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Exponent matrices and Frobenius rings

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Abstract. We give a survey of results connecting the exponent matrices with Frobenius rings. In particular, we prove that for any parmutation $\sigma \in S_n$ there exists a countable set of indecomposable Frobenius semidistributive rings A_m with Nakayama permutation σ.

1. Exponent matrices

Let $M_n(B)$ be the ring of all $n \times n$ matrices over a ring B, a $\mathbb{Z}$ the ring of integers.

An integer matrix $\mathcal{E} = (\alpha_{ij} \in M(\mathbb{Z})$ is called an exponent matrix if

(a) $\alpha_{ij} + \alpha jk \geqslant \alpha_{ik}$ and $\alpha_{ii} = 0$ for all $i, j, k = 1, \ldots, n$;

a reduced exponent matrix if

(b) $\alpha_{ij} + \alpha_{ji} > 0$ for all $i, j = 1, \ldots, n$; $i \neq j$.

Let

$$\mathcal{E}^{(1)} = (\beta_{ij}), \text{ where } \beta_{ij} = \begin{cases} \alpha_{ij}, \text{ if } i \neq j \\ 1, \quad \text{if } i = j \end{cases}$$

and

$$\mathcal{E}^{(2)} = (\gamma_{ij}), \text{ where } \gamma_{ij} = \min_{1 \leqslant k \leqslant n} (\beta_{ik} + \beta_{kj}).$$

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Consider the matrix $[Q] = \mathcal{E}^2 - \mathcal{E}^1 = (q_{ij})$. Let $\beta_{ik} + \beta_{kj}$ $(i \neq j)$ be an integer. If $k = i$, then $\beta_{ii} + \beta_{ij} = \beta_{ii} + \alpha_{ij} = \alpha_{ij} + 1$.

Therefore $\min\limits_{1 \leqslant k \leqslant n} (\beta_{ik} + \beta_{kj}) \leqslant \alpha_{ij} + 1$ and $[Q]$ is $(0,1)$-matrix and it is the adjacency matrix of a simply laced quiver.

Definition 1.1. *A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called Gorenstein if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

The permutation σ is denoted by $\sigma(\mathcal{E})$. Obviously, $\sigma(\mathcal{E})$ has no cycles of length 1.

Gorenstein matrices are closely related to the semiperfect semidistributive rings A with non-zero Jacobson radical and $inj.dim_A A_A = 1$ [1].

Consider a reduced symmetric exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$. It means $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$, i.e., $2\alpha_{ij} > 0$ for $i \neq j$. Thus, all nondiagonal elements of the reduced symmetric matrix are positive, and it defines a finite metric space.

Definition 1.2. *A metric space is a pair (M, d), where M is a set and $d : M \times M \to [0, \infty)$ is a metric, satisfying the following axioms:*

(1) *$d(x, y) = 0$, if and only if $x = y$;*

(2) *$d(x, y) = d(y, x)$ for all $x, y \in M$;*

(3) *$d(x, y) + d(y, x) \geqslant d(x, z)$ for all $x, y, z \in M$.*

If M is finite, then M is a finite metric space. Denote $m_{ij} = d(i, j)$. The matrix $D(M) = (m_{ij}) \in M_n(\mathbb{R})$ is called the distance matrix of M.

Now we give the examples of Gorenstein matrices.

Example 1.1. The following exponent matrix

$$T_{n,\alpha} = \begin{pmatrix} 0 & 0 & \ldots & \ldots & \ldots & 0 \\ \alpha & 0 & 0 & \ldots & \ldots & 0 \\ \alpha & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & & \ddots & \alpha & 0 & 0 \\ \alpha & \ldots & \ldots & \alpha & \alpha & 0 \end{pmatrix}$$

is Gorenstein with $\sigma(T_{n,\alpha}) = (12 \ldots n)$.

Example 1.2. The Cayley table of the Klein four-group $(2) \times (2)$ can be written in the following form:

$$K(4) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

$K(4)$ is Gorenstein with $\sigma(K(4)) = (14)(23)$.

