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RESEARCH ARTICLE

# On one class of algebras

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Abstract. In this paper a $g$-dimonoid which is isomorphic to the free $g$-dimonoid is given and a free $n$-nilpotent $g$-dimonoid is constructed. We also present the least $n$-nilpotent congruence on a free $g$-dimonoid and give numerous examples of $g$-dimonoids.

## 1. Introduction

Recall that a dialgebra (dimonoid) [1] is a vector space (set) with two binary operations $\dashv$ and $\vdash$ satisfying the axioms $(x \dashv y) \dashv z = x \dashv (y \dashv z)$ $(D1)$, $(x \dashv y) \dashv z = x \dashv (y \vdash z)$ $(D2)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)$ $(D3)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)$ $(D4)$, $(x \vdash y) \vdash z = x \vdash (y \vdash z)$ $(D5)$. In our time dimonoids are standard tool in the theory of Leibniz algebras. So, for example, free dimonoids were used for constructing free dialgebras and for studying a cohomology of dialgebras. There exist papers devoted to studying structural properties of dimonoids (see, e.g., [2 – 4]). If in the definition of a dialgebra delete the axioms $(D1)$, $(D3)$, $(D5)$, then we obtain a 0-dialgebra which was considered in [5]. Algebras obtained from the definition of a dimonoid by deleting the axioms $(D2)$ and $(D4)$ were considered in [6]. In the last paper the free object in the corresponding variety was constructed. Observe that dimonoids are closely connected with restrictive bisemigroups considered by B.M. Schein [7]. In [8–11] the notions of interassociativity, respectively,

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strong interassociativity, related semigroups and doppelalgebras which are naturally connected with dimonoids were considered. Another reason for interest in dimonoids is their connection with $n$-tuple semigroups which were used in [12] for studying properties of $n$-tuple algebras of associative type. If in the definition of a dimonoid delete the axiom $(D3)$, then we obtain an algebraic system which is called a $g$-dimonoid (see [13, 14]).

In this paper $g$-dimonoids are studied. In Section 2 we give numerous examples of $g$-dimonoids. In Section 3 we suggest a new concrete representation of a free $g$-dimonoid using the construction of a free $g$-dimonoid from [14]. The main result of this section was announced in [13]. In Section 4 the construction of a free $n$-nilpotent $g$-dimonoid is given. Moreover, here we characterize the least $n$-nilpotent congruence on a free $g$-dimonoid.

## 2. Examples of $g$-dimonoids

In this section we give different examples of $g$-dimonoids.

a) Obviously, any dimonoid is a $g$-dimonoid.

b) Let $X$ be an arbitrary nonempty set, $|X| > 1$ and let $X^*$ be the set of all finite nonempty words in the alphabet $X$. Denoting the first (respectively, the last) letter of a word $w \in X^*$ by $w^{(0)}$ (respectively, by $w^{(1)}$), define operations $\dashv$ and $\vdash$ on $X^*$ by $w \dashv u = w^{(0)}, \quad w \vdash u = u^{(1)}$ for all $w, u \in X^*$. From the proof of Theorem 2 [2] it follows that $(X^*, \dashv, \vdash)$ is a $g$-dimonoid but not a dimonoid.

c) Let $\{D_i\}_{i \in I}$ be a family of arbitrary $g$-dimonoids $D_i$, $i \in I$, and let $\overline{\prod}_{i \in I} D_i$ be a set of all functions $f : I \to \bigcup_{i \in I} D_i$ such that $if \in D_i$ for any $i \in I$.

It is easy to prove the following lemma.

**Lemma 1.** $\overline{\prod}_{i \in I} D_i$ *with multiplications defined by*

$$i(f_1 \dashv f_2) = if_1 \dashv if_2, \ \ i(f_1 \vdash f_2) = if_1 \vdash if_2, \tag{1}$$

*where* $i \in I, f_1, f_2 \in \overline{\prod}_{i \in I} D_i$, *is a $g$-dimonoid.*

The obtained algebra is called the Cartesian product of $g$-dimonoids $D_i$, $i \in I$. If $I$ is finite, then the Cartesian product and the direct product coincide. The Cartesian product of a finite number of $g$-dimonoids $D_1$, $D_2, ..., D_n$ is denoted by $D_1 \times D_2 \times ... \times D_n$. In particular, the Cartesian power of a $g$-dimonoid can be defined as follows. Let $V$ be an arbitrary $g$-dimonoid and $X$ be any nonempty set. Denote by $Map(X; V)$ the set of all maps $X \to V$. Define operations $\dashv$ and $\vdash$ on $Map(X;V)$ by (1) for all $f_1, f_2 \in Map(X;V)$ and $i \in X$. Then $(Map(X;V), \dashv, \vdash)$ is a $g$-dimonoid which is called the Cartesian power of $V$.

d) As usual, $\mathbb{N}$ denotes the set of all positive integers.

Let $F[X]$ be the free semigroup in an alphabet $X$. We denote the length of a word $w \in F[X]$ by $l(w)$. Fix $n \in \mathbb{N}$ and define operations $\dashv$ and $\vdash$ on $F[X] \times \mathbb{N}$ by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, n),$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, l(w_1) + m_2)$$

for all $(w_1, m_1),\ (w_2, m_2) \in F[X] \times \mathbb{N}$. Denote the algebra $(F[X] \times \mathbb{N}, \dashv, \vdash)$ by $X\mathbb{N}_n$.

