On one-sided interval edge colorings of biregular bipartite graphs

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Abstract. A proper edge $t$-coloring of a graph $G$ is a coloring of edges of $G$ with colors $1, 2, \ldots, t$ such that all colors are used, and no two adjacent edges receive the same color. The set of colors of edges incident with a vertex $x$ is called a spectrum of $x$. Any nonempty subset of consecutive integers is called an interval. A proper edge $t$-coloring of a graph $G$ is interval in the vertex $x$ if the spectrum of $x$ is an interval. A proper edge $t$-coloring $\varphi$ of a graph $G$ is interval on a subset $R_0$ of vertices of $G$, if for any $x \in R_0$, $\varphi$ is interval in $x$. A subset $R$ of vertices of $G$ has an $i$-property if there is a proper edge $t$-coloring of $G$ which is interval on $R$. If $G$ is a graph, and a subset $R$ of its vertices has an $i$-property, then the minimum value of $t$ for which there is a proper edge $t$-coloring of $G$ interval on $R$ is denoted by $w_R(G)$. We estimate the value of this parameter for biregular bipartite graphs in the case when $R$ is one of the sides of a bipartition of the graph.

We consider undirected, finite graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. For any vertex $x \in V(G)$, we denote by $N_G(x)$ the set of vertices of a graph $G$ adjacent to $x$. The degree of a vertex $x$ of a graph $G$ is denoted by $d_G(x)$, the maximum degree of a vertex of $G$ by $\Delta(G)$.

For a graph $G$ and an arbitrary subset $V_0 \subseteq V(G)$, we denote by $G[V_0]$ the subgraph of $G$ induced by the subset $V_0$ of its vertices.

2010 MSC: 05C15, 05C50, 05C85.

Key words and phrases: proper edge coloring, interval edge coloring, interval spectrum, biregular bipartite graph.
Using a notation $G(X, Y, E)$ for a bipartite graph $G$, we mean that $G$
has a bipartition $(X, Y)$ with the sides $X, Y$, and $E = E(G)$.

An arbitrary nonempty subset of consecutive integers is called an
interval. An interval with the minimum element $p$ and the maximum
element $q$ is denoted by $[p, q]$.

A function $\varphi : E(G) \to [1, t]$ is called a proper edge $t$-coloring of a
graph $G$, if all colors are used, and no two adjacent edges receive the same
color.

The minimum $t \in \mathbb{N}$ for which there exists a proper edge $t$-coloring of
a graph $G$ is denoted by $\chi'(G)$ [26].

For a graph $G$ and any $t \in [\chi'(G), |E(G)|]$, we denote by $\alpha(G, t)$ the
set of all proper edge $t$-colorings of $G$. Let

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)} \alpha(G, t).$$

If $G$ is a graph, $x \in V(G), \varphi \in \alpha(G)$, then let us set $S_G(x, \varphi) \equiv \{\varphi(e) / e \in E(G), e$ is incident with $x\}$.

We say that $\varphi \in \alpha(G)$ is persistent-interval in the vertex $x_0 \in V(G)$
of the graph $G$ iff $S_G(x_0, \varphi) = [1, d_G(x_0)]$. We say that $\varphi \in \alpha(G)$ is
persistent-interval on the set $R_0 \subseteq V(G)$ iff $\varphi$ is persistent-interval in
$\forall x \in R_0$.

We say that $\varphi \in \alpha(G)$ is interval in the vertex $x_0 \in V(G)$ of the graph
$G$ iff $S_G(x_0, \varphi)$ is an interval. We say that $\varphi \in \alpha(G)$ is interval on the set
$R_0 \subseteq V(G)$ iff $\varphi$ is interval in $\forall x \in R_0$.

We say that a subset $R$ of vertices of a graph $G$ has an $i$-property
iff there exists $\varphi \in \alpha(G)$ interval on $R$; for a subset $R \subseteq V(G)$ with an
$i$-property, the minimum value of $t$ warranting existence of $\varphi \in \alpha(G, t)$
interval on $R$ is denoted by $w_R(G)$.

