} R[\overline{f}, \overline{g}]
$$

where $\widehat{p_2}$ is surjective and $\widehat{p_2}^{-1}[1] = \mathrm{im}\,(\widehat{i_2})$ where 1 is the identity element of Q_2.

An immediate consequence of this result is

Corollary 2.2 If $R(\overline{f}, \overline{g})$ is infinite then $R(f, g)$ is also infinite.

Proof. Since $\widehat{p_2}$ is surjective the result follows. $\square$

In order to study the injectivity of the map $\widehat{i_2}$, for each element $\overline{\alpha} \in Q_2$ let $H_2(\overline{\alpha}) = p_2^{-1}(\overline{\alpha})$, $C_{\overline{\alpha}} = \{\overline{\tau} \in Q_1 | \overline{g}(\overline{\tau})\overline{\alpha}\overline{f}(\overline{\tau}^{-1}) = \overline{\alpha}\}$ and let $R_{\overline{\alpha}}[f', g']$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ by the equivalence relation $\beta \sim g(\tau)\beta f(\tau^{-1})$ where $\beta \in H_2(\overline{\alpha})$ and $\tau \in p_1^{-1}(C_{\overline{\alpha}})$. Finally let $R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ given by the relation $\beta \sim g(\tau)\beta f(\tau^{-1})$ where $\beta \in H_2(\overline{\alpha})$ and $\tau \in G_1$.

Proposition 1.2 in [11] says

Proposition 2.3 Two classes of $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ represent the same class of $R(f, g)$ if and only if they belong to the same orbit by the action of $C_{\overline{\alpha}}$. Further the isotropy subgroup of this action at an element $[\beta]$ is $G_{[\beta]} = p_1(C_\beta) \subset C_{\overline{\alpha}}$ where $\beta \in [\beta]$.

An immediate consequence of this result is

Corollary 2.4 If $C_{\overline{\alpha}}$ is finite and $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ is infinite, for some α, then $R(f, g)$ is also infinite. In particular this is the case if Q_2 is finite.

Proof. The orbits of the action of $C_{\overline{\alpha}}$ on $R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ are finite. So we have an infinite number of orbits. The last part is clear from the first part and the result follows. $\qquad\square$

3.   The cases $B(1, n)$

As pointed out in the introduction, let $n > 1$ and $B(1, n) = \langle a, b : a^{-1}ba = b^n, n > 1\rangle$.
Recall from [4] that the Baumslag-Solitar groups $B(1, n)$ are finitely-generated solvable groups which are not virtually nilpotent.These groups have exponential growth[16] and they are not Gromov hyperbolic . Furthermore, those groups are metabelian and torsion free.

Consider the homomorphisms $|\ |_a : B(1, n) \longrightarrow Z$ which associates for each word $w \in B(1, n)$ the sum of the exponents of a in the word. It is easy to see that this is a well defined map into Z which is surjective.

Proposition 3.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(1, n) \xrightarrow{|\ |_a} Z \longrightarrow 1,$$

where K, the kernel of the map $|\ |_a$, is the set of the elements which have the sum of the powers of a equal to zero. Furthermore, $B(1, n) = K \rtimes Z$ - (semidirect product).

Proof. The first part is clear. The second part follows because Z is free, so the sequence splits. $\qquad\square$

Proposition 3.2 The kernel K coincide with $N\langle b\rangle$ which is the normalizer of $\langle b\rangle$ in $B(1, n)$.

Proof. We have $N\langle b\rangle \subset K$. But the quotient $B/N\langle b\rangle$ has the following presentation: $\bar{a}^{-1}\bar{b}\bar{a} = \bar{b}^n, \bar{b} = 1$. Therefore this group is isomorphic to $\mathbb{Z}$ and the natural projection coincides with the map $|\ |_a$ under the obvious identification of $\mathbb{Z}$ with $B/N\langle b\rangle$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0\to & N\langle b\rangle \to & B(1,n)\to & B/N\langle b\rangle \to & 1 \\
 & \downarrow & \downarrow & \downarrow & \\
0\to & K\to & B(1,n)\to & \mathbb{Z}\to & 1
\end{array}
$$

where the last vertical map is an isomorphism. From the five Lemma the result follows. $\square$