**Lemma 2.** *The algebra $X\mathbb{N}_n$ is a g-dimonoid but not a dimonoid.*

*Proof.* One can directly verify that $X\mathbb{N}_n$ is a $g$-dimonoid. Show that it is not a dimonoid. For all $(w_1, m_1),\ (w_2, m_2), (w_3, m_3) \in X\mathbb{N}_n$ obtain

$$((w_1, m_1) \vdash (w_2, m_2)) \dashv (w_3, m_3) = (w_1 w_2, l(w_1) + m_2) \dashv (w_3, m_3) =$$

$$= (w_1 w_2 w_3, n) \neq (w_1 w_2 w_3, l(w_1) + n) = (w_1, m_1) \vdash (w_2 w_3, n) =$$

$$= (w_1, m_1) \vdash ((w_2, m_2) \dashv (w_3, m_3)).$$ □

e) Let $S$ be an arbitrary semigroup, $a, b \in S$. By $E_S$ denote the set of all idempotents of $S$. Define operations $\dashv$ and $\vdash$ on $S$ by

$$x \dashv y = ax, \quad x \vdash y = by$$

for all $x, y \in S$. Denote the algebra $(S, \dashv, \vdash)$ by $S(a, b)$.

**Lemma 3.** *Let $S$ be an arbitrary right cancellative semigroup, $a, b \in E_S$.*

*(i) If $a$ and $b$ are non-commuting, then $S(a, b)$ is a g-dimonoid but not a dimonoid.*

*(ii) If $a$ and $b$ are commuting, then $S(a, b)$ is a dimonoid.*

*Proof.* (i) The axioms $(D1)$, $(D2)$, $(D4)$, $(D5)$ are checked directly. Besides,

$$(x \vdash y) \dashv z = by \dashv z = aby, \quad x \vdash (y \dashv z) = x \vdash ay = bay$$

for all $x, y, z \in S$. Suppose that $aby = bay$. Then, using the right cancellability, obtain $ab = ba$. Thus, we arrive at a contradiction, i.e., the assumption that $aby = bay$ does not hold. Consequently, $S(a,b)$ is not a dimonoid.

(ii) If $a$ and $b$ are commuting, then, obviously, all axioms of a dimonoid hold. □

f) Let $S$ be an arbitrary semigroup, $a, b \in S$. Define operations $\dashv$ and $\vdash$ on $S$ by

$$x \dashv y = xa, \quad x \vdash y = yb$$

for all $x, y \in S$. Denote the algebra $(S, \dashv, \vdash)$ by $S[a,b]$.

Similarly to Lemma 3, the following lemma can be proved.

**Lemma 4.** *Let $S$ be an arbitrary left cancellative semigroup, $a, b \in E_S$.*

*(i) If $a$ and $b$ are non-commuting, then $S[a,b]$ is a g-dimonoid but not a dimonoid.*

*(ii) If $a$ and $b$ are commuting, then $S[a,b]$ is a dimonoid.*

g) Let $S$ be an arbitrary semigroup, $a, b \in S$. Define operations $\dashv$ and $\vdash$ on $S$ by

$$x \dashv y = axb, \quad x \vdash y = ayb$$

for all $x, y \in S$. Denote the algebra $(S, \dashv, \vdash)$ by $S(a,b]$.

The following lemma could be proved immediately.

**Lemma 5.** *If $a, b \in E_S$, then $S(a,b]$ is a dimonoid.*

h) Let $Y$ be an arbitrary nonempty set, $S = S_Y$ be some monoid defined on the set of finite words in the alphabet $Y$ and $\theta \in S$ be an empty word which is a unit of $S$. Denote the operation on $S$ by $*$ and the length of a word $w \in S$ by $l(w)$. By definition $l(\theta) = 0$, $u^0 = \theta$ for all $u \in S$. Fix elements $a,\ b \in Y$, $k \in \mathbb{N} \cup \{0\}$ and define operations $\dashv$ and $\vdash$ on $S$, assuming

$$u_1 \dashv u_2 = u_1 * a^{l(u_2)+k}, \qquad u_1 \vdash u_2 = u_2 * b^{l(u_1)+k}$$

for all $u_1, u_2 \in S$. The obtained algebra will be denoted by $S_a^b(k)$.

**Lemma 6.** *Let $T$ be the free monoid in the alphabet $Y$. Then for any $a, b \in Y$, $k \in \mathbb{N} \cup \{0\}$ the algebra $T_a^b(k)$ is a g-dimonoid. If $a \neq b$, then it is not a dimonoid.*

*Proof.* Let $u_1, u_2, u_3 \in T_a^b(k)$. In order to prove that $T_a^b(k)$ is a $g$-dimonoid we consider the following cases.

Case 1. Let $u_1 \neq \theta, \quad u_2 \neq \theta, \quad u_3 \neq \theta$. Then

$$u_1 \dashv (u_2 \dashv u_3) = u_1 \dashv (u_2 * a^{l(u_3)+k}) =$$

$$= u_1 * a^{l(u_2 a^{l(u_3)+k})+k} = u_1 * a^{l(u_2)+l(u_3)+2k} =$$

$$= u_1 * a^{l(u_2)+k} * a^{l(u_3)+k} = (u_1 * a^{l(u_2)+k}) \dashv u_3 = (u_1 \dashv u_2) \dashv u_3,$$