Example 1.3. The Cayley table of the elementary Abelian n2-group $(2) \times (2) \times (2)$ is as following

$$K(8) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

$K(8)$ is Gorenstein with $\sigma(K(8)) = (18)(27)(36)(45)$.

The notion of a Latin square was introduced by L. Euler at the end of the XVIII century (see [2] for details).

Definition 1.3. *A Latin square $\mathcal{L}_n$ of order n is a square $n \times n$ matrix of order n, such that its rows and columns are permutations of some set $S = \{s_1, \ldots, s_n\}$.*

In what follows we shall take $S = \{0, 1, \ldots, n-1\}$. So, $\mathcal{L}_n = (\alpha_{ij})$, where $\alpha_{ij} \in \{0, 1, \ldots, n-1\}$.

We say that a Latin square of order n is normalized if its first row is $(0, 1, \ldots, n-1)$ and the first column is $(0, 1, \ldots, n-1)^T$, where T (as exponent index) means transposition. The normalized Latin squares are also called "reduced Latin squares" or "Latin squares of standard form".

We use the following notations:

$$\Gamma_0 = (0), \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$U_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in M_n(\mathbb{Z}), \qquad X_{k-1} = 2^{k-1} U_{2^{k-1}};$$

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \Gamma_{k-1} + X_{k-1} \\ \Gamma_{k-1} + X_{k-1} & \Gamma_{k-1} \end{pmatrix} \text{ for } k = 1, 2, \dots .$$

The matrix $\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of the cyclic group G_1 of order 2 and is a Gorenstein matrix with permutation $\sigma(\Gamma_1) = (12)$.

Proposition 1.4 ([3, §7.6])**.** *Γ_k is an exponent matrix for any natural number k.*

Proposition 1.5 ([3, §7.6])**.** *Γ_k is the Cayley table of the elementary Abelian group G_k of order 2^k.*

Proposition 1.6 ([3, §7.6])**.** *The matrix Γ_k is Gorenstein with permutation*

$$\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k - 1 & 2^k \\ 2^k & 2^k - 1 & 2^k - 2 & \dots & 2 & 1 \end{pmatrix}.$$

Theorem 1.7 ([4] and [3, §7.6])**.** *Suppose that a Latin square $\mathcal{L}_n$ with first row and first column $(0\,1\,\dots\,n-1)$ is an exponent matrix. Then $n = 2^k$ and $\mathcal{L}_n = \Gamma_k$ is the Cayley table of the direct product of k copies of the cyclic group of order 2.*

Conversely, the Cayley table Γ_k of the elementary Abelian group

$$G_k = \mathbb{Z}/(2) \times \dots \times \mathbb{Z}/(2) = (2) \times \dots \times (2)$$

(k factors) of order 2^k is a Latin square and a Gorenstein symmetric matrix with the first row $(0, 1, \dots, 2^k - 1)$ and permutation

$$\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k - 1 & 2^k \\ 2^k & 2^k - 1 & 2^k - 2 & \dots & 2 & 1 \end{pmatrix}.$$

Now we consider the case when a Latin square $\mathcal{L}_n$ with first row and first column $(0\,1\,\dots\,n-1)$ is a distance matrix $D = D(M)$ of a finite metric space $M = \{m_1, \dots, m_n\}$. Obviously, if $\mathcal{L}_n = D(M)$ then $\mathcal{L}_n$ is an exponent matrix. So we obtain the following theorem.

Theorem 1.8. *Suppose that a Latin square $\mathcal{L}_n$ with first row and first column* $(0, 1, \ldots, n-1)$ *is a distance matrix $D = D(M)$ of a finite metric space $M = \{m_1, \ldots, m_n\}$. Then $n = 2^k$ and $\mathcal{L}_n = \Gamma_k$ is the Cayley table of the direct product of k copies of the cyclic group of order 2.*

Conversely, the Cayley table Γ_k of the elementary Abelian group

$$G_k = \mathbb{Z}/(2) \times \ldots \times \mathbb{Z}/(2) = (2) \times \ldots \times (2)$$

(k factors) of order 2^k is a Latin square and the distance matrix $D = D(M)$ of a finite metric space with 2^k elements.

