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RESEARCH ARTICLE

Type of a point in Universal Geometry and in Model Theory

B. Plotkin, E. Plotkin, G. Zhitomirski

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To L. A. Kaluzhnin, a wonderful mathematician and friend

Abstract. The paper is devoted to relations between model theoretic types and logically geometric types. We show that the notion of isotypic algebras can be equally defined through MT-types and LG-types.

1. Informal introduction

The paper is devoted to the centenary of a friend, a wonderful person and an outstanding mathematician Lev Arkadievich Kaluzhnin. The senior author and L. A. Kaluzhnin were friends for many years. It was a time of mathematical, cultural, intellectual conversations, a time wherein spiritual themes lived in a peaceful agreement with jokes, kids and other topics of daily life. L. Kaluzhnin was a sharp mathematician and a wise man. He was Chair of Department of Algebra and Mathematical Logic in Kiev University. He created a scientific school in Kiev. Many of well-known mathematicians are proud to say that they belong to community of L. Kaluzhnin's mathematical "children" and "grandchildren". Among them O. Ganyushkin, Yu. Bodnarchuk, F. Lazebnik, M. Klin, R. Poeschel, V. Sushchanskii, V. Vyshenskii, V. Ustimenko, and many others.

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2. Introduction

The paper deals with relations between model theoretic types and logically geometric types. It can be viewed as a complement to the previous paper of G. Zhitomirski ([15]) devoted to the same subject. We discuss the fact that the notion of isotypic algebras can be equally defined through model theoretic types and logically geometric types. This bilateral insight gives rise to a lot of applications in algebra, geometry and computer science.

3. Introduction to Universal Geometry

In this Section we provide the reader with some account of notions which will be used explicitly and implicitly in Section 4 devoted to types. A more detailed background can be found, for example, in [6], [12], [10], [8], [7], [14], etc.

First of all, speaking of Algebraic Geometry, we mean Universal Geometry, i.e., geometry in an arbitrary variety of algebras Θ. If H is an algebra in Θ and $X = \{x_1, \dots, x_n\}$ is a set of variables, then we have a point $\mu : X \to H$ over H, which also can be written as $\bar{a} = (a_1, \dots, a_n)$, where $a_i = \mu(x_i)$. Passing to a free in Θ algebra $W = W(X)$, we represent the same point as a homomorphism $\mu : W(X) \to H$. Here we are able to speak of a kernel of a point $Ker(\mu)$. It is the equality relation $w \equiv w'$, $w, w' \in W(X)$ which is considered as an element of the algebra of formulas $\Phi(X)$. As for $\Phi(X)$, we assume that it is a Boolean algebra extended by the quantifier operations $\exists x$, $x \in X$ and by all possible equalities $w \equiv w'$, $w, w' \in W(X)$.

We will consider several categories. We fix an infinite set of variables X^0. Let Γ be a system of all finite subsets X in X^0. Denote by Θ^0 the category of all $W(X)$, $X \in \Gamma$, with morphisms as homomorphisms $s : W(Y) \to W(X)$. As usual, such a category can be viewed as a multi-sorted algebra whose domains are objects of Θ^0 and morphisms are multy-sorted operations. Consider also the category (and the multi-sorted algebra) $\widetilde{\Phi}_\Theta$ of all algebras of formulas $\Phi(X)$, $X \in \Gamma$ with the morphisms $s_* : \Phi(Y) \to \Phi(X)$ induced by morphisms $s : W(Y) \to W(X)$.

For each formula $v \in \Phi(Y)$ we have $s_* v = u \in \Phi(X)$. Transitions $W(X) \to \Phi(X)$ and $s \to s_*$ are organized in such a way that they induce a covariant functor $\Theta^0 \to \widetilde{\Phi}_\Theta$.

Now we shall define affine spaces. These are the sets $\mathrm{Hom}(W(X), H)$ of all points $\mu : W(X) \to H$. To every $s : W(Y) \to W(X)$ we associate

$\tilde{s} : \mathrm{Hom}(W(X), H) \to \mathrm{Hom}(W(Y), H)$, acting by $\tilde{s}(\mu) = \mu s : W(Y) \to H$, i.e., $\mu s(w) = \mu(s(w))$ for $\mu : W(X) \to H$.

The correspondence $W(X) \to \mathrm{Hom}(W(X), H)$ and $s \to \tilde{s}$ determines a contravariant functor $\Theta^0 \to \Theta^*(H)$, where $\Theta^*(H)$ is the category of affine spaces. It can be proved that these categories are dual if and only if the algebra H generates the whole variety Θ.