$$u_1 \dashv (u_2 \vdash u_3) = u_1 \dashv (u_3 * b^{l(u_2)+k}) =$$

$$= u_1 * a^{l(u_3 b^{l(u_2)+k})+k} = u_1 * a^{l(u_3)+l(u_2)+2k},$$

$$u_1 \vdash (u_2 \vdash u_3) = u_1 \vdash (u_3 * b^{l(u_2)+k}) =$$

$$= u_3 * b^{l(u_2)+k} * b^{l(u_1)+k} = u_3 * b^{l(u_2)+l(u_1)+2k} =$$

$$= u_3 * b^{l(u_2 b^{l(u_1)+k})+k} = (u_2 * b^{l(u_1)+k}) \vdash u_3 = (u_1 \vdash u_2) \vdash u_3,$$

$$(u_1 \dashv u_2) \vdash u_3 = (u_1 * a^{l(u_2)+k}) \vdash u_3 =$$

$$= u_3 * b^{l(u_1 a^{l(u_2)+k})+k} = u_3 * b^{l(u_1)+l(u_2)+2k}.$$

Case 2. Let $u_1 = u_2 = u_3 = \theta$. Then

$$\theta \dashv (\theta \dashv \theta) = \theta \dashv (\theta * a^{l(\theta)+k}) = \theta \dashv a^k = \theta * a^{l(a^k)+k} = a^{2k} =$$

$$= a^k * a^{l(\theta)+k} = a^k \dashv \theta = (\theta * a^{l(\theta)+k}) \dashv \theta = (\theta \dashv \theta) \dashv \theta,$$

$$\theta \dashv (\theta \vdash \theta) = \theta \dashv (\theta * b^{l(\theta)+k}) = \theta \dashv b^k = \theta * a^{l(b^k)+k} = a^{2k},$$

$$\theta \vdash (\theta \vdash \theta) = \theta \vdash (\theta * b^{l(\theta)+k}) = \theta \vdash b^k = b^k * b^{l(\theta)+k} = b^{2k} =$$

$$= \theta * b^{l(b^k)+k} = b^k \vdash \theta = (\theta * b^{l(\theta)+k}) \vdash \theta = (\theta \vdash \theta) \vdash \theta,$$

$$(\theta \dashv \theta) \vdash \theta = (\theta * a^{l(\theta)+k}) \vdash \theta = a^k \vdash \theta = \theta * b^{l(a^k)+k} = b^{2k}.$$

Case 3. Let $u_1 = \theta, \quad u_2 \neq \theta, \quad u_3 \neq \theta$. Then

$$\theta \dashv (u_2 \dashv u_3) = \theta \dashv (u_2 * a^{l(u_3)+k}) = \theta * a^{l(u_2 a^{l(u_3)+k})+k} = a^{l(u_2)+l(u_3)+2k} =$$

$$= a^{l(u_2)+k} * a^{l(u_3)+k} = (\theta * a^{l(u_2)+k}) \dashv u_3 = (\theta \dashv u_2) \dashv u_3,$$

$$\theta \dashv (u_2 \vdash u_3) = \theta \dashv (u_3 * b^{l(u_2)+k}) = \theta * a^{l(u_3 b^{l(u_2)+k})+k} = a^{l(u_2)+l(u_3)+2k},$$

$$\theta \vdash (u_2 \vdash u_3) = \theta \vdash (u_3 * b^{l(u_2)+k}) = u_3 * b^{l(u_2)+k} * b^{l(\theta)+k} = u_3 * b^{l(u_2)+2k} =$$

$$= u_3 * b^{l(u_2 * b^{l(\theta)+k})+k} = (u_2 * b^{l(\theta)+k}) \vdash u_3 = (\theta \vdash u_2) \vdash u_3,$$

$$(\theta \dashv u_2) \vdash u_3 = (\theta * a^{l(u_2)+k}) \vdash u_3 = u_3 * b^{l(a^{l(u_2)+k})+k} = u_3 * b^{l(u_2)+2k}.$$

Case 4. Let $u_1 = \theta, \quad u_2 \neq \theta, \quad u_3 = \theta$. Then

$$\theta \dashv (u_2 \dashv \theta) = \theta \dashv (u_2 * a^{l(\theta)+k}) = \theta \dashv (u_2 * a^k) = \theta * a^{l(u_2 * a^k)+k} = a^{l(u_2)+2k} =$$

$$= a^{l(u_2)+k} * a^{l(\theta)+k} = (\theta * a^{l(u_2)+k}) \dashv \theta = (\theta \dashv u_2) \dashv \theta,$$

$$\theta \dashv (u_2 \vdash \theta) = \theta \dashv (\theta * b^{l(u_2)+k}) = \theta * a^{l(b^{l(u_2)+k})+k} = a^{l(u_2)+2k},$$

$$\theta \vdash (u_2 \vdash \theta) = \theta \vdash (\theta * b^{l(u_2)+k}) = b^{l(u_2)+k} * b^{l(\theta)+k} = b^{l(u_2)+2k} =$$

$$= \theta * b^{l(u_2 * b^k)+k} = (u_2 * b^{l(\theta)+k}) \vdash \theta = (\theta \vdash u_2) \vdash \theta,$$

$$(\theta \dashv u_2) \vdash \theta = (\theta * a^{l(u_2)+k}) \vdash \theta = \theta * b^{l(a^{l(u_2)+k})+k} = b^{l(u_2)+2k}.$$

The cases $u_1 \neq \theta$, $u_2 = \theta$, $u_3 \neq \theta$; $u_1 \neq \theta$, $u_2 \neq \theta$, $u_3 = \theta$; $u_1 = u_2 = \theta$, $u_3 \neq \theta$; $u_1 \neq \theta$, $u_2 = u_3 = \theta$ are considered in a similar way.

