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Variety of Jordan algebras in small dimensions

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Abstract. The variety $\mathcal{J}or_n$ of Jordan unitary algebra structures on $\mathbf{k}^n$, $\mathbf{k}$ an algebraically closed field with $\operatorname{char}\mathbf{k} \neq 2$, is studied, as well as infinitesimal deformations of Jordan algebras. Also we establish the list of GL_n-orbits on $\mathcal{J}or_n$, $n = 4, 5$ under the action of structural transport. The numbers jor_4 and jor_5 of irreducible components are 3 and 6 respectively; a list of generic structures is included.

1. Introduction

In his survey [1] of the classification theory of finite dimensional associative algebras E. Study gave a list of isomorphism classes of associative algebras up to dimension four. He considered all algebras of a given dimension as an algebraic variety and he showed that for associative algebras of dimension more than three it is impossible to find a generic algebra, or equivalently that corresponding variety has more than one irreducible components. In modern language the problem can be formulated as follows. Let $\mathbf{k}$ be an algebraically closed field of characteristic $\neq 2$, V an n-dimensional $\mathbf{k}$-vector space and $e_1, e_2, \dots, e_n$ a basis of V. In order to endow V with the $\mathbf{k}$-algebra structure we specify n^3 structure constants $c_{ij}^k \in \mathbf{k}$,

$$e_i \cdot e_j = \sum_{k=1}^{n} c_{ij}^k e_k.$$

Equivalently, an algebra structure on V is given by bilinear map, i.e. an element of $V^* \otimes V^* \otimes V$, which we consider together with its natural

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structure of an algebraic variety over $\mathbf{k}$. Moreover for any class of algebras defined by identities, the corresponding multiplication maps form an algebraically closed subset Alg_n of $V^* \otimes V^* \otimes V$. The linear group $\mathrm{GL}(V)$ operates on Alg_n by so-called 'transport of structure' action, which is induced by the natural action of $\mathrm{GL}(V)$ on $V^* \otimes V^* \otimes V$

$$(g, \mathcal{J})(x, y) \;\rightarrow\; g\mathcal{J}(g^{-1}x, g^{-1}y) \tag{1.1}$$

for any $\mathcal{J} \in Alg_n$, $g \in \mathrm{GL}(V)$ and $x, y \in V$. For any $\mathcal{J} \in Alg_n$ we denote by $\mathcal{J}^{\mathrm{GL}(V)}$ the orbit of $\mathcal{J}$ under $\mathrm{GL}(V)$ and we can consider the inclusion diagram of the Zariski closure of orbits of elements in Alg_n. More precisely we define for $\mathcal{J}_1$, $\mathcal{J}_2 \in Alg_n$ that $\mathcal{J}_1$ *deforms* to $\mathcal{J}_2$ if $\mathcal{J}_2$ lies in the Zariski closure of the orbit of $\mathcal{J}_1$ ($\mathcal{J}_1 \rightarrow \mathcal{J}_2$). Algebraic deformation theory was introduced for associative algebras by Gerstenhaber [2], [3], and was extended to Lie algebras by Nijenhius and Richardson [4]. The corresponding varieties of associative algebras $Assoc_n$ and of Lie algebras Lie_n are of fundamental importance in the theory of algebra and their deformations. Flanigan in [9] referred to the study of $Assoc_n$ as 'algebraic geography'.

The geometry of both $Assoc_n$ and Lie_n is rather complicated. It is known that the number of irreducible components increases exponentially with n [5], [6] and their dimensions are at most $\frac{4}{27}n^3 + O(r^{8/3})$ for $Assoc_n$ and $\frac{2}{27}n^3 + O(r^{8/3})$ for Lie_n [6]. However, a complete description of the orbits and degeneration partial order is only known for $n \leq 5$ in associative case [5] and $n \leq 4$ for Lie algebras [7]. Many but not all components of $Assoc_n$ are orbit closures [9]. An algebra $\mathcal{J}$ such that the closure of $\mathcal{J}^{\mathrm{GL}(V)}$ is a component is called *rigid*. Every semisimple associative or Lie algebra is known to be rigid [2], [6].

In this paper we introduce the variety of Jordan unitary algebras $\mathcal{J}or_n$. We extend some properties and facts mentioned above for $Assoc_n$ and Lie_n to the case of Jordan algebras. In Section 2, we consider infinitesimal deformations of Jordan algebras. We show that semisimple Jordan algebras are rigid and also prove analogue of the "straightening-out theorem" of Flanigan [10]. In Section 3, we define an algebraic variety $\mathcal{J}or_n$ and write down explicitly some routine properties as well as some invariants. Finally in Section 4, we will use them to establish the list of GL_n-orbits on $\mathcal{J}or_n$ for $n = 4, 5$.

2. Infinitesimal deformations of Jordan algebras

Recall, that a *Jordan* **k**-*algebra* is an algebra $\mathcal{J}$ with a multiplication "$\cdot$", satisfying the following relations for any $a, b \in \mathcal{J}$

$$\begin{aligned} a \cdot b &= b \cdot a \\ ((a \cdot a) \cdot b) \cdot a &= (a \cdot a) \cdot (b \cdot a). \end{aligned} \tag{2.2}$$

In particular any commutative associative algebra is Jordan.