Further on we will work with an individual affine space $\mathrm{Hom}(W(X), H)$. Let $Bool(W(X), H)$ be its Boolean, that is the Boolean algebra of all subsets A in $\mathrm{Hom}(W(X), H)$. We want to equip this Boolean algebra with quantifier operations and equalities. First of all, define $B = \exists x A$, where $A \in Bool(W(X), H)$, setting: $\mu \in B$ if we have $\nu \in A$, such that $\mu(x') = \nu(x')$ for each $x' \in X$, $x' \neq x$. Universal quantifier $\forall x$ is defined via $\forall x A = \neg(\exists x(\neg A))$.

For every equality $w \equiv w'$ in $\Phi(X)$ determine the set $[w \equiv w']_H$ in $Bool(W(X), H)$. It is the set of all points $\mu : W(X) \to H$, satisfying the formula $w \equiv w'$, that is, $w^\mu \equiv w'^\mu$. This means that $(w, w') \in Ker(\mu)$. The elements $[w \equiv w']_H$ are called equalities in $Bool(W(X), H)$.

Boolean algebra $Bool(W(X), H)$ with additional operations called quantifiers and equalities provides the example of an extended Boolean algebra. Denote the constructed extended Boolean algebras of sets by $Hal^X_\Theta(H)$. One can define extended Boolean algebras axiomatically. For instance, the algebra $\Phi(X)$ has the same signature of operations as $Hal^X_\Theta(H)$ and also gives an example of an extended Boolean algebra.

Algebras $\Phi(X)$ and $Hal^X_\Theta(H)$ are defined in such a way that for every $H \in \Theta$ we have the value homomorphism $Val^X_H : \Phi(X) \to Hal^X_\Theta(H)$. It takes equalities to equalities, i.e., $Val^X_H(w \equiv w') = [w \equiv w']_H$. One can show that for any $u \in \Phi(X)$ the set of all points satisfying the formula u is the set $Val^X_H(u)$. In particular, the point $\mu : W(X) \to H$ satisfies the formula $u = s_* v$, $v \in W(Y)$, $s : W(Y) \to W(X)$, if and only if the point $\tilde{s}(\mu) = \mu s$ satisfies the formula v.

Let us pass to the category $Hal_\Theta(H)$. Objects of this category are algebras $Hal^X_\Theta(H)$. Morphisms have the form

$$\widetilde{s} : Hal^X_\Theta(H) \to Hal^Y_\Theta(H),$$

where $s : W(Y) \to W(X)$. Here, for $A \subset \mathrm{Hom}(W(X), H)$ we have $\widetilde{s}(A) = B \subset \mathrm{Hom}(W(Y), H)$, where B is the set of all $\widetilde{s}(\mu) = \mu s$, $\mu \in A$.

Let us redenote the category $Hal_\Theta(H)$ by $\overleftarrow{Hal_\Theta}(H)$ for some reason. Then the category $\overrightarrow{Hal_\Theta}(H)$ has the same objects as $\overleftarrow{Hal_\Theta}(H)$ and opposite morphisms s_* defined by $s_* = \widetilde{s}^{-1} : Hal^Y_\Theta(H) \to Hal^X_\Theta(H)$. More precisely, if $B = Val_H^Y(T_2) \subset Hal_\Theta^Y(H)$, then define $s_*(B) = \widetilde{s}^{-1}(B) = A \subset Hal_\Theta^X(H)$ as the set of points μ such that $\widetilde{s}(\mu)$ lies in B. According to the definition, s_* acts on equalities by the rule $s_*[w \equiv w']_H = [sw \equiv sw']_H$ and preserves Boolean operations. We come up with the diagram

$$\begin{array}{ccc} \Phi(Y) & \xrightarrow{s_*} & \Phi(X) \\ {\scriptstyle Val_H^Y}\downarrow & & \downarrow{\scriptstyle Val_H^X} \\ Hal_\Theta^Y(H) & \xrightarrow{s_*=\widetilde{s}^{-1}} & Hal_\Theta^X(H). \end{array} \tag{3.1}$$

Commutativity of 3.1 means that if $v \in \Phi(Y)$, $u = s_*v \in \Phi(X)$, $A = Val_H^X(u)$, $B = Val_H^Y(v)$, then $Val_H^X(s_*v) = s_*Val_H^Y(v)$. The latter equality represents the fact that $Val_H : \widetilde{\Phi}_\Theta \to \overrightarrow{Hal_\Theta}$ is the homomorphism of multy-sorted Halmos algebras. In fact, we have also anti-homomorphism $\widetilde{\Phi}_\Theta \to \overleftarrow{Hal_\Theta}$.