Thus, $T_a^b(k)$ is a $g$-dimonoid.

Finally, show that $T_a^b(k)$ is not a dimonoid when $a \neq b$. For $u_1 \neq \theta$, $u_2 \neq \theta$ and $u_3 \neq \theta$ we have

$$(u_1 \vdash u_2) \dashv u_3 = (u_2 * b^{l(u_1)+k}) \dashv u_3 = u_2 * b^{l(u_1)+k} * a^{l(u_3)+k} =$$

$$= u_2 b^{l(u_1)+k} a^{l(u_3)+k} \neq u_2 a^{l(u_3)+k} b^{l(u_1)+k} =$$

$$= u_2 * a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(u_3)+k}) = u_1 \vdash (u_2 \dashv u_3)$$

and so, the axiom $(D3)$ of a dimonoid does not hold. □

The following lemma gives an answer on the question when $S_a^b(k)$ is a dimonoid.

**Lemma 7.** *Let $M$ be the free commutative monoid in the alphabet $Y$. For any $a, b \in Y$, $k \in \mathbb{N} \cup \{0\}$ algebras $M_a^b(k)$ and $T_a^a(k)$ are dimonoids.*

*Proof.* From Lemma 6 it follows that $M_a^b(k)$ satisfies the axioms $(D1)$, $(D2)$, $(D4)$, $(D5)$. Show that the axiom $(D3)$ also holds.

Let $u_1, u_2, u_3 \in M_a^b(k)$. Consider the following eight cases.

Case 1. Let $u_1 \neq \theta, \quad u_2 \neq \theta, \quad u_3 \neq \theta$. Then

$$(u_1 \vdash u_2) \dashv u_3 = (u_2 * b^{l(u_1)+k}) \dashv u_3 = u_2 * b^{l(u_1)+k} * a^{l(u_3)+k} =$$

$$= u_2 * a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(u_3)+k}) = u_1 \vdash (u_2 \dashv u_3).$$

Case 2. Let $u_1 = u_2 = u_3 = \theta$. Then

$$(\theta \vdash \theta) \dashv \theta = (\theta * b^{l(\theta)+k}) \dashv \theta = b^k \dashv \theta = b^k * a^{l(\theta)+k} = b^k * a^k =$$

$$= a^k * b^k = a^k * b^{l(\theta)+k} = \theta \vdash a^k = \theta \vdash (\theta * a^{l(\theta)+k}) = \theta \vdash (\theta \dashv \theta) .$$

Case 3. Let $u_1 = \theta, \quad u_2 \neq \theta, \quad u_3 \neq \theta$. Then

$$(\theta \vdash u_2) \dashv u_3 = (u_2 * b^{l(\theta)+k}) \dashv u_3 = u_2 * b^{l(\theta)+k} * a^{l(u_3)+k} = u_2 * b^k * a^{l(u_3)+k} =$$

$$= u_2 * a^{l(u_3)+k} * b^k = u_2 * a^{l(u_3)+k} * b^{l(\theta)+k} = \theta \vdash (u_2 * a^{l(u_3)+k}) = \theta \vdash (u_2 \dashv u_3) .$$

Case 4. Let $u_1 \neq \theta, \quad u_2 = \theta, \quad u_3 \neq \theta$. Then

$$(u_1 \vdash \theta) \dashv u_3 = (\theta * b^{l(u_1)+k}) \dashv u_3 = b^{l(u_1)+k} * a^{l(u_3)+k} =$$

$$= a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (\theta * a^{l(u_3)+k}) = u_1 \vdash (\theta \dashv u_3) .$$

Case 5. Let $u_1 \neq \theta, \quad u_2 \neq \theta, \quad u_3 = \theta$. Then

$$(u_1 \vdash u_2) \dashv \theta = (u_2 * b^{l(u_1)+k}) \dashv \theta = u_2 * b^{l(u_1)+k} * a^{l(\theta)+k} = u_2 * b^{l(u_1)+k} * a^k =$$

$$= u_2 * a^k * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(\theta)+k}) = u_1 \vdash (u_2 \dashv \theta) .$$

Case 6. Let $u_1 = u_2 = \theta, \quad u_3 \neq \theta$. Then

$$(\theta \vdash \theta) \dashv u_3 = (\theta * b^{l(\theta)+k}) \dashv u_3 = b^k \dashv u_3 = b^k * a^{l(u_3)+k} =$$

$$= a^{l(u_3)+k} * b^k = a^{l(u_3)+k} * b^{l(\theta)+k} = \theta \vdash (\theta * a^{l(u_3)+k}) = \theta \vdash (\theta \dashv u_3) .$$

Case 7. Let $u_1 \neq \theta, \quad u_2 = u_3 = \theta$. Then

$$(u_1 \vdash \theta) \dashv \theta = (\theta * b^{l(u_1)+k}) \dashv \theta = b^{l(u_1)+k} * a^{l(\theta)+k} = b^{l(u_1)+k} * a^k =$$

$$= a^k * b^{l(u_1)+k} = u_1 \vdash a^k = u_1 \vdash (\theta * a^{l(\theta)+k}) = u_1 \vdash (\theta \dashv \theta) .$$

Case 8. Let $u_1 = \theta, \quad u_2 \neq \theta, \quad u_3 = \theta$. Then

$$(\theta \vdash u_2) \dashv \theta = (u_2 * b^{l(\theta)+k}) \dashv \theta = u_2 * b^k * a^{l(\theta)+k} = u_2 * b^k * a^k =$$

$$= u_2 * a^k * b^k = u_2 * a^k * b^{l(\theta)+k} = \theta \vdash (u_2 * a^{l(\theta)+k}) = \theta \vdash (u_2 \dashv \theta) .$$

Thus, $M_a^b(k)$ is a dimonoid.
A proof is the same for $T_a^a(k)$. □

Note that independence of axioms of a $g$-dimonoid follows from independence of axioms of a dimonoid (see [2], Theorem 2).