2. Semiperfect rings

Recall that a ring A is called local, if it has a unique maximal right ideal $\mathfrak{M}$.

An idempotent $e \in A$ is called local if the ring eAe is local. Clearly, a local idempotent is always a primitive idempotent. We shall say that idempotents can be lifted modulo an ideal $\mathcal{I}$ of a ring A if from the fact that $g^2 - g \in \mathcal{I}$, where $g \in A$, it follows that there exists an idempotent $e^2 = e \in A$ such that $e - g \in I$ (or that the same $g - e \in \mathcal{I}$).

Show that in a local ring an idempotents can be lifted modulo a unique maximal ideal R. Consider $g^2 - g \in R$. So $g(g-1) \in R$. If $g \in R$, then $g - 0 = g \in R$. If $g \notin R$, then $g - 1 \in R$.

Proposition 2.1. *Idempotents can be lifted modulo any nil-ideal I of a ring A.*

Proof. Let $g^2 - g \in I$. Set $r = g^2 - g$, $g_1 = g + r - 2gr$. Obviously, $rg = g^3 - g^2 = g(g^2 - g) = gr$. Calculating $g_1^2 - g_1$, we obtain $g_1^2 = (g + r - 2gr)^2 = (g+r)^2 + 4g^2r^2 - 4gr(g+r) = g^2 + 2gr + r^2 + 4g^2r^2 - 4g^2r - 4gr^2$. Therefore, $g_1^2 - g_1 = 2gr + r^2 + 4g^2r^2 - 4g^2r - 4gr^2 + 2gr = r^2 + 4g^2r^2 + 4gr - 4g^2r - 4gr^3 = r^2 - 4r^3 - 4r^2 = r^2(4r - 3)$.

Let $r_1 = r^2(4r - 3) \in I$ and $g_1 = g_1 + r_1 - 2g_1r_1$. We obtain that $r_2 = g_2^2 - g_2 = r_1^2(4r_1 - 3)$, i.e., in the expression of r_2 the element r^4 enters a factor. Since $r^k = 0$ for some integer $k > 0$, we obtain that $r_n = 0$ for some n, i.e., $g_n^2 = g_n$. We have that $g_1 - g \in I$ and $g_i - g_{i-1} \in I$ for $i = 1, 2, \ldots, n$ and we obtain $g_n = e$ is idempotent and $g - e \in I$ $\square$

Theorem 2.2 (Hopkins-Levitsky Theorem)**.** *The Jacobson radical R of a right Artinian ring A is nilpotent* [5], [6].

Therefore, any right Artinian ring is semiperfect.

Lemma 2.3. *Let a ring A have two decompositions into a direct sum of right ideals: $A = e_1A \oplus \ldots \oplus e_nA = f_1A \oplus \ldots \oplus f_nA$ (where $1 = e_1 + \ldots + e_n = f_1 + \ldots + f_n$ are two decompositions of $1 \in A$ into a sum of pairwise orthogonal idempotents), and suppose $e_iA \simeq f_iA$ $(i = 1, \ldots, n)$, after renumbering if necessary. Then there is an invertible element $a \in A$ such that $f_i = ae_ia^{-1}$.*