We recall that the groups $B(1,n)$ are metabelian. Further $B(1,n)$ is isomorphic to $\mathbb{Z}[1/n] \rtimes_\theta \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{Z}[1/n]$ is given by $\theta(1)(x) = x/n$. To see this, first observe that the map defined by $\phi(a) = (0,1)$ and $\phi(b) = (1,0)$ extends to a unique homomorphism $\phi : B \to \mathbb{Z}[1/n] \rtimes \mathbb{Z}$ which is clearly surjective. It suffices to show that this homomorphism is injective. Given a word $w = a^{r_1}b^{s_1}...a^{r_t}b^{s_t}$ such that $r_1 + ... + r_t = 0$, so a word on K, using the relation of the group this word is equivalent to $b^{s_1/n^{r_1}} b^{s_2/n^{r_1+r_2}} ....b^{s_{t-1}/n^{r_1+...+r_{t-1}}}.b^{s_t}$. If we apply ϕ to this element and assume that it is in the kernel of ϕ, we obtain that the sum of the powers $s_1/n^{r_1} + s_2/n^{r_1+r_2} + ... + s_{t-1}/n^{r_1+...+r_{t-1}} + s_t$ is zero. But this means that w is the trivial element, hence ϕ restricted to K is injective. Therefore the result follows.

Proposition 3.3 Any homomorphism $\phi : B(1,n) \to B(1,n)$ is a homomorphism of the short exact sequence given in Proposition 3.2.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianized of $B(1,n)$. The abelianized of $B(1,n)$, denoted by $B(1,n)_{ab}$, is isomorphic to $Z_{n-1} + Z$. The torsion elements of $B(1,n)_{ab}$ form a subgroup isomorphic to Z_{n-1} which is invariant under any homomorphism. The preimage of this subgroup under the projection to the abelianized $B(1,n) \to B(1,m)_{ab}$ is exactly the subgroup $N(b)$, i.e. the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$ and the result follows. $\square$

Theorem 3.4 For any injective homomorphism of $B(1,n)$ we have that the Reidemeister number is infinite.

Proof. Let ϕ be a homomorphism. By Propositon 3.3 it is a homomorphism of short exact sequence. The induced map on the quotient is a nontrivial endomomorphism of Z. If the induced homomorphism $\bar{\phi}$ is the identity, by Corollary 2.2 the number of Reidemeister classes is infinite. Since ϕ is also injective, we can assume that $\bar{\phi}$ is multiplication

by $k \neq 0, 1$. Now we claim that there is no injective homomorphism of $B(1, n)$ such that the induced on the quotient is multiplication by k with $k \neq 0, 1$. When we apply the homomorphism ϕ to the relation $a^{-1}ba = b^n$, using the description of $B(1, n) \to Z[1/n] \rtimes Z$ we get: $a^{-k}\phi(b)a^k = (n^k\phi(b), 0) = (n\phi(b), 0)$, which implies that either $n^{1-k} = 1$ or $\phi(b) = 0$. So the result follows.   □

4.   The case $B(m, n)$, $1 < m < n$

The groups in this section are more complicated than the ones in the previous section. Nevertheless in order to get the results we will use a similar procedure as the one in section 3. As pointed out in the introduction let $1 < m < n$ and $B(m, n) = \langle a, b : a^{-1}b^m a = b^n \rangle$.
Recall that such groups are non-virtually solvable.

Consider the homomorphisms $|\ |_a : B(m, n) \longrightarrow \mathbb{Z}$ which associates to each word $w \in B(m, n)$ the sum of the powers of a in the word. It is easy to see that this is a well defined map into $\mathbb{Z}$ which is surjective.

Proposition 4.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(m, n) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where K, the kernel of the map $|\ |_a$, is the set of the elements which have the sum of the powers of a equals to zero. Furthermore, $B(m, n) = K \rtimes \mathbb{Z}$ - semidirect product where the action is given with respect to some fixed section.

Proof. The first part is clear. The second part follows because $\mathbb{Z}$ is free so the sequence splits. Since the kernel K is not abelian the action is defined with respect to a specific section (see [2]) and the result follows.   □

Proposition 4.2 The kernel K coincides with $N\langle b \rangle$ which is the normalizer of $\langle b \rangle$ in $B(m, n)$.