Notice that the problem of deciding whether the set of all vertices of
an arbitrary graph has an $i$-property is $NP$-complete [7, 8, 17]. Unfortu-
nately, even for an arbitrary bipartite graph (in this case the interest is
strengthened owing to the application of an $i$-property in timetablings
[6, 17]) the problem keeps the complexity of a general case [3, 12, 25]. Some
positive results were obtained for graphs of certain classes with numerical
or structural restrictions [9, 11, 13–15, 17, 19–22, 28, 29]. The examples of
bipartite graphs whose sets of vertices have not an $i$-property are given
in [6, 13, 16, 23, 25].

The subject of this research is a parameter $w_R(G)$ of a bipartite graph
$G = G(X, Y, E)$ in the case when $R$ is one of the sides of the bipartition
of $G$ (the exact value of this parameter for an arbitrary bipartite graph is not known as yet). We obtain an upper bound of the parameter being discussed for biregular [2–5, 24] bipartite graphs, and the exact values of it in the case of the complete bipartite graph $K_{m,n}$ ($m \in \mathbb{N}, n \in \mathbb{N}$) as well.

The terms and concepts that we do not define can be found in [27].

First we recall some known results.

**Theorem 1** ([7, 8, 17]). If $R$ is one of the sides of a bipartition of an arbitrary bipartite graph $G = G(X, Y, E)$, then: 1) there exists $\varphi \in \alpha(G, |E|)$ interval on $R$, 2) for $\forall t \in [w_R(G), |E|]$, there exists $\psi_t \in \alpha(G, t)$ interval on $R$.

**Theorem 2** ([1, 7, 8]). Let $G = G(X, Y, E)$ be a bipartite graph. If for $\forall e = (x, y) \in E$, where $x \in X, y \in Y$, the inequality $d_G(y) \leq d_G(x)$ is true, then $\exists \varphi \in \alpha(G, \Delta(G))$ persistent-interval on $X$.

**Corollary 1** ([1, 7, 8]). Let $G = G(X, Y, E)$ be a bipartite graph. If $\max_{y \in Y} d_G(y) \leq \min_{x \in X} d_G(x)$, then $\exists \varphi \in \alpha(G, \Delta(G))$ persistent-interval on $X$.

**Remark 1.** Note that Corollary 1 follows from the result of [10].

Let $H = H(\mu, \nu)$ be a $(0, 1)$-matrix with $\mu$ rows, $\nu$ columns, and with elements $h_{ij}$, $1 \leq i \leq \mu, 1 \leq j \leq \nu$. The $i$-th row of $H$, $i \in [1, \mu]$, is called collected, iff $h_{ip} = h_{iq} = 1, t \in [p, q]$ imply $h_{it} = 1$, and the inequality $\sum_{j=1}^{\nu} h_{ij} \geq 1$ is true. Similarly, the $j$-th column of $H$, $j \in [1, \nu]$, is called collected, iff $h_{pj} = h_{qj} = 1, t \in [p, q]$ imply $h_{ij} = 1$, and the inequality $\sum_{i=1}^{\mu} h_{ij} \geq 1$ is true. If all rows and all columns of $H$ are collected, then for $i$-th row of $H$, $i \in [1, \mu]$, we define the number $\varepsilon(i, H) \equiv \min \{j/h_{ij} = 1\}$.

$H$ is called a collected matrix (see Figure 1), iff all its rows and all its columns are collected, $h_{11} = h_{\mu\nu} = 1$, and $\varepsilon(1, H) \leq \varepsilon(2, H) \leq \cdots \leq \varepsilon(\mu, H)$.

$H$ is called a $b$-regular matrix ($b \in \mathbb{N}$), iff for $\forall i \in [1, \mu], \sum_{j=1}^{\nu} h_{ij} = b$. $H$ is called a $c$-compressed matrix ($c \in \mathbb{N}$), iff for $\forall j \in [1, \nu], \sum_{i=1}^{\mu} h_{ij} \leq c$.