Proof. Let $\varphi : e_iA \to f_iA$ be the isomorphism. Then $\varphi(e_ia) = \varphi(e_i)a$ and $\varphi(e_i) = \varphi(e_i^2) = \varphi(e_i)e_i$. We have $\varphi(e_i) = f_ia$. So $\varphi(e_i) = f_iae_i$ and endomorphism φ is realized by multiplication on the left side by element $a_i = f_iae_i$. Then $f_ia_i = a_ie_i = a_i$. Consider the elements $b_i \in e_iaf_i$ realizing the inverse isomorphisms. We set $b = b_1 + \ldots + b_n$ and $a = a_1 + \ldots + a_n$. Obviously, $e_ib = b_i = bf_i$. Moreover, $a_ib_i = f_i$, $b_ia_i = e_i$. Clearly, $ab = \sum\limits_{i=1}^{n} a_ib_i = \sum\limits_{i=1}^{n} f_i = 1$ and $ba = \sum\limits_{i=1}^{n} b_ia_i = \sum\limits_{i=1}^{n} e_i = 1$, i.e., $b = a^{-1}$. Since $ae_i = f_ia$, we have $f_i = ae_ia^{-1}$. □

We will use the following notations and definitions.

Let M be a module over a ring A with the Jacobson radical R. Denote by rad M the intersection of all its maximal submodules. By convention, if M does not have maximal submodules we define $radM = M$. This submodule is called the radical of the module M.

Recall that a right ideal $\mathfrak{M}$ in a ring A is called maximal in A if there is no right ideal J, different from $\mathfrak{M}$ and A such that $\mathfrak{M} \subset I \subset A$. In nonzero ring with identity always there exist maximal proper right ideals.

The following proposition is often useful.

Proposition 2.4. *Let M be a right A-module, and let R be the Jacobson radical of the ring A. Then $MR \subset radM$.*

Theorem 2.5. *A ring A is semiperfect if and only if it can be decomposed into a direct sum of right ideals each of which has exactly one maximal submodule.*

Proof. Let $\bar{A} = A/R = \bar{e}_1\bar{A} \oplus \ldots \oplus \bar{e}_n\bar{A}$ be a decomposition of $\bar{A}$ into a direct sum of a simple $\bar{A}$-modules. Write $\bar{e}_i\bar{A} = U_i$. Since A is semiperfect, for each idempotent $\bar{e}_i$ there is an idempotent e_i such that $e_i + R = \bar{e}_i$. Let $P_i = e_iA$. Obviously, $P_iR = e_iAR \subset R$. Therefore $P_i \cap R = P_iR$ Consequently, $P_i + R/R \simeq P_i/P_iR \simeq U_i$ and every module P_i has exactly one maximal submodule P_iR. Let $P = \bigoplus\limits_{i=1}^{n} P_i$. Obviously, there is an epimorphism $\varphi : P \to \bar{A}$. Denote by π the natural projection $A \to \bar{A}$. Since the module P is projective, there exists a homomorphism $\psi : P \to A$ such that $\pi\psi = \varphi$. Obviously, $Im\psi + R = A$. By Nakayama's lemma $Im\psi = A$. Since the module A is projective, we have $P \simeq Im\psi \oplus Ker\psi = A \oplus Ker\psi$. We want show that $X = Ker\psi = 0$. Consider P/PR. Then $P/PR \simeq \bar{A}$ and, on the other hand $P/PR = \bar{A} \oplus X/XR$. By the Krull-Schmidt theorem for semisimple modules we obtain that the module X/XR is equals to zero. By Nakayama's lemma $X = 0$, i.e., X is finitely generated as the image P. Therefore $Ker\psi = 0$ and A decomposes into a direct sum of right ideals $\psi(e_i A)$, each of which has exactly one maximal submodule.

Conversely, let $A_A = P_1 \oplus \ldots \oplus P_n$ be a decomposition of right regular A-module A into a direct sum of right A-modules, each of which has exactly one maximal submodule $radP_i$ $(i = 1, \ldots, n)$. Consequently, $R = rad\, P_1 \oplus \ldots \oplus rad\, P_n$ and $\bar{A} = A/R$ is a right semisimple ring. Let $1 = f_1 + \ldots + f_n$ be a decomposition of the identity of the ring A into a sum of pairwise orthogonal idempotents such that $P_i = f_i A$ $(i = 1, \ldots, n)$. □