Proof. Similar to Propositon 3.2.   □

Propositon 4.3 Any homomorphism $\phi : B(m, n) \to B(m, n)$ is a homomorphism of the short exact sequence given in Proposition 4.1.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianized of $B(1, n)$. The abelianized of $B(m, n)$, denoted by $B(m, n)_{ab}$, is isomorphic to $\mathbb{Z}_{n-m} + \mathbb{Z}$. The torsion elements of $B(m, n)_{ab}$ form a subgroup isomorphic to $\mathbb{Z}_{n-m}$ which is invariant under any homomorphism. The preimage of this subgroup under the projection to the abelianized

$B(1, n) \to B(1, m)_{ab}$ is exactly the subgroup $N(b)$, i.e. the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$ and the result follows. $\qquad\square$

In order to have a homomorphism ϕ of $B(m, n)$ which has finite Reidemeister number, the induced map on the quotient $\mathbb{Z}$ must be different from the identity by the same argument used in the proof of Theorem 3.4. So we will assume this from now on that ϕ is not the identity.

Now we will give a presentation of the group K. The group K is generated by the elements $g_i = a^{-i}ba^i$ $i \in Z$ and they satisfy the following relations: $\{1 = a^{-j}(a^{-1}b^m ab^{-n})a^j = g_{j+1}^m g_j^{-n}\}$ for all integers j. This presentation is a consequence of the Bass-Serre theory, see [5] Theorem 27 page 211.

Now we will prove the main result of this section. Denote by K_{ab} the abelianized of K.

Theorem 4.4 For any injective homomorphism of $B(m, n)$ we have that the Reidemeister number is infinite.

Proof. Let us consider the short exact sequence obtained from the short exact sequence given in Proposition 4.1 by taking the quotient with the commutators subgroup of K, i.e.

$$0 \longrightarrow K_{ab} \longrightarrow B(m, n)/[K, K] \longrightarrow \mathbb{Z} \longrightarrow 1.$$

So we obtain a short exact sequence where the kernel K_{ab} is abelian. From the presentation of K we obtain a presentation of K_{ab} given as follows: It is generated by the elements g_i $i \in Z$ and they satisfy the following relations: $\{1 = g_{j+1}^m g_j^{-n}, g_i g_j = g_j g_i\}$ for all integers i, j. This is the same as the quotient of the free abelian group generated by the elements g_i, $i \in Z$(so the direct sum of $Z's$ indexed by Z), module the subgroup generated by the relations $\{1 = g_{j+1}^m g_j^{-n}\}$. So an element can be regarded as an equivalence class of a sequence of integers indexed by Z, where the elements of the sequence are zero but for finite number. By abuse of notation we denote the induced homomorphism on $B(m, n)/[K, K]$ also by ϕ. As in Theorem 3.5, we apply the injective homomorphism ϕ to the relation $a^{-1}b^m a = b^n$ regarded as elements of $B(m, n)/[K, K]$ and we obtain $\phi(a^{-1}(b)^m a) = \phi(b^n) = \phi(b)^n$. Since the original homomorphism is injective it follows that the induced homomorphisms on the quotient and on K are also injective. But this implies that the homomorphism on K_{ab} is also injective and it follows that ϕ is injective on $B/[K, K]$. As in Theorem 3.5, we can assume that $\phi(a) = a^k\theta$ for $\theta \in K_{ab}$ and $k \neq 0, 1$, otherwise either the map would not be injective or it follows immediatly that the Reidemeister number is infinite, respectively. Since the kernel

of the extension is abelian the equation becomes $\theta^{-1}a^{-k}\phi((b)^m)a^k\theta = a^{-k}\phi((b)^m)a^k = \phi(b^n) = n\phi(b)$. The element $\phi(b)$ is the equivalence class of a sequence of integers indexed in Z (this comes from the presentation of K_{ab}) where only a finite number of elements of the sequence is different from zero. So let $\phi(b) = (n_{i_1},...,n_{i_r})$ where $i_1 < i_2 < ... < i_r$ and $n_{i_s} \neq 0$ for all $s = 1,...,r$, and let $t = i_r - i_1$. In the group K_{ab} we have identifications and $\xi_l^n = \xi_{l-1}^m$ (this follows from the presentation of the group). So if we take the power $b^{mn^{t+1}}$, then $\phi(b^{mn^{t+1}}) = \phi(b^{n^t})^{mn}$. From the relations it follows that $\phi(b^{n^t}) = \phi(b)^{n^t} = (n^t n_{i_1},...,n^t n_{i_r}) = b_0$ for some $b_0 \in Z_{i_1}$. Now consider the equation

$$a^{-k}\phi((b^{n^t})^{m^k})a^k = \phi(b^{n^{t+k}}) = (\phi(b^{n^t}))^{n^k} = (b_0)^{n^k}.$$