Let us define the Galois correspondence between sets of formulas T in $\Phi(X)$ and sets of points A in $\mathrm{Hom}(W(X), H)$. For each point $\mu : W(X) \to H$ denote by $LKer(\mu)$ the logical kernel of point μ. It consists of the formulas $u \in \Phi(X)$ such that $\mu \in Val_H^X(u)$. One can say that μ satisfies every formula from $LKer(\mu)$. Logical kernel of a point is always a Boolean ultrafilter in $\Phi(X)$ which is invariant with respect to existential quantifier and is not invariant with respect to universal quantifier.

Let now T be a set of formulas in $\Phi(X)$. Determine the set $A = T_H^L$ in $\mathrm{Hom}(W(X), H)$ by the rule: a point $\mu : W(X) \to H$ is contained in A if and only if $T \subset LKer(\mu)$. In other words, $A = \bigcap_{u\in T} Val_H^X(u)$. Every set A of such kind is called definable.

Let, further, $A \subset \mathrm{Hom}(W(X), H)$ be given. We set: $T = A_H^L = \bigcap_{\mu\in A} LKer(\mu)$. In other words, $u \in T$ if and only if $A \subset Val_H^X(u)$. Here T is a (Boolean) filter in the algebra $\Phi(X)$, and we have a Boolean algebra $\Phi(X)/T$. A filter T of such kind is called H-closed.

Let us pass now to the types of points in Model Theory in an arbitrary Θ.

4. Definitions of types

The notion of a type is one of the key notions of Model Theory. In what follows we will distinguish between model theoretical types (MT-types) and logically geometric types (LG-types). Both kinds of types are oriented towards some algebra $H \in \Theta$, where Θ is a fixed variety of algebras.

Generally speaking, a type of a point $\mu : W(X) \to H$ is a logical characteristic of the point μ. Model-theoretical idea of a type and its definition is described in many sources, see, in particular, [1], [3], [5]. We consider this idea from the perspective of algebraic logic (cf., for example, [12]) and give all the definitions in the corresponding terms.

Proceed from the algebra of formulas $\Phi(X^0)$, where X^0 is an infinite set of variables. It is obtained from the algebra of pure first-order formulas over equalities $w \equiv w'$, $w, w' \in W(X^0)$ by Lindenbaum-Tarski algebraization approach (see, for example, [6], [7]). $\Phi(X^0)$ is an X^0-extended Boolean algebra, which means that $\Phi(X^0)$ is a Boolean algebra with quantifiers $\exists x$, $x \in X^0$ and equalities $w \equiv w'$, where $w, w' \in W(X^0)$. Here, $W(X^0)$ is the free over X^0 algebra in Θ. All these equalities generate the algebra $\Phi(X^0)$. Besides, the semigroup $End(W(X^0))$ acts on the Boolean algebra $\Phi(X^0)$ and we can speak of a *polyadic algebra* $\Phi(X^0)$ (see [2]). However, the elements $s \in End(W(X^0))$ and the corresponding s_* are not included in the signature of the algebra $\Phi(X^0)$.

Since $\Phi(X^0)$ is a one-sorted algebra, one can speak, as usual, about free and bound occurrences of the variables in the formulas $u \in \Phi(X^0)$.

Remark 4.1. One can replace the variety Θ by the variety Θ^H, where H is a fixed algebra of constants (see [7] for details). Then we can assume that elements of $\Phi(X)$ and $\Phi(X^0)$ may contain constants from H.

Define further X-special formulas in $\Phi(X^0)$, $X = \{x_1, \dots, x_n\}$. Take $X^0 \backslash X = Y^0$.

Definition 4.2. A formula $u \in \Phi(X^0)$ is X*-special* if each of its free variables occurs in X and each bound variable belongs to Y^0.

A formula $u \in \Phi(X^0)$ is *closed* if it does not have free variables. Only finite number of variables occur in each formula.

Denoting an X-special formula u as $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$ we solely mean that the set X consists of variables x_i, $i = 1, \dots n$, and those of them who occur in u, occur freely.

Definition 4.3. Let H be an algebra from Θ. An X-type (over H) is a set of X-special formulas in $\Phi(X^0)$, consistent with the elementary theory of the algebra H.

We call such type an X-MT-type (Model Theoretic type) over H. An X-MT-type is called *complete* if it is maximal with respect to inclusion. Any complete X-MT-type is a Boolean ultrafilter in the algebra $\Phi(X^0)$.

Hence, for every X-special formula $u \in \Phi(X^0)$, either u or its negation belongs to a complete type.

Definition 4.4. An X-LG-type (Logically Geometric type) (over H) is a Boolean ultrafilter in the corresponding $\Phi(X)$, which contains the elementary theory $Th^X(H)$.

So, any X-MT-type lies in the one-sorted algebra $\Phi(X^0)$. Any X-LG-type lies in the domain $\Phi(X)$ of the multi-sorted algebra $\widetilde{\Phi}$.