We shall show that for any idempotent $\bar{e}^2 = \bar{e} \in \bar{A}$ there is an idempotent $e \in A$ such that $e + R = \bar{e}$. The right ideal $\bar{e}\bar{A}$ is semisimple as $\bar{A}$-submodule of the semisimple module $\bar{A}$. Therefore there is a decomposition of $\bar{1} \in A$ into a sum of pairwise orthogonal idempotents $\bar{1} = \bar{e}_1 + \ldots + \bar{e}_s + \ldots + \bar{e}_n$ such that $\bar{e} = \bar{e}_1 + \ldots + \bar{e}_s$ and all modules $\bar{e}_i\bar{A}$ are simple. On the other hand, let $\bar{1} = \bar{f}_1 + \ldots + \bar{f}_n$ be a decomposition of $\bar{1} \in \bar{A}$ into a sum of pairwise orthogonal idempotents such that the modules $\bar{f}_i\bar{A}$ are simple. By Krull-Schmidt theorem for semisimple modules for an an appropriate renumeration we have $\bar{e}_i\bar{A} \simeq \bar{f}_i\bar{A}$ $(i = 1, \ldots, n)$. By lemma 2.3 there exists an element $\bar{a} \in \bar{A}$ such that $\bar{e}_i = \bar{a}^{-1}\bar{f}_i\bar{a}$ $(i = 1, \ldots, n)$. Let $\bar{a}$ be the image of a and $\bar{a}^{-1}$ be the image of $b = a^{-1}$. Obviously, $ab = 1 + r$, where $r \in R$. Since $(1 + r)x = 1$ we have $x = 1 - r_1$, where $r_1 \in R$. Therefore, $b(1 - r_1) = bx = 1$. So, $bx = a^{-1}$. $bx = b - br_1$ and the image of the element a^{-1} under the epimorphism π coincides with $\bar{a}^{-1}$. Then $\pi(e) = \sum \pi(a^{-1}f_i a) = \sum \bar{a}^{-1}\bar{f}_i\bar{a} = \bar{e}$. The theorem is proved.

Theorem 2.6 (B. J. Mueller)**.** *A ring A is semiperfect if and only if $1 \in A$ can be decomposed into a sum of a finite number of pairwise orthogonal local idempotents.*

Proof. Let a ring A be semiperfect. By previous theorem $A = e_1 A \oplus \ldots \oplus e_n A$, where $e_1, \ldots, e_n$ are idempotents and each right ideal $P_i = e_i A$ $(i = 1, \ldots, n)$ has exactly one maximal submodule. Then $Hom(P_i, P_i) \simeq e_i A e_i$ and for any $\psi : P_i \to P_i$ either $Im\,\psi = P_i$ or $Im\,\psi \subset P_i R$. In the first case, since $Im\,\psi \simeq P_i$ is projective we have $P_i \simeq Im\,\psi \oplus Ker\,\psi$, which implies $Ker\,psi = 0$ and so ψ is an automorphism. In the second case, ψ is noninvertible element and, obviously, all noninvertible elements form an ideal. Therefore, the ring e_iAe_i is local for $i = 1, \ldots, n$.