But

$$\phi(a^{-1}(b^{n^t})^{m^k})a) = \phi((b^{n^{t+1}})^{m^{k-1}}) =$$
$$= (\phi(b^{nt}))^{nm^{k-1}} = (b_0)^{nm^{k-1}}.$$

So we have either $n^k = nm^{k-1}$, which implies $m = n$, or b_0 is the trivial element. Since neither cases can happen, the result follows. $\square$

5.   The case $B(n,n)$, $n > 1$

As it was point out before if $n = 1$ the group is $\mathbb{Z} + \mathbb{Z}$. Then we have even an automorphism that has a finite number of Reidemeister classes. For $n > 1$ we describe the groups $B(n,n)$ as certain extensions in order to study the automorphims. At the end we show that any automorphism have infinite Reidemeister number.

These groups, in contrast with the other Baumslag-Solitar groups already considered, have subgroups isomorphic to $\mathbb{Z} + \mathbb{Z}$. Let us point out that for $n = 2$ this is not the fundamental group of the Klein bottle. There is a surjection from $B(2,2)$ into the fundamental group of the Klein bottle.

We start by describing these groups. Let $|\ |_b : B(n,n) \to \mathbb{Z}$ be the homomorphism which associates to a word the sum of the powers of b which appears in the word. This is a well defined surjective homomorphism and we have:

Proposition 5.1 There is a splitting short exact sequence:

$$0 \to F \to B(n,n) \to \mathbb{Z} \to 1,$$

where F is the free group in n generators $x_1,...,x_n$ and the last map is $|\ |_b$. Further, the action of the generator $1 \in Z$ is the automorphism of F which sends x_j to x_{j+1} for $j < n$, and x_n to x_1.

Proof. Let $F \rtimes \mathbb{Z}$ be the semi-direct product of F by $\mathbb{Z}$ where F is the free group on the set $x_1, ..., x_n$ and the action is given by the automorphism of F which maps x_j to x_{j+1} for $j < n$ and x_n to x_1. We will show that $B(n,n)$ is isomorphic to $F \rtimes \mathbb{Z}$. For, consider the map $\psi : \{a, b\} \rightarrow F \rtimes \mathbb{Z}$, which sends a to x_1 and b to $1 \in Z$. This map extends to a homomorphism $B(n,n) \rightarrow F \rtimes Z$, which we also denote by ψ, since the relation which defines the group $B(n,n)$ is preserved by the map. Also ψ is a homomorphism of short exact sequences. The map restricted to the kernel of $| \ |_b$ is surjective to the free group F. Also the kernel admits a set of generators with cardinality n. So the map restrict to the kernel is an isomorphism and the result follows. $\qquad\square$

Proposition 5.2 The center of $B(n,n)$ is the subgroup generated by b^n and any injective homomorphism $\phi : B(n,n) \rightarrow B(n,n)$ leaves the center invariant.

Proof. For the first part, from Proposition 5.1 we know that $B(n,n)$ is of the form $F \rtimes \mathbb{Z}$. Let $(w, b^r) \in F \rtimes \mathbb{Z}$ be in the center and $(v, 1) \in F \rtimes \mathbb{Z}$ where v is an arbitrary element of F. We have $(w, b^r)(v, 1) = (w.b^r(v), b^r)$ and $(v, 1)(w, b^r) = (v.w, b^r)$. We can assume that w is a word written in the reduced form which starts with $x_i^{n_i}$ for some $1 \leq i \leq n$. Let r_0 be the integer $0 \leq r_0 \leq n - 1$ congruent to n. Now we consider several cases. I- Let $r_0 = 0$. Then take $v = x_{i+1}$ if $i < n$ or $v = x_1$ if $i = n$. We claim that $w.b^r(v) \neq v.w$, so the elements do not commute. To see that they do not commute observe first that $v.w$ is in the reduced form. If $w.b^r(v)$ is not reduced they can not be equal. If it is reduced, also they can not be equal since they start with different letters. The argument above does not work if $w = 1$, but this is the case where the element is in the center. II- Let $r_0 \neq 0$ and $w \neq 1$. Then take $v = x_i^{n_i}$. Again $v.w$ is in the reduced form which stars with $x_i^{2n_i}$. If $w.b^r(v)$ is not reduced they can not be equal. If it is reduced, also they can not be equal since they start with different power of x_i, even if the word contains only one letter since $b^r(v)$ is not a power of x_i(r is not congruent to 0 mod n). III- Let $r_0 \neq 0$ and $w = 1$. Then $r = kn + r_0$ and from the relation of the group follows $a^{-1}b^r a = a^{-1}b^{kn+r_0}ra = b^{kn}a^{-1}b^{r_0}a$. But $a^{-1}b^{r_0}a = b^{r_0}$ implies $b^{r_0}ab^{-r_0} = a$ which in terms of the notation of the Propositon 5.1 means $x_1 = x_{r_0}$ which is a contradition. So the result follows.