Let V be a vector space associated to $\mathcal{J}$ and $K = \mathbf{k}((t))$ be the quotient field of the ring of formal series in one variable t. If we put $V_K = V \otimes_{\mathbf{k}} K$ then any bilinear function $f : V \times V \to V$, in particular a multiplication in $\mathcal{J}$ can be extended to a function $V_K \times V_K \to V_K$. Let f_t be a bilinear function $V_K \times V_K \to V_K$ of the following for

$$f_t(a, b) = a \cdot b + tF_1(a, b) + t^2 F_2(a, b) + \ldots,$$

where each F_i is a bilinear function defined over $\mathbf{k}$. We suppose that f_t is a Jordan product i.e. it satisfies (2.2). We may consider the algebra $\mathcal{J}_t = (V_K, f_t)$ as the generic element of a 'one parameter family of deformations of $\mathcal{J}$' as in [2]. The condition (2.2) for f_t is equivalent to having for all $a, b \in V$ and all $\nu = 0, 1, 2$

$$\begin{gathered} F_\nu(a, b) = F_\nu(b, a) \\ \sum_{\substack{c\lambda+\mu+\gamma=\nu \\ \lambda>0\ \mu>0\ \gamma>0}} F_\gamma(F_\mu(a, a), F_\lambda(b, a)) - F_\gamma(F_\mu(F_\lambda(a, a), b), a) = 0 \end{gathered} \tag{2.3}$$

In particular for $\nu = 0$ we obtain (2.2) for original multiplication and for $\nu = 1$ the relations (2.3) could be written in the form:

$$\begin{aligned} F_1(a^2, b \cdot a) - F_1(a^2 \cdot b, a) + a^2 \cdot F_1(b, a) - F_1(a^2, b) \cdot a + \\ + F_1(a, b) \cdot (b \cdot a) - (F_1(a, a) \cdot b) \cdot a = 0 \end{aligned} \tag{2.4}$$

Due to Theorem II.8.12, [8] any function satisfying (2.4) defines a null extension of a Jordan algebra $\mathcal{J}$ by Jordan bimodule $\mathcal{J}$, i.e. any such function is 2-cocycle of Jordan algebra $\mathcal{J}$ with coefficients in $\mathcal{J}$.

From [2], two one parameter families of the deformations f_t and g_t are called *equivalent* if $g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b)$, where Φ_t is an automorphism of V_K of the form:

$$\Phi_t(x) = x + t\phi_1(x) + t^2\phi_2(x) + \ldots. \tag{2.5}$$

Moreover if $g_t(a, b) = a \cdot b + tG_1(a, b) + \ldots$ and f_t and g_t are equivalent then $G_1(a, b) = F_1(a, b) + a \cdot \phi_1(b) + b \cdot \phi_1(a) - \phi_1(a \cdot b)$. If g_t is equivalent to $f_t(x, y) = x \cdot y$ it is called a *trivial* deformation. It is obvious that the obtained algebra $\mathcal{J}_t = (V_K, g_t)$ is isomorphic to $\mathcal{J} \otimes_{\mathbf{k}} K$.

Proposition 2.1. Any one-parameter family f_t of deformations of $\mathcal{J}$ is equivalent to $g_t(a,b) = a \cdot b + t^n F_n(a,b) + t^{n+1}F_{n+1}(a,b) + \dots$, with $F_n(a,b)$ being the non-split extension of $\mathcal{J}$ by Jordan bimodule $\mathcal{J}$.

Proof. Let $f_t(a,b) = a \cdot b + t^n F_n(a,b) + \dots$. Writing (2.3) for $\nu = n$ we obtain that F_n defines null extension of $\mathcal{J}$ by $\mathcal{J}$. Suppose that F_n corresponds to split extension, i.e. as in [8], II.8 $F_n(a,b) = \phi(a \cdot b) - \phi(a) \cdot b - a \cdot \phi(b)$ for some $\phi : \mathcal{J} \to \mathcal{J}$. Choosing $\Phi_t(a) = a + t^n \phi(a)$ we obtain that

$$\Phi_t^{-1} f_t(\Phi_t a, \Phi_t b) = ab + t^{n+1}F_{n+1}(a,b) + \dots .$$

Since

$$\begin{aligned} f_t(\Phi_t a, \Phi_t b) &= f_t(a + t^n\phi(a), b + t^n\phi(b)) = f_t(a,b) + t^n f_t(\phi(a),b) + t^n(a,\phi(b)) \\ &+t^{2n} f_t(\phi(a),\phi(b)) = ab + t^n F_n(a,b) + t^n\phi(a)b + t^n a\phi(b) + t^{n+1}(\dots) \end{aligned}$$

and

$$\Phi_t(ab + t^{n+1}F_{n+1}(a,b) + \dots) = ab + t^n\phi(ab) + t^{n+1}(\dots).$$

□

As we already mentioned in the introduction that one of the important problems in geometric classification is to describe *rigid* algebras (i.e. algebras which have only trivial deformations). Using last result we obtain the following class of rigid algebras.

Corollary 2.2. If any null extension of $\mathcal{J}$ by $\mathcal{J}$ is split then $\mathcal{J}$ is rigid.

In particular, from [12] follows that any semisimple Jordan algebra is rigid. Indeed, in this case the set of equivalence classes of null extension of $\mathcal{J}$ by any Jordan bimodule M is trivial, in other words any extension is split.