## 3. Free $g$-dimonoids

In this section we construct a $g$-dimonoid which is isomorphic to the free $g$-dimonoid of an arbitrary rank and consider separately free $g$-dimonoids of rank 1.

A nonempty subset $A$ of a $g$-dimonoid $(D, \dashv, \vdash)$ is called a $g$-subdimonoid, if for any $a, b \in D$, $a$, $b \in A$ implies $a \dashv b$, $a \vdash b \in A$.

Note that the class of all $g$-dimonoids is a variety as it is closed under taking of homomorphic images, $g$-subdimonoids and Cartesian products. A $g$-dimonoid which is free in the variety of all $g$-dimonoids is called a free $g$-dimonoid.

In order to prove the main result of this section we need the construction of a free $g$-dimonoid from [14].

Let $e$ be an arbitrary symbol. Consider the following sets:

$$I^1 = \{e\}, \;\; I^n = \{(\varepsilon_1, \dots, \varepsilon_{n-1}) \,|\, \varepsilon_k \in \{0,1\}, 1 \leqslant k \leqslant n-1\}, \; n > 1,$$

$$I = \bigcup_{n \geqslant 1} I^n.$$

If $l = 0$, we will regard the sequence $\varepsilon_1, \dots, \varepsilon_l$ without brackets as empty, and the sequence $(\varepsilon_1, \dots, \varepsilon_l)$ with brackets as $e$. Define operations $\dashv$ and $\vdash$ on $I$ by

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv (\theta_1, \dots, \theta_{m-1}) = (\varepsilon_1, \dots, \varepsilon_{n-1}, \underbrace{1, 1, \dots, 1}_{m}),$$

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash (\theta_1, \dots, \theta_{m-1}) = (\theta_1, \dots, \theta_{m-1}, \underbrace{0, 0, \dots, 0}_{n}).$$

By Lemma 3 from [14] $(I, \dashv, \vdash)$ is a $g$-dimonoid. Observe that $e \dashv e = (1)$, $e \vdash e = (0)$ and $(I, \dashv, \vdash)$ is not a dimonoid.

Let $X$ be an arbitrary nonempty set and $F[X]$ be the free semigroup in the alphabet $X$. Define operations $\dashv$ and $\vdash$ on $FG = \{(w, \varepsilon) \,|\, w \in F[X], \, \varepsilon \in I^{l(w)}\}$ by

$$(w_1, \varepsilon) \dashv (w_2, \xi) = (w_1 w_2, \varepsilon \dashv \xi),$$

$$(w_1, \varepsilon) \vdash (w_2, \xi) = (w_1 w_2, \varepsilon \vdash \xi)$$

for all $(w_1, \varepsilon), (w_2, \xi) \in FG$. The algebra $(FG, \dashv, \vdash)$ is denoted by $FG[X]$. By Theorem 4 from [14] $FG[X]$ is the free $g$-dimonoid.

Using notations from Section 2, introduce the set

$$XT_a^b(k) = \{(w, u) \in F[X] \times T_a^b(k) \,|\, l(w) - l(u) = 1\}.$$

If $s = 1$, we will regard the sequence $y_1 y_2 ... y_{s-1} \in T_a^b(k)$ as $\theta$.

The main result of this section is the following.

**Theorem 1.** *The $g$-dimonoid $XT_a^b(1)$ is free if $|Y| = 2$ and $a \neq b$.*

*Proof.* By Lemma 1 $F[X] \times T_a^b(k)$ is a $g$-dimonoid. It is not difficult to check that $XT_a^b(1)$ is a $g$-subdimonoid of $F[X] \times T_a^b(1)$.

Let $|Y| = 2$ and $a \neq b$. Let us show that $XT_a^b(1)$ is free. Take $(x_1 x_2 ... x_s, y_1 y_2 ... y_{s-1}) \in XT_a^b(1)$, where $x_i \in X$, $1 \leqslant i \leqslant s$, $y_j \in Y$, $1 \leqslant j \leqslant s-1$, and define a map

$$\pi : XT_a^b(1) \to FG[X] :$$

$$(x_1 x_2 ... x_s, y_1 y_2 ... y_{s-1}) \mapsto (x_1 x_2 ... x_s, y_1 y_2 ... y_{s-1})\pi,$$

assuming

$$(x_1 x_2 ... x_s, y_1 y_2 ... y_{s-1})\pi = (x_1 x_2 ... x_s, (\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_{s-1})),$$

where

$$\widetilde{y}_i = \begin{cases} 1, \ y_i = a, \\ 0, \ y_i = b \end{cases}$$

for all $1 \leqslant i \leqslant s-1$, $s \neq 1$, and $(\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_{s-1})$ is $e$ for $s = 1$. Show that $\pi$ is an isomorphism.