Conversely, let $\pi : A \to \bar{A}$ be the natural projection of the ring A on the ring $\bar{A} = A/R$. (R is the Jacobson radical of the ring A). Let $\pi(a) = \bar{a}$ and e be a local idempotent of the ring A. We shall show that the module $\pi(eA) = \bar{e}\bar{A}$ is simple. We may consider that the ring A is not local (a local ring is semiperfect). Consider $(\bar{1} - \bar{e})\bar{A}$. Since it is a proper right ideal in the ring $\bar{A}$, it is contained in a maximal right ideal $\tilde{I}$ of the ring $\bar{A}$. We shall show that $\bar{e}\bar{A} \cap \tilde{I} = 0$. If $(\bar{e}\bar{A} \cap \tilde{I}) \neq 0$, then $(\bar{e}\bar{A} \cap \tilde{I})^2 \neq 0$. The ring A/R is semiprimitive ring and therefore it has no nilpotent right ideals. Then there is an element $\bar{e}\bar{a} \in \tilde{I}$ and $\bar{e}\bar{a}\bar{e}\bar{a} \neq 0$. So $\bar{e}\bar{a}\bar{e} \neq 0$. Since eAe is a local ring and $rad\,(eAe) = eRe$, we conclude that the ring $\bar{e}\bar{A}\bar{e}$ is a division ring. Therefore, there is an element $\bar{e}\bar{x}\bar{e} \in \bar{e}\bar{A}\bar{e}$ such that $\bar{e}\bar{a}\bar{e}\bar{x}\bar{e} = \bar{e}$ and $\bar{e} \in \tilde{I}$. Thus $\bar{1} \in \tilde{I}$, a contradiction. We obtain, that $\bar{e}\bar{A} \cap \tilde{I} = 0$ and $\bar{A} = \bar{e}\bar{A} \oplus \tilde{I}$. Since $\tilde{I}$ is the maximal ideal in the ring A, the module $\bar{e}\bar{A}$ is simple. The theorem is proved. □

3. Distributive modules

Recall that a module M is called distributive if for all submodules K, L, N of M we have $K \cap (L + N) = K \cap L + K \cap N$. Clearly, a submodule and a quotient module of a distributive module is distributive. A module is called semidistributive if it is a direct sum of distributive modules. A ring A is called right (left) semidistributive if the right (left) regular module A_A ($_AA$) is semidistributive. A right and left semidistributive ring is called semidistributive. Obviously, every uniserial module is a distributive module and every serial module is a semidistributive module.

Example 3.1. Let D be a division ring and

$$A = \left\{ \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \ \middle| \ d_{ij} \in D \right\}.$$

Obviously, A is the semidistributive ring, which is left serial, but not right serial.

We write $SPSD$-ring for semiperfect semidistributive ring.

4. Semiprime right Noetherian $SPSD$-rings

Definition 4.1. *A ring A is called semimaximal if it is a semiperfect semiprime right Noetherian ring such that for each local idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessary commutative), i.e., all principal endomorphism rings of A are discrete valuation rings.*

The following is a decomposition theorem for semiprime right Noetherian $SPSD$-rings.

Theorem 4.2. *The following conditions for a semiperfect semiprime right Noettherian ring A are equivalent:*

(a) *the ring A is semidistributive;*

(b) *the ring A is a direct product of a semisimple Artinian ring and a semimaximal rings.*

Theorem 4.3. *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \ldots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \ldots & \pi^{\alpha_{2n}}\mathcal{O} \\ \ldots & \ldots & \ldots & \ldots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \ldots & \mathcal{O} \end{pmatrix},$$

where $n \geqslant 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and the α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k ($\alpha_{ii} = 0$ for any i).

Definition 4.4. *A ring A is called a tiled order if it is a prime Noetherian $SPSD$-ring with nonzero Jacobson radical.*

Our definition of a tiled order is a generalization of the definition of a tiled order over a discrete valuation ring in the sense V. A. Jategaonkar [7], [8].

We'll use the following notation $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of a ring A, i.e., $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the e_{ij} are the matrix units. If a tiled order A is reduced, i.e., A/R is a direct product of division rings, then the matrix $\mathcal{E}(A)$ also reduced, i.e., $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$ ($i \neq j$).

It is well-known the following proposition.

Proposition 4.5 ([9, vol II, proposition 22.1.A]). *Let P be nonzero projective A-module and $rad\, A$ be the Jacobson radical of A. Then $rad\, P = Prad\, A \neq P$.*

Using this proposition and Muller's theorem, we obtain the description of projective modules over a semiperfect ring.