We denote the MT-type of a point $\mu : W(X) \to H$ by $Tp^H(\mu)$, while the LG-type of the same point is, by definition, its logical kernel $LKer(\mu)$.

Definition 4.5. Let a point $\mu : W(X) \to H$, with $a_i = \mu(x_i)$, be given. An X-special formula $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$ belongs to the type $Tp^H(\mu)$ if the formula $v = u(a_1, \dots, a_n; y_1, \dots, y_m)$ is satisfied in the algebra H.

The type $Tp^H(\mu)$ consists of all X-special formulas satisfied on μ. It is a complete X-MT-type over H.

By definition, the formula $v = u(a_1, \dots, a_n; y_1, \dots, y_m)$ is closed. Thus, if it is satisfied on a point, then it is satisfied on the whole affine space $\mathrm{Hom}(W(X), H)$.

Note also that in our definition of an X-MT-type the set of free variables in the formula u is not necessarily the whole $X = \{x_1, \dots, x_n\}$ and can be a part of it. In particular, the set of free variables can be empty. In this case the formula u belongs to the type if it is satisfied in H.

Beforehand, the algebra $\widetilde{\Phi}$ was built basing on the set Γ of all finite subsets of the set X^0. In fact, one can take the system $\Gamma^* = \Gamma \bigcup X^0$ instead of Γ and construct the corresponding multi-sorted algebra. Then, to each homomorphism $s : W(X^0) \to W(X)$ it corresponds a morphism $s_* : \Phi(X^0) \to \Phi(X)$ and, vice versa, $s : W(X) \to W(X^0)$ induces $s_* : \Phi(X) \to \Phi(X^0)$. In this setting the extended Boolean algebra $Hal_\Theta^{X^0}(H)$ and the homomorphism $Val_H^{X^0} : \Phi(X^0) \to Hal_\Theta^{X^0}(H)$ are defined in the usual way. A point $\mu : W(X^0) \to H$ satisfies $u \in \Phi(X^0)$ if $\mu \in Val_H^{X^0}(u)$.

Remark 4.6. One should underline several distinctions between one-sorted and multi-sorted cases. If we consider a sublagebra $\Phi(X) \subset \Phi(X^0)$, then we mean an identical embedding. In the multi-sorted case $\Phi(X)$ can be mapped in $\Phi(X^0)$ by quite different ways. A particular map is determined by a choice of a morphism s_*. This is why we will distinguish below embeddings by special morphisms s_*.

Besides that, $\Phi(X^0)$, treated as a one-sorted algebra, has the signature of a polyadic algebra. On the other hand, $\Phi(X^0)$, treated as a domain of $\widetilde{\Phi}$, has the signature of a multi-sorted Halmos algebra. This means that the elements of the form $s_*(w \equiv w')$ are present in the second case while they are not present in the first one.

5. Transition from $Tp^H(\mu)$ to $LKer(\mu)$.

We would like to relate the X-MT-type of a point to its LG-type.

Definition 5.1. Given an infinite set X^0 and a finite subset $X = \{x_1, \dots, x_n\}$, a homomorphism $s : W(X^0) \to W(X)$ is called special if $s(x) = x$ for each $x \in X$, i.e., s is identical on the set X. Homomorphism s gives rise to the morphism of extended Boolean algebras

$$s_* : \Phi(X^0) \to \Phi(X).$$

Theorem 5.2 ([13], cf., [15]). *For each special homomorphism s, each special formula $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$ in $\Phi(X^0)$ and every point $\mu : W(X) \to H$, we have $u \in Tp^H(\mu)$ if and only if $s_*u \in LKer(\mu)$. Here, u is considered in one-sorted algebra $\Phi(X^0)$, while s_*u lies in the domain $\Phi(X)$ of the multi-sorted $\widetilde{\Phi} = (\Phi(X),\ X \in \Gamma^*)$.*

This theorem can be viewed as a criterion which relates one-sorted and multi-sorted cases.

6. Correspondence between $u \in \Phi(X)$ and $\tilde{u} \in \Phi(X^0)$

Definition 6.1. A formula $u \in \Phi(X)$ is called X-*correct*, if there exists an X-special formula $\tilde{u}$ in $\Phi(X^0)$ such that for every point $\mu : W(X) \to H$ we have $u \in LKer(\mu)$ if and only if $\tilde{u} \in Tp^H(\mu)$.

Now, we shall formulate the principal theorem. This theorem is implicit in [15]. Here we need to formulate it explicitly and provide a proof. We will first notice that all equalities are correct and then show that the system of all correct formulas over all sorts X is closed in the signature of algebra $\widetilde{\Phi}$.