For all

$$(x_1 x_2 \ldots x_s, \ y_1 y_2 \ldots y_{s-1}), (a_1 a_2 \ldots a_m, \ b_1 b_2 \ldots b_{m-1}) \in XT_a^b(1),$$

where $a_i \in X, 1 \leqslant i \leqslant m$, $b_j \in Y, 1 \leqslant j \leqslant m-1$, obtain

$$((x_1 x_2 \ldots x_s, \ y_1 y_2 \ldots y_{s-1}) \dashv (a_1 a_2 \ldots a_m, \ b_1 b_2 \ldots b_{m-1}))\,\pi =$$

$$= (x_1 x_2 \ldots x_s a_1 a_2 \ldots a_m, \ y_1 y_2 \ldots y_{s-1} * a^m)\,\pi =$$

$$= \left( x_1 x_2 \ldots x_s a_1 a_2 \ldots a_m, \ (\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_{s-1}, \underbrace{\widetilde{a}, \widetilde{a}, \ldots, \widetilde{a}}_{m}) \right) =$$

$$= \left( x_1 x_2 \ldots x_s a_1 a_2 \ldots a_m, \ (\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_{s-1}, \underbrace{1, 1, \ldots, 1}_{m}) \right) =$$

$$= (x_1 x_2 \ldots x_s, \ (\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_{s-1})) \dashv \left( a_1 a_2 \ldots a_m, \ (\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_{m-1}) \right) =$$

$$= (x_1 x_2 \ldots x_s, \ y_1 y_2 \ldots y_{s-1})\,\pi \dashv (a_1 a_2 \ldots a_m, \ b_1 b_2 \ldots b_{m-1})\,\pi,$$

$$((x_1x_2\dots x_s,\ y_1y_2\dots y_{s-1}) \vdash (a_1a_2\dots a_m,\ b_1b_2\dots b_{m-1}))\,\pi =$$

$$= (x_1x_2\dots x_s a_1a_2\dots a_m,\ b_1b_2\dots b_{m-1} * b^s)\,\pi =$$

$$= \left(x_1x_2\dots x_s a_1a_2\dots a_m,\ (\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_{m-1}, \underbrace{\widetilde{b}, \widetilde{b}, \dots, \widetilde{b}}_{s})\right) =$$

$$= \left(x_1x_2\dots x_s a_1a_2\dots a_m,\ (\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_{m-1}, \underbrace{0, 0, \dots, 0}_{s})\right) =$$

$$= (x_1x_2\dots x_s,\ (\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_{s-1})) \vdash \Big(a_1a_2\dots a_m,\ (\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_{m-1})\Big) =$$

$$= (x_1x_2\dots x_s,\ y_1y_2\dots y_{s-1})\,\pi \vdash (a_1a_2\dots a_m,\ b_1b_2\dots b_{m-1})\,\pi.$$

So, $\pi$ is a homomorphism. Obviously, $\pi$ is a bijection and thus, $\pi$ is an isomorphism. Hence we obtain that $XT_a^b(1)$ is the free $g$-dimonoid. $\square$

The following lemma gives one property of $S_a^b(k)$.

**Lemma 8.** *If $S_a^b(k)$ is a dimonoid, then $a^k$ and $b^k$ are commuting in $S$.*

*Proof.* Let $S_a^b(k)$ be a dimonoid. Then

$$(\theta \vdash \theta) \dashv \theta = (\theta * b^{l(\theta)+k}) \dashv \theta = b^k \dashv \theta = b^k * a^{l(\theta)+k} = b^k * a^k,$$

$$\theta \vdash (\theta \dashv \theta) = \theta \vdash (\theta * a^{l(\theta)+k}) = \theta \vdash a^k = a^k * b^{l(\theta)+k} = a^k * b^k$$

and, using the axiom $(D3)$, obtain $b^k * a^k = a^k * b^k$. $\square$

Now we construct a $g$-dimonoid which is isomorphic to the free $g$-dimonoid of rank 1.

Let $|Y| = 2$, $a \neq b$. Define operations $\dashv$ and $\vdash$ on

$$\widetilde{\mathbb{N}T}_a^b(1) = \{(m, u) \in \mathbb{N} \times T_a^b(1) \mid m - l(u) = 1\}$$

by

$$(m_1, u_1) \dashv (m_2, u_2) = (m_1 + m_2, u_1 * a^{l(u_2)+1}),$$

$$(m_1, u_1) \vdash (m_2, u_2) = (m_1 + m_2, u_2 * b^{l(u_1)+1})$$

for all $(m_1, u_1), (m_2, u_2) \in \widetilde{\mathbb{N}T}_a^b(1)$. By Lemma 1 $(\mathbb{N}, +) \times T_a^b(1)$ is a $g$-dimonoid. An immediate verification shows that operations $\dashv$ and $\vdash$ are well-defined. Thus, $(\widetilde{\mathbb{N}T}_a^b(1), \dashv, \vdash)$ is a $g$-subdimonoid of $(\mathbb{N}, +) \times T_a^b(1)$. Denote it by $\mathbb{N}T_a^b(1)$.

**Lemma 9.** *The free $g$-dimonoid of rank 1 is isomorphic to the $g$-dimonoid $\mathbb{N}T_a^b(1)$.*

*Proof.* Let $X = \{r\}$. An easy verification shows that a map

$$\xi : XT_a^b(1) \to \mathbb{N}T_a^b(1),$$

defined by $\omega\xi = (k, u) \Leftrightarrow \omega = (r^k, u)$, is an isomorphism. □

## 4. Free $n$-nilpotent $g$-dimonoids

In this section we construct a free $n$-nilpotent $g$-dimonoid of an arbitrary rank and consider separately free $n$-nilpotent $g$-dimonoids of rank 1. We also characterize the least $n$-nilpotent congruence on a free $g$-dimonoid.