Finally let us discuss the structure of the deformed algebra $\mathcal{J}_t$ and compare it with that of $\mathcal{J}$ (or to be precise, with that of $\mathcal{J}_K$). Any finite dimensional Jordan algebra admits Levi-Maltsev decomposition $\mathcal{J} = \mathcal{S}(\mathcal{J}) + \operatorname{Rad}\mathcal{J}$, where $\mathcal{S}(\mathcal{J})$ is a semisimple subalgebra of $\mathcal{J}$ and $\operatorname{Rad}\mathcal{J}$ its radical. If f_t is the generic element of a one parameter family of deformation of $\mathcal{J}$ then, as in [9], one can construct a deformation equivalent to the given one such that the new deformation preserves the original multiplication in $\mathcal{S}(\mathcal{J}_K)$ as well as action of $\mathcal{S}(\mathcal{J}_K)$ on $\operatorname{Rad}\mathcal{J}$. Indeed, suppose that we are given the generic element of a one parameter family of deformations of $\mathcal{J}$ algebra $\mathcal{J}_t$ given by multiplication $f_t(x,y) = xy + tF_1(x,y) + t^2F_2(x,y)\dots$.

Theorem 2.3. There exists a one parameter family of deformations of $\mathcal{J}$ with the generic element $g_t(x,y)$ which is equivalent to f_t such that the radical of algebra $\mathcal{J}_t = (V_K, g_t)$ is $\operatorname{Rad}\mathcal{J}_t = R \otimes_k K$, where R is a nilpotent ideal in $\mathcal{J}$. Moreover, for all $x, y \in \mathcal{S}(\mathcal{J}_K)$ $g_t(x,y) = x \cdot y$ and for all $x \in \mathcal{S}(\mathcal{J}_K)$, $z \in \operatorname{Rad}\mathcal{J}_K$ $g_t(x,z) = x \cdot z$.

Proof. First we consider a **k**-basis of $\operatorname{Rad}(\mathcal{J}_t, f_t) = \{\xi_1, \xi_2, \dots, \xi_r\}$. We can always choose $\xi_i = z_i + a_1^i t + a_2^i t^2 + \dots$ with $z_i, a_k^i \in \mathcal{J}$ such that $z_1, \dots, z_r$ are linearly independent. Moreover, by definition we can choose ξ_i in a such way that z_i are **k**-linearly independent. Indeed, if for any $\xi \in \operatorname{Rad}\mathcal{J}_t$ $\xi = z + t(\dots)$ we always can write a linear combination $z = \alpha_1 z_1 + \alpha_2 z_2 + \dots \alpha_s z_s$ for some $s < r$ then

$$\operatorname{Rad}\mathcal{J}_t \ni t^{-1}(\xi - \alpha_1\xi_1 - \dots \alpha_k\xi_k) = b + t(\dots)$$

and b is again a linear combination of $z_1, \dots, z_s$. Repeating it we obtain that any ξ is a linear combination of $\xi_1, \dots, \xi_s$ what contradicts to the fact that $\dim \operatorname{Rad}(f_t) = r > k$. Since ξ_i belongs to the radical $\operatorname{Rad}(\mathcal{J}_t, f_t)$, z_i also generates a nilpotent ideal and in particular z_i belongs to the radical $\operatorname{Rad}\mathcal{J}$. Now extend $z_1, \dots, z_r$ to a **k**-basis of $\mathcal{J}$ and define Φ_t in $\mathcal{J}_K$ in the form $id + t\phi_1 + \dots$ such that $\Phi_t(\xi_i) = z_i$. The multiplication $\Phi_t \cdot f_t \cdot (\Phi_t^{-1} \times \Phi_t^{-1})$ satisfies the first part of the theorem.

To prove the second part we introduce a deformation of the homomorphism, as in [9]. If $f : A \to B$ is a homomorphism of associative algebras then a deformation of f is given by K-algebra homomorphism $f_t : A_K \to B_K$ in the form $f_t = f + tF_1 + t^2F_2 + \dots$ with $F_i : A \to B$ **k**-linear. The deformations f_t is called trivial if there exists an automorphism $\beta_t : B_K \to B_K$ such that $f_t = \beta_t \cdot f$. We say that f is rigid if all the deformation of f_t are trivial.

Further, for any Jordan algebra $\mathcal{I}$ we denote by $U(\mathcal{I})$ its universal enveloping algebra. The associative algebra $U(\mathcal{I})$ is a factor algebra of the associative free algebra $F(\mathcal{I})$ by the ideal generated by right multiplication maps R_a, $a \in \mathcal{I}$, see [8], II.8. We define an application: $g : U(\mathcal{S}(\mathcal{J})) \otimes U(\mathcal{S}(\mathcal{J})) \to \operatorname{End}(U(\mathcal{J}))$, $g(x,x')(y) = x * y * x'$ for all $x, x' \in U(\mathcal{S}(\mathcal{J}))$ and $y \in U(\mathcal{J})$, where $U(\mathcal{S}(\mathcal{J}))$, $U(\mathcal{J})$ are universal enveloping algebras for $\mathcal{S}(\mathcal{J})$ and $\mathcal{J}$ respectively and $*$ is a multiplication in $U(\mathcal{J})$. Now we consider the universal enveloping algebra $U(\mathcal{J}_t)$ of $\mathcal{J}_t$ with multiplication

$$m(a,b) = t^{-r}F_{-r}(a,b) + \dots + F_0(a,b) + tF_1(a,b) + \dots,$$

for $a, b \in U(\mathcal{J}_t)$. The multiplication in associative free algebra $F(\mathcal{J}_t)$ is **k**-linear therefore for elements of $\mathcal{J}$ F_i is 0 for $i < 0$. Using **k**-linearity we obtain that $m(a,b) = a*b + tF_1(a,b) + \dots$. Finally we define