**Lemma 1** ([18]). If a collected $n$-regular ($n \in \mathbb{N}$) matrix $P = P(m, w)$ with elements $p_{ij}$ ($1 \leq i \leq m, 1 \leq j \leq w$) is $n$-compressed, then $w \geq \left\lceil \frac{m}{n} \right\rceil \cdot n$.

**Proof.** We use induction on $\left\lceil \frac{m}{n} \right\rceil$.

If $\left\lceil \frac{m}{n} \right\rceil = 1$, the statement is trivial.
Now assume that $\left\lceil \frac{m}{n} \right\rceil = \lambda_0 \geq 2$, and the statement is true for all collected $n'$-regular $n'$-compressed matrixes $P'(m',w')$ with $\left\lceil \frac{m'}{n'} \right\rceil \leq \lambda_0 - 1$.

First of all let us prove that $\varepsilon(n + 1, P) \geq n + 1$. Assume the contrary: $\varepsilon(n + 1, P) \leq n$. Since $P$ is a collected $n$-regular matrix, we obtain $\sum_{i=1}^{m} p_{in} \geq \sum_{i=1}^{n+1} p_{in} \geq n + 1$, which is impossible because $P(m, w)$ is an $n$-compressed matrix. This contradiction shows that $\varepsilon(n + 1, P) \geq n + 1$.

Now let us form a new matrix $P'(m - n, w - (\varepsilon(n + 1, P) - 1))$ by deleting from the matrix $P$ the elements $p_{ij}$, which satisfy at least one of the inequalities $i \leq n$, $j \leq \varepsilon(n + 1, P) - 1$.

It is not difficult to see that $P'(m - n, w - (\varepsilon(n + 1, P) - 1))$ is a collected $n$-regular $n$-compressed matrix with $\left\lceil \frac{m-n}{n} \right\rceil = \lambda_0 - 1$. By the induction hypothesis, we have

$$w - (\varepsilon(n + 1, P) - 1) \geq \left\lceil \frac{m-n}{n} \right\rceil \cdot n,$$

which means that

$$w \geq (\lambda_0 - 1)n + \varepsilon(n + 1, P) - 1 \geq (\lambda_0 - 1)n + n = \lambda_0 n = \left\lceil \frac{m}{n} \right\rceil \cdot n. \quad \square$$

Now, for arbitrary positive integers $m, l, n, k$, where $m \geq n$ and $ml = nk$, let us define the class $Bip(m, l, n, k)$ of biregular bipartite graphs:

$$Bip(m, l, n, k) \equiv \left\{ G = G(X, Y, E) \mid \begin{array}{l} |X| = m, |Y| = n, \\
\text{for } \forall x \in X, d_G(x) = l, \\
\text{for } \forall y \in Y, d_G(y) = k. \end{array} \right\}$$

**Remark 2.** Clearly, if $G \in Bip(m, l, n, k)$, then $\chi'(G) = k$. 
Theorem 3. If $G = G(X, Y, E) \in \text{Bip}(m, l, n, k)$, then $w_Y(G) = k$, $w_X(G) \leq l \cdot \left[ \frac{m}{n} \right]$.

Proof. The equality follows from Remark 2. Let us prove the inequality.

Let $X = \{x_1, \ldots, x_m\}$. For $\forall r \in [1, \left\lceil \frac{m}{n} \right\rceil]$, define $X_r = \{x_{(r-1)l+1}, \ldots, x_{rl}\}$. Define $X_{1+\left\lceil \frac{m}{n} \right\rceil} = X \setminus \left( \bigcup_{i=1}^{\left\lceil \frac{m}{n} \right\rceil} X_i \right)$. For $\forall r \in [1, \left\lceil \frac{m}{n} \right\rceil]$, define $Y_r = \bigcup_{x \in X_r} N_G(x)$. Define $Y_{1+\left\lceil \frac{m}{n} \right\rceil} = \bigcup_{x \in X_{1+\left\lceil \frac{m}{n} \right\rceil}} N_G(x)$. For $\forall r \in [1, \left\lceil \frac{m}{n} \right\rceil]$, define $G_r \equiv G[X_r \cup Y_r]$.