An element 0 of a $g$-dimonoid $(D, \dashv, \vdash)$ will be called zero, if $x \star 0 = 0 = 0 \star x$ for all $x \in D$ and $\star \in \{\dashv, \vdash\}$.

A $g$-dimonoid $(D, \dashv, \vdash)$ with zero will be called nilpotent, if for some $n \in \mathbb{N}$ and any $x_i \in D$, $1 \leqslant i \leqslant n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leqslant j \leqslant n$, any parenthesizing of

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} \tag{2}$$

gives $0 \in D$. The least such $n$ we shall call the nilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a nilpotent $g$-dimonoid of nilpotency index $\leqslant k$ is said to be $k$-nilpotent.

Note that from (2) it follows that operations of any 1-nilpotent $g$-dimonoid coincide and it is a zero semigroup.

It is not difficult to see that the class of all $n$-nilpotent $g$-dimonoids is a subvariety of the variety of all $g$-dimonoids. A $g$-dimonoid which is free in the variety of $n$-nilpotent $g$-dimonoids will be called a free $n$-nilpotent $g$-dimonoid.

Fix $n \in \mathbb{N}$ and, using notations from Section 3, assume

$$G_n = \{(w, u) \in XT_a^b(1) \mid l(w) \leqslant n\} \cup \{0\} \quad (|Y| = 2,\ a \neq b).$$

Define operations $\prec$ and $\succ$ on $G_n$ by

$$(w_1, u_1) \prec (w_2, u_2) = \begin{cases} \left(w_1w_2, u_1 * a^{l(u_2)+1}\right), & l(w_1w_2) \leqslant n, \\ 0, & l(w_1w_2) > n, \end{cases}$$

$$(w_1, u_1) \succ (w_2, u_2) = \begin{cases} \left(w_1w_2, u_2 * b^{l(u_1)+1}\right), & l(w_1w_2) \leqslant n, \\ 0, & l(w_1w_2) > n, \end{cases}$$

$$(w_1,u_1)\star 0 = 0\star (w_1,u_1) = 0\star 0 = 0$$

for all $(w_1,u_1), (w_2,u_2) \in G_n\backslash\{0\}$ and $\star \in \{\prec,\succ\}$. The algebra $(G_n,\prec,\succ)$ will be denoted by $G_n(X)$.

**Theorem 2.** *$G_n(X)$ is the free $n$-nilpotent $g$-dimonoid.*

*Proof.* Prove that $G_n(X)$ is a $g$-dimonoid. Let $(w_1,u_1),(w_2,u_2)$, $(w_3,u_3) \in G_n\backslash\{0\}$. If $l(w_1w_2) > n$ or $l(w_2w_3) > n$, then the proof is straightforward. The fact that axioms of a $g$-dimonoid hold when $l(w_1w_2w_3) \leqslant n$ follows from Theorem 1. In the case $l(w_1w_2) \leqslant n$, $l(w_2w_3) \leqslant n$ and $l(w_1w_2w_3) > n$ we have

$$((w_1,u_1) *_1 (w_2,u_2)) *_2 (w_3,u_3) = 0 = (w_1,u_1) *_1 ((w_2,u_2) *_2 (w_3,u_3))$$

for $*_1, *_2 \in \{\prec,\succ\}$. The proofs of the remaining cases are obvious. Thus, $G_n(X)$ is a $g$-dimonoid.

For any $(w_i,u_i) \in G_n\backslash\{0\}$, $1 \leqslant i \leqslant n+1$, and $*_j \in \{\prec,\succ\}$, $1 \leqslant j \leqslant n$, any parenthesizing of

$$(w_1,u_1) *_1 (w_2,u_2) *_2 \ldots *_n (w_{n+1},u_{n+1})$$

gives 0, hence $G_n(X)$ is nilpotent. Moreover, for any $(x_i,\theta) \in G_n\backslash\{0\}$, where $x_i \in X$, $1 \leqslant i \leqslant n$,

$$(x_1,\theta) \prec (x_2,\theta) \prec \ldots \prec (x_n,\theta) = (x_1x_2\ldots x_n, a^{n-1}) \neq 0.$$

It means that $G_n(X)$ has nilpotency index $n$.

Let us show that $G_n(X)$ is free in the variety of $n$-nilpotent $g$-dimonoids.

The $g$-dimonoid $(\mathcal{G}(X),\dashv,\vdash)$ which is isomorphic to $FG[X]$ from Section 3 was constructed in [14]. The corresponding isomorphism $(\mathcal{G}(X),\dashv,\vdash) \to FG[X]$ is denoted by $\sigma$ (see [14], Theorem 4). In the last paper for an arbitrary $g$-dimonoid $(\mathcal{D},\dashv,\vdash)$ the homomorphism $\psi_0$ from $(\mathcal{G}(X),\dashv,\vdash)$ to $(\mathcal{D},\dashv,\vdash)$ was given. We will call $\psi_0$ as a canonical homomorphism. Observe that $\psi_0$ sends an arbitrary term with elements $x_1,...,x_n$ to the product of some $n$ elements from $\mathcal{D}$.