$g_t : U(\mathcal{S}(\mathcal{J}_t)) \otimes U(\mathcal{S}(\mathcal{J}_t)) \to \mathrm{End}(U(\mathcal{J}_t))$, $g_t(w,w')(v) = m(m(w,v),w')$ for all $w, w' \in U(\mathcal{S}(\mathcal{J}_t))$ and $v \in U(\mathcal{J}_t)$. The homomorphism g_t is a deformation of g. On the other hand $U(\mathcal{S}(\mathcal{J}))$ is a semisimple algebra and therefore from [10] it follows that g_t must be trivial. Therefore we obtain an automorphism $\beta_t : U(\mathcal{J}_t) \to U(\mathcal{J}_t)$ such that $g_t = \beta_t \cdot g$ which induce a $K-$linear automorphism Ω_t of $\mathcal{J}_K$ in the form $\Omega_t(x) = x + t\omega_1(x) + \dots$ such that the composition $\Omega_t^{-1} \cdot f_t \cdot (\Omega_t \times \Omega_t)$ satisfies the second part of the theorem. We denote this multiplication by g_t. □

Roughly speaking, this theorem proves that the radical of $\mathcal{J}_K$ shrinks under deformation to that of $\mathcal{J}_t$ while a semisimple part of $\mathcal{J}_t$ contains semisimple part of $\mathcal{J}_K$ and absorb part of its radical. In particular, remark that the dimension of the radical does not increase under deformation.

3. Algebraic variety $\mathcal{J}or_n$

As we already mentioned in introduction, $\mathcal{J}or_n$ is an algebraic subvariety in affine space $\mathbb{A}^{n^3} \simeq V^* \otimes V^* \otimes V$. For any chosen set of structure constants $c_{ij}^k \in \mathbf{k}$ the product defined by it defines a Jordan algebra if it satisfies the identities (2.2), i.e.

$$c_{ij}^k = c_{ji}^k,$$

$$\sum_{a=1}^n c_{ij}^a \sum_{b=1}^n c_{kl}^b c_{ab}^p - \sum_{a=1}^n c_{kl}^a \sum_{b=1}^n c_{ja}^b c_{ib}^p + \sum_{a=1}^n c_{lj}^a \sum_{b=1}^n c_{ki}^b c_{ab}^p - \sum_{a=1}^n c_{ki}^a \sum_{b=1}^n c_{ja}^b c_{lb}^p + \sum_{a=1}^n c_{kj}^a \sum_{b=1}^n c_{il}^b c_{ab}^p - \sum_{a=1}^n c_{il}^a \sum_{b=1}^n c_{ja}^b c_{kb}^p = 0, \tag{3.6}$$

for all $i, j, k, l, p \in \{1, 2, \dots, n\}$. The first one guaranties that the product is commutative and the second one provided by linearization of Jordan identity. Moreover, since we consider only unitary algebras we can choose e_1 as the identity element of $\mathcal{J}$. In addition to (3.6) last condition translates into

$$c_{1i}^j = c_{i1}^j = \delta_{ij} \qquad i, j = 1, \dots, n. \tag{3.7}$$

Equations (3.7) and (3.6) cut out an algebraic variety $\mathcal{J}or_n$ in $\mathbf{k}^{n^3} = V^* \otimes V^* \otimes V$. A point $(c_{ij}^k) \in \mathcal{J}or_n$ represents n-dimensional unitary $\mathbf{k}$-algebra $\mathcal{J}$, along with a particular choice of basis (which gives the structural constants c_{ij}^k). A change of basis in $\mathcal{J}$ gives rise to a possible different point of $\mathcal{J}or_n$ or equivalently $\mathrm{GL}(V)$ operates on Alg_n. The set of different $\mathrm{GL}(V)$-orbits of this action is in one-to-one correspondence with the isomorphism classes of n-dimensional Jordan algebras. Recall from the introduction that $\mathcal{J}_1$ is called a deformation of $\mathcal{J}_2$

$(\mathcal{J}_1 \to \mathcal{J}_2)$ if $\mathcal{J}_2^{GL(V)} \subset \mathcal{J}or_n$ is contained in the Zariski closure of the orbit $\mathcal{J}_1^{GL(V)}$.

Lemma 3.1. If $\mathcal{J} \in \mathcal{J}or_n$ and $\mathcal{I}$ is a subvariety of $\mathcal{J}or_n$ then $\mathcal{J} \in \overline{\mathcal{I}}$ implies:

$$n^2 - \dim(\mathrm{Aut}(\mathcal{J})) \le \dim(\mathcal{I}).$$

In particular for $\mathcal{J}_1 \in \mathcal{J}or_n$ and $\mathcal{J} \to \mathcal{J}_1$ we have

$$\dim \mathrm{Aut}(\mathcal{J}) \le \dim \mathrm{Aut}(\mathcal{J}_1).$$

Proof. From [13], p. 98 follows that $\dim \mathcal{I} \le \dim \mathcal{J}^{GL(V)}$. To finish the proof use $\mathcal{J}^{GL(V)} = GL(V)/\mathrm{Aut}(\mathcal{J})$. □

This lemma gives a partial order on the set of GL-orbits of Jordan algebras. The following lemma is the basic tool for construction of deformation between two algebras.

Lemma 3.2. If there exists a curve Γ in $\mathcal{J}or_n$ which generically lies in $\mathcal{I}$ and which cuts $\mathcal{J}^{GL(V)}$ in special point than $\mathcal{J} \in \bar{\mathcal{I}}$.