Theorem 6.2 (cf., [15]). *For every $X = \{x_1, \dots, x_n\}$, every formula $u \in \Phi(X)$ is correct.*

Proof. First of all, each equality $w \equiv w'$, $w, w' \in W(X)$ is a correct formula. Indeed, define $\widetilde{(w \equiv w')}$ by $\widetilde{(w \equiv w')} = (w \equiv w')$.

Take two correct formulas u and v, both from $\Phi(X)$. Show that $u \wedge v$, $u \vee v$ and $\neg u$ are also correct. We have $\widetilde{u}$ and $\widetilde{v}$. Define

$$\widetilde{u \wedge v} = \widetilde{u} \wedge \widetilde{v},$$
$$\widetilde{u \vee v} = \widetilde{u} \vee \widetilde{v},$$
$$\widetilde{\neg u} = \neg \widetilde{u}.$$

By definition, we have $u \in LKer(\mu)$ if and only if $\widetilde{u} \in T_p^H(\mu)$ for every point $\mu : W(X) \to H$. The same is true with respect to v and $\neg u$. Let $u \vee v \in LKer(\mu)$ and, say, $u \in LKer(\mu)$. Then $\widetilde{u} \in Tp^H(\mu)$, and, hence, $\widetilde{u} \vee \widetilde{v} = \widetilde{u \vee v} \in Tp^H(\mu)$. Conversely, let $\widetilde{u \vee v} = \widetilde{u} \vee \widetilde{v} \in Tp^H(\mu)$. Suppose that $\widetilde{u} \in Tp^H(\mu)$. Then $u \in LKer(\mu)$, that is, $u \vee v \in LKer(\mu)$. The similar proofs work for the correctness of the formulas $u \wedge v$ and $\neg u$. In the latter case one should use the completeness property of a type: $\neg u \in Tp^H(\mu)$ if and only if $u \notin Tp^H(\mu)$.

Our next aim is to check that if the formula $u \in \Phi(X)$ is correct, then the formula $\exists x u \in \Phi(X)$ is also correct.

Beforehand, note that it is hard to define free and bounded variables in the algebra $\Phi(X)$. This is because of the multi-sorted nature of $\Phi(X)$ and presence of the formulas including operations of the type s_* in it. So, the syntactical definition of $\exists x u \in \Phi(X)$ is a sort of problem and we will proceed from the semantical definition of this formula.

Recall that a point $\mu : W(X) \to H$ satisfies the formula $\exists x u \in \Phi(X)$ if and only if there exists a point $\nu : W(X) \to H$ such that $u \in LKer(\nu)$ and μ coincides with ν for every variable $x' \neq x$, $x' \in X$.

Indeed, a point $\mu : W(X) \to H$ satisfies $\exists x u \in \Phi(X)$ if $\mu \in Val_H^X(\exists x u) = \exists x(Val_H^X(u))$ (see Section 3). Denote the set $Val_H^X(u)$ in $Hal_\Theta^X(H) = Bool(W(X), H)$ by A. Then μ belongs to $\exists x A$. Using the definition of existential quantifiers in $Hal_\Theta^X(H)$ (Section 3) and the fact that $u \in LKer(\nu)$ if and only if $\nu \in Val_H^X(u)$, we arrive to the definition above.

Since u is correct, there exists an X-special formula $\widetilde{u} \in \Phi(X^0)$,

$$\widetilde{u} = \widetilde{u}(x_1, \ldots, x_n, y_1, \ldots, y_m),\ x_i \in X,\ y_i \in Y^0 = (X^0 \setminus X),$$

such that $\widetilde{u} \in Tp^H(\mu)$ if and only if $u \in LKer(\mu)$, where $\mu : W(X) \to H$.

Define

$$\widetilde{\exists x u} = \exists x \widetilde{u}.$$

The formula $\exists x\widetilde{u}$ is not X-special since x is bound (we assume that x coincides with one of x_i, say, x_n). Take a variable $y \in Y^0$, such that y is different from each $x_i \in X$, $i = 1, \dots, n$, and y_j, $j = 1, \dots, m$.

Define $\exists y\widetilde{u}_y$ to be a formula which coincides with $\exists x\widetilde{u}$ modulo replacement of x by y. So, $\exists y\widetilde{u}_y = v(x_1, \dots, x_{n-1}, y, y_1, \dots, y_m)$ has one less free variable and one more bound variable than $\exists x\widetilde{u}$.

Consider endomorphism s of $W(X^0)$ taking $s(x)$ to y and leaving all other variables from X^0 unchanged. Let s_* be the corresponding automorphism of the one-sorted Halmos algebra $\Phi(X^0)$. Then $s_*(\exists x\widetilde{u}) = \exists s_*(x)s_*(\widetilde{u}) = \exists y\widetilde{u}_y$.