For the second part we have to show that if an element commutes with $im(\phi)$ then it is of the form b^{kn} for some $k \neq 0$. Since $\phi(b^n)$ has the property which commutes with $im(\phi)$, the result will follows from the claim above. First observe that the image of ϕ is a non-abelian subgroup and $\phi(b^n)$ is of the form $(w, b^{k_1 n})$ where k_1 is possible zero. It is not

hard to see that for an element of the form $(w, b^{k_1 n})$ where $w \neq 1$ its centralizer is $\mathbb{Z} + \mathbb{Z}$, namely genereted by b^n and the highest root of w in F. So there is an element in the image which doesn't commute with $(w, b^{k_1 n})$ unless $w = 1$. But in this case k_1 has to be different from zero since the homomorphism is injective and the result follows. $\qquad \square$

Now we consider the group which is the quotient of $F \rtimes \mathbb{Z}$ by the center, where the center is the subgroup $< b^n >$. This quotient is isomorphic to $F \rtimes \mathbb{Z}_n$ where we denote the image of the generator b in $\mathbb{Z}$ by $\bar{b}$ in $\mathbb{Z}_n$.

Propositon 5.3 Any homomorphism of the group $F \rtimes Z_n$ such that the restriction to F is an automorphism has infinite Reidemeister number.

Proof. We know that F is the free group in the letters $x_1, ..., x_n$ and let $\theta : F \rtimes \mathbb{Z} \to \mathbb{Z}_n$ be the homomorphism defined by $\theta(x_i) = 1$ and $\theta(\bar{b}) = 0$. The kernel of this homomorphism defines a subgroup of $F \rtimes \mathbb{Z}$ of index n which is isomorphic to $F' \rtimes \mathbb{Z}_n$ where F' is the kernel of the homomorphism θ restricted to F. Now we claim that F' is invariant with respect to any homomorphism, i.e. F' is characteristic. Let $(w, \bar{1})$ be an arbitrary element of the subgroup F' with $w \neq 1$. First observe that $\theta(\phi(x_i)) = \theta(\phi(x_1))$ for all i. This folows by induction. Since $x_{i+1} = \bar{b}.x_i.\bar{b}^{-1}$ follows $\theta(\phi(x_{i+1})) = \theta(\phi(\bar{b})).\theta(\phi(x_i)).\theta(\phi(\bar{b}^{-1})) = \theta(\phi(x_i))$. Therefore $\theta(\phi(w, \bar{1})) = \theta((w, \bar{1}))\theta(\phi(x_1))$ and follows that the subgroup is invariant. Based on this the given homomorphism ϕ provides a map of the short exact sequence

$$0 \to F' \to F \rtimes \mathbb{Z}_n \to \mathbb{Z}_n + \mathbb{Z}_n \to 0$$

where the restriction to the kernel is an automorphism of a free group of finite rank. So by the Corollary 2.4 the result follows. $\qquad \square$

Now we proof the main result.

Theorem 5.4 Any automorphism ϕ of $B(m, n)$ has infinite Reidemeister number.

Proof. Given an automorphism from Propositon 5.2 it induces a surjective homomorphism on $F \rtimes Z_n$, which we denote by $\bar{\phi}$. Let $(w, \bar{1}) \in F \rtimes \mathbb{Z}_n$ be an element of the kernel. So the image of the element $(w, 1)$ by ϕ belongs $n\mathbb{Z}$. Any multiple of $(w, \bar{1})$ also belongs to the kernel. Since ϕ is injective, the restriction to $< b^n >$ is injective so the image is a non-trivial subgroup. So some non-trivial multiple of $\phi(w, 1)$ belongs to the image of ϕ', where ϕ' is the restriction of ϕ to the subgroup $n\mathbb{Z}$. This contradict the fact that ϕ is injective unless the multiple of $(w, 1)$ is trivial. Since F is torsion free follows that $(w, 1)$ must be zero and we conclude that

the restriction of $\bar{\phi}$ to F is injective. Now from Proposition 5.3 the result follows. □

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