The proof follows directly from the definition. To illustrate the lemma let us consider $\mathcal{J}or_2$ the variety of unitary 2-dimensional Jordan algebras. There are only two non-isomorphic unitary Jordan algebras in this dimension both are associative algebras $\mathcal{J}_1 = \mathbf{k} \times \mathbf{k}$ and $\mathcal{J}_2 = \mathbf{k}[x]/x^2$. We choose a basis e_1, e_2 corresponding to primitive idempotents of $\mathcal{J}_1$. And consider the transformation $f_1 = e_1 + e_2$ and $f_2 = te_2$, for some parameter t. Then for any $t \in \mathbf{k}^*$ the new algebra $\mathcal{J}'$ is isomorphic to $\mathcal{J}_1$. For $t = 0$ we get the following multiplication $f_1^2 = f_1$, $f_2^2 = tf_2 = 0$ and $f_1 \cdot f_2 = f_2$ which is $\mathcal{J}_2$. Hence $\mathcal{J}_1 \to \mathcal{J}_2$, all algebras of $\mathcal{J}or_2$ belong to the closure of $\mathcal{J}_1^{GL(V)}$ and therefore $\mathcal{J}or_2$ is irreducible subvariety of $\mathbb{A}^8_{\mathbf{k}}$ of dimension 4. Let $Comm_n$ denote the algebraic variety of commutative associative unitary algebras of dimension n. As we just showed $\mathcal{J}or_2 = Comm_2$. In general, any commutative associative algebra is Jordan algebra and therefore $Comm_n$ is a closed subvariety in $\mathcal{J}or_n$. If $\mathcal{J}_1 \in \mathcal{J}or_n$ is associative algebra and $\mathcal{J}_1 \to \mathcal{J}_2$ then $\mathcal{J}_2$ is also associative.

Proposition 3.3. The following sets are closed in Zariski topology in $\mathcal{J}or_n$:

1. $\{\mathcal{J} \in \mathcal{J}or_n | \dim \mathrm{Rad}\, \mathcal{J} \ge s\}$

2. $\{\mathcal{J} \in \mathcal{J}or_n | \dim \mathcal{J}^{\mathrm{m}} \ge s\}$

for all positive integers m, s.

Proof. The proof for the first set is given in [11]. To prove the second statement choose a basis $\{e_1, \dots, e_n\}$ of algebra $\mathcal{J} \in \mathcal{J}or_n$ and consider the set Ω of all commutative words of the length m in n variables $\{e_1, \dots, e_n\}$. We denote by $P_{n,m}$ its cardinality. Any such word can be written

$$w_l(e_1, \dots, e_n) = f_1^l e_1 + \dots + f_n^l e_n,$$

where f_i^l is a polynomial in structure constants c_{ij}^k of $\mathcal{J}$. Further let A be the matrix of dimension $P_{m,n} \times n$ where to every word from Ω corresponds the line $(f_1^l, \dots, f_n^l)$. $\dim \mathcal{J}^m \leq r$ The fact that $\dim \mathcal{J}^m \leq r$ is equivalent to the fact that all minors of degree $s+1$ are zeros, therefore we obtain a finite number of equations for structure constants and consequently the set defined by these identities is Zariski closed. □

4. Varieties $\mathcal{J}or_4$ and $\mathcal{J}or_5$

In this section we study the variety $\mathcal{J}or_n$ for $n = 4, 5$. Let $Comm_n$ be a variety define by n-dimensional commutative associative algebras with the identity. Obviously, $Comm_n$ is a closed subset in $\mathcal{J}or_n$ and therefore is affine subvariety of $\mathcal{J}or_n$. In [14] Mazzola proved that for $n \leq 6$ $Comm_n$ is irreducible affine variety and its only component is a Zariski closure of the orbit of semisimple associative commutative algebra $\mathbf{k} \times \dots \times \mathbf{k}$. Thus to complete a geometric classification for $\mathcal{J}or_n$ for $n \leq 6$ it is enough to deal with non-associative algebras.

Example 4.1. Consider 3-dimensional unitary non-associative Jordan algebras. In fact, there are only two non-isomorphic non-associative Jordan algebras in $\mathcal{J}or_3$: simple Jordan algebra $\mathcal{J}_1 = \{e_1, e_2, a\}$ with e_1, e_2 orthogonal idempotents and $e_1 \cdot a = e_2 \cdot a = \frac{1}{2} a$, $a^2 = e_1 + e_2$ and a Jordan algebra with one-dimensional radical $\mathcal{J}_2 = \{e_1, e_2, a\}$ with the same multiplication table except for $a^2 = 0$. Consider the following transformation of $\mathcal{J}_1$: $f_1 = e_1$, $f_2 = e_2$ and $f_3 = ta$. For $t = 0$ we obtain the multiplication table of $\mathcal{J}_2$ and therefore $\mathcal{J}_1 \to \mathcal{J}_2$. Hence $\mathcal{J}or_3$ consists of two irreducible components: one comes from Jordan associative algebra $\mathbf{k} \times \mathbf{k} \times \mathbf{k}$ and the other one comes from $\mathcal{J}_1$.

4.1. $\mathcal{J}$or$_4$

Let first $\mathcal{J}$ be a 4-dimensional Jordan algebra. From [15] we obtain the following list of non-isomorphic Jordan, non-associative unitary algebras of dimension 4. Here $c := c_1 + c_2$.