Redefine

$$\widetilde{\exists xu} = \exists y\widetilde{u}_y.$$

Thus, in order to check that $\exists xu$ is correct, we need to verify that for every $\mu : W(X) \to H$ the formula $\exists xu$ lies in $LKer(\mu)$ if and only if $\exists y\widetilde{u}_y \in Tp^H(\mu)$.

Let $\exists xu$ lie in $LKer(\mu)$. Thus, there exists a point $\nu : W(X) \to H$ such that $u \in LKer(\nu)$ and μ coincides with ν for every variable $x' \neq x$, $x' \in X$.

Consider $X_y = \{x_1, \dots, x_{n-1}, y\}$. We have points $\mu : W(X) \to H$, $\mu' : X_y \to H$ where $\mu'(x_i) = \mu(x_i) = a_i$, and $\mu'(y)$ is an arbitrary element b in H. We have also $\nu : W(X) \to H$ and $\nu' : X_y \to H$, where $\nu'(x_i) = \nu(x_i)$, and $\nu'(y) = \nu(x_n)$. So, ν and ν' have the same image. Denote it by $(a_1, a_2, \dots, a_{n-1}, a_n)$, $a_i \in H$, i.e., $\nu'(y) = a_n$.

Take

$$\widetilde{u}_y = \widetilde{u}(x_1, \dots, x_{n-1}, y, y_1, \dots, y_m).$$

Since the formula $\widetilde{u}(a_1, \dots, a_{n-1}, b, y_1, \dots, y_m)$ is closed for any b, then either it is satisfied on any point μ', or no one of μ' satisfies this formula. We can take $b = a_n$, that is, $\mu' = \nu'$. Since ν and ν' have the same image and u is correct, the point ν' satisfies $\widetilde{u}_y$. Then ν' satisfies $\exists y\widetilde{u}_y$. Hence, $\exists y\widetilde{u}(x_1, \dots, x_{n-1}, y, y_1, \dots, y_m)$ is satisfied on any μ' regardless of the choice of b. This means that $\exists y\widetilde{u}_y \in T_p^H(\mu')$ for every μ'. We can take μ' to be μ. Then $\widetilde{\exists xu} \in Tp^H(\mu)$.

Conversely, let $\widetilde{\exists xu} \in Tp^H(\mu)$. Take a point $\nu : W(X) \to H$ such that $\nu(x_i) = \mu(x_i)$, $i = 1, \dots, n-1$, $\nu(x_n) = \mu(y)$. We have $\widetilde{u} \in T_p^H(\nu)$. Since $\widetilde{u}$ is correct, then u lies in $LKer(\nu)$. The points μ and ν coincide on all x_i, $i \neq n$. Thus, $\exists xu$ belongs to $LKer(\mu)$.

It remains to check that the operation s_* respects correctness of formulas. Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$ and a morphism

$s : W(Y) \to W(X)$ be given. Take the corresponding $s_* : \Phi(Y) \to \Phi(X)$. Given $v \in \Phi(Y)$ consider $u = s_* v$ in $\Phi(X)$. We shall show that if v is Y-correct then u is X-correct.

First of all, take $\mu : W(X) \to H$, $\nu : W(Y) \to H$ such that $\mu s = \nu$. Then $u \in LKer(\mu)$ if and only if $v \in LKer(\nu)$.

Indeed, $u = s_* v \in LKer(\mu)$ means that $\mu \in Val_H^X(s_* v) = s_* Val_H^Y(v)$ and, thus, $\mu s \in Val_H^Y(v)$. Hence, for $\nu = \mu s$ we have $v \in LKer(\nu)$. Conversely, let $v \in LKer(\nu)$ and $\mu s = \nu \in Val_H^Y(v)$. We have $\mu \in s_* Val_H^Y(v) = Val_H^X(s_* v) = Val_H^X(u)$ and $u \in LKer(\mu)$.

Note that morphism $s_* : \Phi(Y) \to \Phi(X)$ is a homomorphism of Boolean algebras. Suppose that $v \in \Phi(Y)$ is correct. This means that $\widetilde{v}$ is chosen in such a way that $v \in LKer(\nu)$ if and only if $\widetilde{v} \in Tp(\nu)$.

We have

$$\widetilde{v} = \widetilde{v}(y_1, \dots, y_m, z_1, \dots, z_t),$$

where all z_i are bound and belong to $Z = \{z_1, \dots, z_t\}$. All free variables in $\widetilde{v}$ belong to Y (it is assumed that not necessarily all variables from Y occur in $\widetilde{v}$). In this sense $\widetilde{v}$ is Y-special. Since $v \in \Phi(Y)$ is correct then $v \in LKer(\nu)$ if and only if $\widetilde{v} \in Tp(\nu)$.