Consider the sequence $G_1, G_2, \ldots, G_{\left\lceil \frac{m}{n} \right\rceil}$ of subgraphs of the graph $G$.

From Corollary 1, we obtain that for $\forall i \in [1, \left\lceil \frac{m}{n} \right\rceil]$, there is $\varphi_i \in \alpha(G_i, l)$ persistent-interval on $X_i$.

Clearly, for $\forall e \in E(G)$, there exists the unique $\xi(e)$, satisfying the conditions $\xi(e) \in [1, \left\lceil \frac{m}{n} \right\rceil]$ and $e \in E(G_{\xi(e)})$.

Define a function $\psi : E(G) \rightarrow [1, l \cdot \left\lceil \frac{m}{n} \right\rceil]$. For an arbitrary $e \in E(G)$, set $\psi(e) \equiv (\xi(e) - 1) \cdot l + \varphi_{\xi(e)}(e)$.

It is not difficult to see that $\psi \in \alpha(G, l \cdot \left\lceil \frac{m}{n} \right\rceil)$ and $\psi$ is interval on $X$. Hence, $w_X(G) \leq l \cdot \left[ \frac{m}{n} \right]$.

\(\square\)

Theorem 4. Let $R$ be an arbitrary side of a bipartition of the complete bipartite graph $G = K_{m,n}$, where $m \in \mathbb{N}$, $n \in \mathbb{N}$. Then

$$w_R(G) = (m + n - |R|) \cdot \left\lceil \frac{|R|}{m + n - |R|} \right\rceil.$$ 

Proof. Without loss of generality we can assume that $G$ has a bipartition $(X, Y)$, where $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, and $m \geq n$.

Case 1. $R = Y$. In this case the statement follows from Theorem 3; thus $w_Y(G) = m$.

Case 2. $R = X$.

The inequality $w_X(G) \leq n \cdot \left\lceil \frac{m}{n} \right\rceil$ follows from Theorem 3. Let us prove that $w_X(G) \geq n \cdot \left\lceil \frac{m}{n} \right\rceil$.

Consider an arbitrary proper edge $w_X(G)$-coloring $\varphi$ of the graph $G$, which is interval on $X$.

Clearly, without loss of generality, we can assume that

$$\min(S_G(x_1, \varphi)) \leq \min(S_G(x_2, \varphi)) \leq \ldots \leq \min(S_G(x_m, \varphi)).$$

Let us define a $(0, 1)$-matrix $P(m, w_X(G))$ with $m$ rows, $w_X(G)$ columns, and with elements $p_{ij}, 1 \leq i \leq m, 1 \leq j \leq w_X(G)$. For $\forall i \in [1, m]$, and for $\forall j \in [1, w_X(G)]$, set

$$p_{ij} = \begin{cases} 1, & \text{if } j \in S_G(x_i, \varphi) \\ 0, & \text{if } j \not\in S_G(x_i, \varphi) \end{cases}.$$
It is not difficult to see that \( P(m, w_X(G)) \) is a collected \( n \)-regular \( n \)-compressed matrix. From Lemma 1, we obtain \( w_X(G) \geq n \cdot \lceil \frac{m}{n} \rceil \). □

From Theorems 1 and 3, taking into account the proof of Case 2 of Theorem 4, we also obtain

**Corollary 2.** If \( G \in Bip(m, l, n, k) \), then

1) for \( \forall t \in \left[l \cdot \left\lfloor \frac{m}{l} \right\rfloor, ml\right] \), there exists \( \varphi_t \in \alpha(G,t) \) interval on \( X \),

2) for \( \forall t \in [k, nk] \), there exists \( \psi_t \in \alpha(G,t) \) interval on \( Y \).

**References**


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Received by the editors: 17.12.2012
and in final form 10.02.2015.