1. dim $\mathrm{Rad}\mathcal{J} = 0$:

$$\mathcal{J}_1 = \begin{array}{c|cccc} & c_1 & c_2 & a & c_3 \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & 0 \\ c_2 & 0 & c_2 & \frac{1}{2}a & 0 \\ a & \frac{1}{2}a & \frac{1}{2}a & c & 0 \\ c_3 & 0 & 0 & 0 & c_3 \end{array} \qquad \mathcal{J}_2 = \begin{array}{c|cccc} & c_1 & c_2 & a & b \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & \frac{1}{2}b \\ c_2 & 0 & c_2 & \frac{1}{2}a & \frac{1}{2}b \\ a & \frac{1}{2}a & \frac{1}{2}a & 0 & \frac{1}{2}c \\ b & \frac{1}{2}b & \frac{1}{2}b & \frac{1}{2}c & 0 \end{array}$$

dim Aut $\mathcal{J}_1 = 1$ dim Aut $\mathcal{J}_2 = 1$

2. dim $\mathrm{Rad}\mathcal{J} = 1$:

$$\mathcal{J}_3 = \begin{array}{c|cccc} & c_1 & c_2 & c_3 & b \\ \hline c_1 & c_1 & 0 & 0 & \frac{1}{2}b \\ c_2 & 0 & c_2 & 0 & \frac{1}{2}b \\ c_3 & 0 & 0 & c_3 & 0 \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & 0 \end{array} \qquad \mathcal{J}_4 = \begin{array}{c|cccc} & c_1 & c_2 & a & b \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & \frac{1}{2}b \\ c_2 & 0 & c_2 & \frac{1}{2}a & \frac{1}{2}b \\ a & \frac{1}{2}a & \frac{1}{2}a & c & 0 \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & 0 \end{array}$$

dim Aut $\mathcal{J}_3 = 2$ dim Aut $\mathcal{J}_4 = 4$

3. dim $\mathrm{Rad}\mathcal{J} = 2$:

$$\mathcal{J}_5 = \begin{array}{c|cccc} & c_1 & c_2 & a & b \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & b \\ c_2 & 0 & c_2 & \frac{1}{2}a & 0 \\ a & \frac{1}{2}a & \frac{1}{2}a & 0 & 0 \\ b & b & 0 & 0 & 0 \end{array} \qquad \mathcal{J}_6 = \begin{array}{c|cccc} & c_1 & c_2 & a & b \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & b \\ c_2 & 0 & c_2 & \frac{1}{2}a & 0 \\ a & \frac{1}{2}a & \frac{1}{2}a & b & 0 \\ b & b & 0 & 0 & 0 \end{array}$$

dim Aut $\mathcal{J}_5 = 3$ dim Aut $\mathcal{J}_6 = 2$

$$\mathcal{J}_5 = \begin{array}{c|cccc} & c_1 & c_2 & a & b \\ \hline c_1 & c_1 & 0 & \frac{1}{2}a & \frac{1}{2}b \\ c_2 & 0 & c_2 & \frac{1}{2}a & \frac{1}{2}b \\ a & \frac{1}{2}a & \frac{1}{2}a & 0 & 0 \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & 0 \end{array}$$

dim Aut $\mathcal{J}_7 = 6$

4. When dim $\mathrm{Rad}\mathcal{J} = 3$ all 4-dimensional Jordan unitary algebras are associative.

Theorem 4.2. The irreducible components of $\mathcal{J}or_4$ are the Zariski closures of the orbits of algebras $\Omega = \{\mathcal{J}_1, \mathcal{J}_2\}$.

Proof. The proof consists of two parts. First, we show that any Jordan algebra $\mathcal{J}$ from the above list is dominated by either algebra $\mathcal{J}_1$ or $\mathcal{J}_2$. Second, we will show that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are rigid. To show that $\mathcal{J}_1 \to \mathcal{J}_3$ we construct transformation of $\mathcal{J}_1$ as $C_{1t} = c_1$, $C_{2t} = c_2$,

$C_{3t} = c_3$, $A_t = ta$. Then it is clear that the structure constants of this basis specialize to those of $\mathcal{J}_3$. Analogously, $\mathcal{J}_6 \to \mathcal{J}_5$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = ta$, $B_t = b$; $\mathcal{J}_2 \to \mathcal{J}_4$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a + b$, $B_t = tb$; $\mathcal{J}_4 \to \mathcal{J}_7$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = ta$, $B_t = b$; and $\mathcal{J}_1 \to \mathcal{J}_6$ with $C_{1t} = c_1 + c_3$, $C_{2t} = c_2$, $A_t = ta$, $C_{3t} = t^2 c$.

The second part of the proof is trivial in this case since both $\mathcal{J}_1$ and $\mathcal{J}_2$ are semisimple and therefore rigid by Corollary 2.3. □

Therefore we obtain that $\mathcal{J}or_4 \setminus Comm_4$ is affine variety with 2 irreducible components. Finally, $\mathcal{J}or_4$ consists of three irreducible components, two provided by non-associative Jordan semisimple algebra and one by the associative commutative semisimple algebra $\mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}$.

4.2. $\mathcal{J}$or$_5$

Now let $\mathcal{J}$ be a 5-dimensional Jordan unitary algebra. Again, from [15] we obtain the following list of non-associative Jordan unitary algebras of dimension 5. Here $c := c_1 + c_2$.