Let $(P,\dashv',\vdash')$ be an arbitrary $n$-nilpotent $g$-dimonoid, $\alpha$ be the canonical homomorphism from $(\mathcal{G}(X),\dashv,\vdash)$ to $(P,\dashv',\vdash')$ and $\mu = \pi\sigma^{-1}\alpha$ (see Section 3). Obviously, $\mu$ is a homomorphism from $XT_a^b(1)$, where $|Y| = 2$, $a \neq b$, to $(P,\dashv',\vdash')$. Define a map

$$\delta : G_n(X) \to (P,\dashv',\vdash') : \omega \mapsto \omega\delta,$$

assuming

$$\omega\delta = \begin{cases} \omega\mu, & \omega \in G_n \backslash \{0\}, \\ 0, & \omega = 0. \end{cases}$$

Show that $\delta$ is a homomorphism.

Let $\omega_1 = (x_1x_2...x_s, y_1y_2...y_{s-1})$, $\omega_2 = (a_1a_2\dots a_m,\ b_1b_2\dots b_{m-1}) \in G_n\backslash\{0\}$, where $x_i \in X, 1 \leqslant i \leqslant s, y_j \in Y, 1 \leqslant j \leqslant s-1, a_i \in X, 1 \leqslant i \leqslant m$, $b_j \in Y$, $1 \leqslant j \leqslant m-1$. Assume $s + m \leqslant n$. As $\omega_1 \prec \omega_2 \in G_n\backslash\{0\}$, then

$$(\omega_1 \prec \omega_2)\delta = (\omega_1 \prec \omega_2)\mu = (\omega_1 \dashv \omega_2)\mu = \omega_1\mu \dashv' \omega_2\mu = \omega_1\delta \dashv' \omega_2\delta.$$

Analogously, $(\omega_1 \succ \omega_2)\delta = \omega_1\delta \vdash' \omega_2\delta$. Taking into account the previous arguments, in the remaining cases the equalities

$$(\omega_1 \prec \omega_2)\delta = (\omega_1 \succ \omega_2)\delta = 0 = \omega_1\delta{\vdash'}\omega_2\delta = \omega_1\delta{\dashv'}\omega_2\delta$$

hold. Thus, $\delta$ is a homomorhism.

The proof is complete. □

Now we construct a $g$-dimonoid which is isomorphic to the free $n$-nilpotent $g$-dimonoid of rank 1.

Assume $|Y| = 2$, $a \neq b$. For any $n \in \mathbb{N}$ let

$$\widetilde{\mathbb{L}}_n = \{(m,u) \in \mathbb{N} \times T_a^b(1) \,|\, m - l(u) = 1, m \leqslant n\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on $\widetilde{\mathbb{L}}_n$ by the rule

$$(m_1,u_1) \dashv (m_2,u_2) = \begin{cases} \left(m_1+m_2, u_1 * a^{l(u_2)+1}\right), & m_1+m_2 \leqslant n, \\ 0, & m_1+m_2 > n, \end{cases}$$

$$(m_1,u_1) \vdash (m_2,u_2) = \begin{cases} \left(m_1+m_2, u_2 * b^{l(u_1)+1}\right), & m_1+m_2 \leqslant n, \\ 0, & m_1+m_2 > n, \end{cases}$$

$$(m_1,u_1) \star 0 = 0 \star (m_1,u_1) = 0 \star 0 = 0$$

for all $(m_1,u_1), (m_2,u_2) \in \widetilde{\mathbb{L}}_n\backslash\{0\}$ and $\star \in \{\dashv, \vdash\}$. An immediate verification shows that axioms of a $g$-dimonoid hold concerning operations $\dashv$ and $\vdash$. So, $(\widetilde{\mathbb{L}}_n, \dashv, \vdash)$ is a $g$-dimonoid. Denote it by $\mathbb{L}_n$.

**Lemma 10.** *If $|X| = 1$, then $G_n(X) \cong \mathbb{L}_n$.*

*Proof.* Let $X = \{r\}$. An easy verification shows that a map $\varrho : G_n(X) \to \mathbb{L}_n$, defined by

$$\omega\varrho = \begin{cases} (k,u), & \omega = (r^k, u), \\ 0, & \omega = 0, \end{cases}$$

is an isomorphism. □

We finish this section with the description of the least $n$-nilpotent congruence on a free $g$-dimonoid.

If $f : D_1 \to D_2$ is a homomorphism of $g$-dimonoids, then the corresponding congruence on $D_1$ will be denoted by $\Delta_f$. If $\rho$ is a congruence on a $g$-dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an $n$-nilpotent $g$-dimonoid, then we say that $\rho$ is an $n$-nilpotent congruence.

Let $XT_a^b(1)$ be the free $g$-dimonoid ($|Y| = 2$, $a \neq b$) (see Section 3). Fix $n \in \mathbb{N}$ and define a relation $\kappa(n)$ on $XT_a^b(1)$ by

$$(w_1, u_1)\kappa(n)(w_2, u_2) \text{ if and only if}$$
$$(w_1, u_1) = (w_2, u_2) \text{ or } l(w_1) > n,\, l(w_2) > n.$$

**Theorem 3.** *The relation $\kappa(n)$ on the free $g$-dimonoid $XT_a^b(1)$ is the least $n$-nilpotent congruence.*

*Proof.* Define a map $\tau : XT_a^b(1) \to G_n(X)$ by

$$(w, u)\,\tau = \begin{cases} (w, u)\,, & l(w) \leqslant n, \\ 0, & l(w) > n, \end{cases} \qquad (w, u) \in XT_a^b(1).$$

Similarly to the proof of Theorem 4 from [4], the facts that $\tau$ is a surjective homomorphism and $\Delta_\tau = \kappa(n)$ can be proved. □

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