We will define the formula $\widetilde{u}$ and show that in our situation $\widetilde{u} \in Tp^H(\mu)$ if and only if $\widetilde{v} \in Tp^H(\nu)$.

Consider $Z' = \{z'_1, \dots, z'_t\}$, where all z'_i do not belong to X. Take the free algebras $W(X \cup Z')$ and $W(Y \cup Z)$. Define homomorphism $s' : W(Y \cup Z) \to W(X \cup Z')$ extending $s : W(Y) \to W(X)$ by $s'(z_i) = z'_i$ (we are able to do that because of the axioms of Halmos algebras, see, for instance, [7]). Take $Z^0 = \{z_0\}$, where the variable z_0 lies outside X, Y, Z, Z'. The commutative diagram of homomorphisms takes place:

$$\begin{array}{ccc} W(Y \cup Z) & \xrightarrow{s'} & W(X \cup Z') \\ {\scriptstyle s^1}\downarrow & & \downarrow{\scriptstyle s^2} \\ W(Y \cup Z^0) & \xrightarrow{s} & W(X \cup Z^0). \end{array}$$

Here s^1 and s^2 are special homomorphisms which act identically on Y and X, respectively, such that they send all variables from Z and Z' to the same z_0. The corresponding commutative diagram of morphisms of algebras of formulas is as follows:

$$\begin{array}{ccc} \Phi(Y \cup Z) & \xrightarrow{s'_*} & \Phi(X \cup Z') \\ {\scriptstyle s^1_*}\downarrow & & \downarrow{\scriptstyle s^2_*} \\ \Phi(Y \cup Z^0) & \xrightarrow{s_*} & \Phi(X \cup Z^0). \end{array}$$

This diagram is commutative due to the fact that the product of morphisms of algebras of formulas corresponds to the product of homomorphisms of free algebras. Apply the diagram to Y-special formula $\widetilde{v}$ which belongs to the algebra $\Phi(Y \cup Z)$. Then, $s^2_* {s'}_* \widetilde{v} = s_* s^1_* \widetilde{v}$. Assume that $\widetilde{u} = {s'}_* \widetilde{v}$. Here, $\widetilde{u}$ is an X-special formula, contained in the algebra $\Phi(X \cup Z')$. We need to prove that for any point $\mu : W(X) \to H$ the inclusion $\widetilde{u} \in Tp^H(\mu)$ holds if and only if $u \in LKer(\mu)$.

Let $u \in LKer(\mu)$. We use the criterion from Theorem 5.2: $\widetilde{u} \in Tp^H(\mu)$ if and only if $s^2_* \widetilde{u} \in LKer(\mu)$. Let us prove the latter inclusion. The similar criterion is valid for the formula $\widetilde{v}$. Since the formula v is correct, then $\widetilde{v} \in Tp^H(\nu)$, where $\nu = \mu s$. Hence, $s^1_* \widetilde{v} \in LKer(\nu)$, which means that the point ν belongs to the set $Val^Y_H(s^1_* \widetilde{v})$. Since $\nu = \mu s$, then $\mu \in Val^X_H(s_* s^1_* \widetilde{v}) = Val^X_H(s^2_* {s'}_* \widetilde{v}) = Val^X_H(s^2_* \widetilde{u})$. This leads to the inclusion $s^2_* \widetilde{u} \in LKer(\mu)$, which gives $\widetilde{u} \in Tp^H(\mu)$.

The same reasoning in the opposite direction shows that the inclusion $\widetilde{u} \in Tp^H(\mu)$ is equivalent to that of $\widetilde{v} \in Tp^H(\nu)$.

It is worth to recall that we started from the fact $u \in LKer(\mu)$ if and only if $v \in LKer(\nu)$. But $v \in LKer(\nu)$ because of the correctness of the formula v. Thus, $u \in LKer(\mu)$. Hence, the transition from u to $\widetilde{u}$ guarantees the correctness of the formula u.

Therefore, the set of all correct X-formulas, for various X, respects all operations of the multi-sorted algebra $\widetilde{\Phi}$. Since $\widetilde{\Phi}$ is generated by equalities, which are correct, the subalgebra of all correct formulas in $\widetilde{\Phi}$ coincides with $\widetilde{\Phi}$. Thus, every $u \in \widetilde{\Phi}(X)$ for every X is correct. □

7. *LG*- and *MT*-isotypeness of algebras

The following important theorem (see [15]) illuminates the notion of isotypeness of algebras.