1. dim Rad$\mathcal{J} = 0$:

$\mathcal{J}_1 =$

	c_1	c_2	a	c_3	c_4
c_1	c_1	0	$\frac{1}{2}a$	0	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	c	0	0
c_3	0	0	0	c_3	0
c_4	0	0	0	0	c_4

dim Aut $\mathcal{J}_1 = 1$

$\mathcal{J}_2 =$

	c_1	c_2	a	b	c_3
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	0
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	$\frac{1}{2}c$	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	$\frac{1}{2}c$	0	0
c_3	0	0	0	0	c_3

dim Aut $\mathcal{J}_2 = 3$

$\mathcal{J}_3 =$

	c_1	c_2	a	b	d
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	0
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	c	0	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	0	c	0
d	$\frac{1}{2}d$	$\frac{1}{2}d$	0	0	c

dim Aut $\mathcal{J}_3 = 6$

2. dim Rad$\mathcal{J} = 1$:

$\mathcal{J}_4 =$

	c_1	c_2	a	c_3	c_4
c_1	c_1	0	$\frac{1}{2}a$	0	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	0	0
c_3	0	0	0	c_3	0
c_4	0	0	0	0	c_4

dim Aut $\mathcal{J}_4 = 2$

$\mathcal{J}_5 =$

	c_1	c_2	a	c_3	b
c_1	c_1	0	$\frac{1}{2}a$	0	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	c	0	0
c_3	0	0	0	c_3	b
c_4	0	0	0	b	0

dim Aut $\mathcal{J}_5 = 2$

$\mathcal{J}_6 =$

	c_1	c_2	a	b	c_3
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	0
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	c	0	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	0	0	0
c_3	0	0	0	0	c_3

dim Aut $\mathcal{J}_6 = 4$

$\mathcal{J}_7 =$

	c_1	c_2	a	b	d
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}d$
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}d$
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	$\frac{1}{2}c$	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	$\frac{1}{2}c$	0	0
d	$\frac{1}{2}d$	$\frac{1}{2}d$	0	0	0

dim Aut $\mathcal{J}_7 = 7$

3. dim Rad$\mathcal{J} = 2$:

$\mathcal{J}_8 =$

	c_1	c_2	a	c_3	b
c_1	c_1	0	$\frac{1}{2}a$	0	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	0	0
c_3	0	0	0	c_3	b
c_4	0	0	0	b	0

dim Aut $\mathcal{J}_8 = 3$

$\mathcal{J}_9 =$

	c_1	c_2	a	b	c_3
c_1	c_1	0	$\frac{1}{2}a$	b	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	b	0	0
b	b	0	0	0	0
c_3	0	0	0	0	c_3

dim Aut $\mathcal{J}_9 = 2$

$\mathcal{J}_{10} =$

	c_1	c_2	a	b	c_3
c_1	c_1	0	$\frac{1}{2}a$	b	0
c_2	0	c_2	$\frac{1}{2}a$	0	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	0	0
b	b	0	0	0	0
c_3	0	0	0	0	c_3

dim Aut $\mathcal{J}_{10} = 3$

$\mathcal{J}_{11} =$

	c_1	c_2	a	b	c_3
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	0
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	0
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	0	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	0	0	0
c_3	0	0	0	0	c_3

dim Aut $\mathcal{J}_{11} = 6$

$\mathcal{J}_{12} =$

	c_1	c_2	c_3	a	b
c_1	c_1	0	0	$\frac{1}{2}a$	$\frac{1}{2}b$
c_2	0	c_2	0	$\frac{1}{2}a$	0
c_3	0	0	c_3	0	$\frac{1}{2}b$
a	$\frac{1}{2}a$	$\frac{1}{2}a$	0	0	0
b	$\frac{1}{2}b$	0	$\frac{1}{2}b$	0	0

dim Aut $\mathcal{J}_{12} = 3$

$\mathcal{J}_{13} =$

	c_1	c_2	a	b	d
c_1	c_1	0	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}d$
c_2	0	c_2	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}d$
a	$\frac{1}{2}a$	$\frac{1}{2}a$	c	0	0
b	$\frac{1}{2}b$	$\frac{1}{2}b$	0	0	0
d	$\frac{1}{2}d$	$\frac{1}{2}d$	0	0	0

dim Aut $\mathcal{J}_{13} = 6$

4. dim Rad$\mathcal{J} = 3$:

$$\mathcal{J}_{14} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & 0 & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & b & 0 & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & a \end{array} \qquad \mathcal{J}_{14}^{0} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & 0 & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & b & 0 & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & 0 \end{array}$$

$\dim \operatorname{Aut} \mathcal{J}_{14} = 4$ $\qquad$ $\dim \operatorname{Aut} \mathcal{J}_{14}^{0} = 5$

$$\mathcal{J}_{15} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & 0 & \frac{1}{2}d \\ a & a & 0 & b & 0 & 0 \\ b & b & 0 & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & b \end{array} \qquad \mathcal{J}_{15}^{0} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & 0 & \frac{1}{2}d \\ a & a & 0 & b & 0 & 0 \\ b & b & 0 & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & 0 \end{array}$$

$\dim \operatorname{Aut} \mathcal{J}_{15} = 3$ $\qquad$ $\dim \operatorname{Aut} \mathcal{J}_{15}^{0} = 4$

$$\mathcal{J}_{16} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & 0 & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & 0 & b & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & a+b \end{array} \qquad \mathcal{J}_{16}^{(1,0)} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & 0 & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & 0 & b & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & a \end{array}$$

$\dim \operatorname{Aut} \mathcal{J}_{16} = 2$ $\qquad$ $\dim \operatorname{Aut} \mathcal{J}_{16}^{(1,0)} = 3$

$$\mathcal{J}_{16}^{0} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & 0 & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & 0 & b & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & 0 \end{array} \qquad \mathcal{J}_{17} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & \frac{1}{2}b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & \frac{1}{2}b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & b \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & b & 0 & 0 \end{array}$$

$\dim \operatorname{Aut} \mathcal{J}_{16}^{0} = 4$ $\qquad$ $\dim \operatorname{Aut} \mathcal{J}_{17} = 5$

$$\mathcal{J}_{18} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & \frac{1}{2}b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & \frac{1}{2}b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & a & a \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & a & a \end{array} \qquad \mathcal{J}_{19} = \begin{array}{c|ccccc} & c_1 & c_2 & a & b & d \\ \hline c_1 & c_1 & 0 & a & \frac{1}{2}b & \frac{1}{2}d \\ c_2 & 0 & c_2 & 0 & \frac{1}{2}b & \frac{1}{2}d \\ a & a & 0 & 0 & 0 & 0 \\ b & \frac{1}{2}b & \frac{1}{2}b & 0 & 0 & 0 \\ d & \frac{1}{2}d & \frac{1}{2}d & 0 & 0 & 0 \end{array}$$

$\dim \operatorname{Aut} \mathcal{J}_{18} = 4$ $\qquad$ $\dim \operatorname{Aut} \mathcal{J}_{19} = 7$

5. For $\dim \operatorname{Rad} \mathcal{J} = 4$ all Jordan algebras are associative commutative.

The geometric classification in this case is the following:

Theorem 4.3. The irreducible components of $\mathcal{J}or_5$ are the Zariski closures of the orbits of algebras: $\Omega = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_{18}, \mathcal{J}_{12}\}$.

Proof: Again, we show that for any Jordan algebra $\mathcal{J}$ from the above list exist an algebra $\tilde{\mathcal{J}} \in \Omega$ such that $\tilde{\mathcal{J}}$ is a deformation of $\mathcal{J}$.

First, we establish deformations in the above list of algebras by constructing transformations which specialize the structural constant of one algebra into another one.

1. $\mathcal{J}_1 \to \mathcal{J}_4$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3$, $C_{4t} = c_4$, $A_t = ta$;
2. $\mathcal{J}_1 \to \mathcal{J}_5$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3 + c_4$, $C_{4t} = tc_4$, $A_t = a$;
3. $\mathcal{J}_4 \to \mathcal{J}_8$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3 + c_4$, $C_{4t} = tc_4$, $A_t = a$;
4. $\mathcal{J}_2 \to \mathcal{J}_6$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3$, $A_t = a + b$, $B_t = ta + b$;
5. $\mathcal{J}_6 \to \mathcal{J}_{11}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3$, $A_t = ta$, $B_t = b$;
6. $\mathcal{J}_3 \to \mathcal{J}_7$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = \frac{1}{2}a + b$, $B_t = \frac{1}{2}a - bi$, $D_t = td$;
7. $\mathcal{J}_7 \to \mathcal{J}_{13}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a + b$, $B_t = ta - bi$, $D_t = d$;
8. $\mathcal{J}_9 \to \mathcal{J}_{10}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $C_{3t} = c_3$, $A_t = ta$, $B_t = b$;
9. $\mathcal{J}_{14} \to \mathcal{J}_{14}^0$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a$, $B_t = b$, $D_t = td$;
10. $\mathcal{J}_{15} \to \mathcal{J}_{15}^0$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a$, $B_t = b$, $D_t = td$;
11. $\mathcal{J}_{16} \to \mathcal{J}_{16}^{1,0}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = t^2 a$, $B_t = b$, $D_t = td$;
12. $\mathcal{J}_{16}^{1,0} \to \mathcal{J}_{16}^0$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a$, $B_t = b$, $D_t = td$;
13. $\mathcal{J}_{18} \to \mathcal{J}_{19}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = a$, $B_t = b$, $D_t = td$;
14. $\mathcal{J}_{15} \to \mathcal{J}_{14}$ with $C_{1t} = c_1$, $C_{2t} = c_2$, $A_t = ta$, $B_t = b$, $D_t = d$;
15. $\mathcal{J}_1 \to \mathcal{J}_{16}$ with $C_{1t} = c_1 + c_3$, $C_{2t} = c_2 + c_4$, $A_t = ta$, $C_{3t} = t^2 c_1$, $C_{4t} = t^2 c_2$;
16. $\mathcal{J}_1 \to \mathcal{J}_9$ with $C_{1t} = c_1 + c_3$, $C_{2t} = c_2$, $A_t = ta$, $C_{3t} = t^2 c_1 + c_2$, $C_{4t} = c_4$;

17. $\mathcal{J}_9 \to \mathcal{J}_{17}$ with $C_{1t} = c_1 + c_3$, $C_{2t} = c_2$, $A_t = a + c_1$, $B_t = b$, $C_{3t} = c_3 + tb$.

Now we prove that each algebra in Ω has only trivial transformations. First, we observe that it suffices to consider only deformation between non-associative algebras. Further, since $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are semisimples, they are rigid by Corollary 2.2. As the dimension of the radical does not increase under deformation we get that $\mathcal{J}_1, \mathcal{J}_2$ are the only candidates to deform into $\mathcal{J}_{12}$ e $\mathcal{J}_{18}$. But $\mathcal{J}_{12}$ can not be deformed in $\mathcal{J}_1$, for $\mathcal{J}_1$ contain a associative subalgebra of dimension 4. By Theorem (2.3) the algebra $\mathcal{J}_{12}$ could not be deformed in $\mathcal{J}_2$ since the action of semisimple part of radical is preserved under deformation. Finally, the algebra $\mathcal{J}_{18}$ is quadratic when both $\mathcal{J}_1$ and $\mathcal{J}_2$ are not.

We obtain that $\mathcal{J}or_5 \backslash Comm_5$ is an affine algebraic variety consisting of 5 irreducible components. Then $\mathcal{J}or_5$ is an algebraic variety with six irreducible components, each of them is the closure of one of the Jordan algebras from $\{\mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}, \Omega\}$.

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