Theorem 7.1 ([15]). *Let the points $\mu : W(X) \to H_1$ and $\nu : W(X) \to H_2$ be given. Then*

$$Tp^{H_1}(\mu) = Tp^{H_2}(\nu)$$

if and only if

$$LKer(\mu) = LKer(\nu).$$

Proof. We will use Theorem 6.2. Let the points $\mu : W(X) \to H_1$ and $\nu : W(X) \to H_2$ be given and let $Tp^{H_1}(\mu) = Tp^{H_2}(\nu)$. Take $u \in LKer(\mu)$. Then $\widetilde{u} \in Tp^{H_1}(\mu)$ and, thus, $\widetilde{u} \in Tp^{H_2}(\nu)$. Hence, $u \in LKer(\nu)$. The same is true in the opposite direction.

Let, conversely, $LKer(\mu) = LKer(\nu)$. Take an arbitrary X-special formula u in $Tp^{H_1}(\mu)$. Take a special homomorphism $s : W(X^0) \to W(X)$. The morphism $s_* : \Phi(X^0) \to \Phi(X)$ corresponds to s. Then, using Theorem 5.2, the formula $u \in Tp^H(\mu)$ is valid if and only if $s_*u \in LKer(\mu)$. Therefore, $s_*u \in LKer(\nu)$. Then, $u \in Tp^H(\nu)$. □

Definition 7.2. Given X, denote by $S^X(H)$ the set of all MT-types over an algebra H. Algebras H_1 and H_2 are called MT-*isotypic* if $S^X(H_1) = S^X(H_2)$ for any $X \in \Gamma$.

Definition 7.3. Two algebras H_1 and H_2 are called LG-isotypic if for every X and every point $\mu : W(X) \to H_1$ there exists a point $\nu : W(X) \to H_2$ such that $LKer(\mu) = LKer(\nu)$ and, conversely, for every point $\nu : W(X) \to H_2$ there exists a point $\mu : W(X) \to H_1$ such that $LKer(\nu) = LKer(\mu)$.

If we denote by $L^X(H)$ the set of all MT-types over an algebra H, then Definition 7.3 means that two algebras H_1 and H_2 are LG-isotypic if and only if $L^X(H_1) = L^X(H_2)$ for any $X \in \Gamma$.

Corollary 7.4. *Algebras H_1 and H_2 in the variety Θ are MT-isotypic if and only if they are LG-isotypic.*

So, it doesn't matter which type (LG-type or MT-type) is used in the definition of isotypeness.

Recall that (see, for example, [8], [9]),

Definition 7.5. Algebras H_1 and H_2 are LG-equivalent, if for every X and every set of formulas T in $\Phi(X)$ holds $T^{LL}_{H_1} = T^{LL}_{H_2}$.

Then,

Theorem 7.6 ([15])**.** *Algebras H_1 and H_2 are LG-equivalent if and only if they are LG-isotypic.*

Corollary 7.7. *Algebras H_1 and H_2 in the variety Θ are isotypic if and only if they are LG-equivalent.*

If algebras H_1 and H_2 are isotypic then they are locally isomorphic. This means that if A is a finitely generated subalgebra in H, then there exists a subalgebra B in H_2 which is isomorphic to A. The same is true in the direction from H_2 to H_1.

On the other hand, local isomorphism of H_1 and H_2 does not imply their isotypeness: the groups F_n and F_m, $m, n > 1$ are locally isomorphic, but they are isotypic only for $n = m$.

Isotypeness implies elementary equivalence of algebras, but the same example with F_n and F_m shows that the converse is false.

Recall here the following problems (see [13])

Problem 1. Suppose that H_1 and H_2 are two finitely generated isotypic algebras. Are they always isomorphic?

In particular,

Problem 2. Let Θ be the variety of commutative and associative algebras over a field. Let an algebra $H \in \Theta$ be isotypic to an n-generated polynomial algebra. Are they isomorphic?

8. *MT*-saturated and *LG*-saturated algebras

Definition 8.1. An algebra $H \in \Theta$ is called *LG-saturated* if for every $X \in \Gamma$ each ultrafilter T in $\Phi(X)$ containing $Th^X(h)$ has the form $T = LKer(\mu)$ for some $u : W(X) \to H$.

The standard notion of saturation defined in Model Theory will be called MT-saturation. MT-saturation of an algebra H means that for any X-type T there is a point $\mu : W(X) \to H$ such that $T \subset Tp^H(\mu)$.

Theorem 8.2 ([13])**.** *If algebra H is LG-saturated, then H is MT-saturated.*

We do not know whether MT-saturation implies LG-saturation.

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Contact information

Boris Plotkin
Institute of Mathematics, Hebrew University, 91904, Jerusalem, Israel
E-Mail(s): borisov@math.huji.ac.il

Eugene Plotkin, G. Zhitomirski
Department of Mathematics, Bar Ilan University, Ramat Gan, Israel
E-Mail(s): plotkin@macs.biu.ac.il, zhitomg@012.net.il

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