Theorem 4.6. *Any indecomposable projective module over a semiperfect ring A is finitely generated, and has exactly one maximal submodule. There is one-to-one correspondence between pairwise nonisomorphic indecomposable projective A-modules $P_1, \ldots, P_s$ and mutually nonisomorphic simple A-modules? which are given by the following correspondence: $P_i \to P_i/P_iR = U_i$ and $U_i \to P(U_i)$.*

Remark. A submodule N of a module M is called small if the equality $N + X = M$ implies $X = M$ for any submodule X of M.

A projective module P is called a projective cover of M and it is denoted by $P(M)$ if there is an epimorphism $\varphi : P \to M$ such that $Ker\, \varphi$ is a small submodule in P. The following is a method of constructing a projective cover for an arbitrary finitely generated module M over a semiperfect ring A. Let M - be a finitely generated A-module. Clearly, M/MR is a module over the semisimple Artinian ring $\bar{A} = A/R$. Therefore $M/MR \simeq \bigoplus\limits_{j=1}^{s} U_j^{m_j}$, where $U_1, \ldots, U_s$ are mutually nonisomrphic simple $\bar{A}$-modules. Lifting the idempotents we obtain that $U_j = e_jA/e_jR$, where $e_j^2 = e_j$. Denote $P_j = e_jA$. Then the projective cover $P(M)$ of M is $P(M) = \bigoplus\limits_{j=1}^{s} P_j^{m_j}$.

5. Quiver of a semiperfect ring

Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring A into a direct sum of pairwise nonisomorphic indecomposable projectives. Let $P = P_1 \oplus \ldots \oplus P_s$ and $B = End_A\, P$. By the Morita theorem, the category of right A-modules is equivalent to the category of right B-modules. Obviously, $B/rad\, B$ is a direct product of division rings. The ring B is called the basic ring of the ring A.

Let A be a semiperfect right Noetherian ring and $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition A as above. Let P_iR is the unique maximal submodule of P_i. Consider the projective cover $P(P_iR)$ of P_iR. Let $P(P_iR) = \bigoplus_{j=1}^{s} P_j^{t_{ij}}$ $(i = 1, \dots, s)$. We assign to the modules $P_1, \dots, P_s$ points $1, \dots, s$ on the plane and join point i with j by t_{ij} arrows. The so constructed graph is called the right quiver of the semiperfect right Noetherian ring A and will be denoted by $Q(A)$.

Analogously, one can define the left quiver $Q'(A)$ of a left Noetherian semiperfect ring.

Note, that the quiver of a semiperfect right Noetherian ring does not change by switching to its basic ring and $Q(A) = Q(A/R^2)$.

Definition 5.1. *Let A be a semiperfect ring such that A/R^2 is a right Artinian ring. The quiver of the ring A/R^2 is the quiver of the ring A and is denoted by $Q(A)$.*

The following proposition gives the description of the Jacobson radical of semiperfect ring A (see [10, Chapter 11]).

Proposition 5.2. *Let $A = P_1^{n_1} \oplus ... \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring A into a direct sum of principal right A-modules and let $1 = f_1 + ... + f_s$ be a corresponding decomposition of the identity of A into a sum of pairwise orthogonal idempotents, i.e., $f_iA = P_i^{n_i}$. Then the Jacobson radical of the ring A has a two-sided Peirce decomposition of the following form:*

$$R = \begin{pmatrix} R_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & R_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & R_{nn} \end{pmatrix},$$

where $R_{ii} = rad(f_iAf_i)$, $A_{ij} = f_iAf_j$ for $i, j = 1, ..., n$.

Lemma 5.3 (Annihilation lemma)**.** *Let $1 = f_1 + \dots + f_s$ be a canonical decomposition of $1 \in A$. For every simple right A-module U_i and for each f_j we have $U_if_j = \delta_{ij}U_i$, $i, j = 1, \dots, s$. Similarly, for every simple left A-module V_i and for each f_j, $f_jV_i = \delta_{ij}V_i$, $i, j = 1, \dots, s$.*

Lemma 5.4 (Q-Lemma)**.** *The simple module U_k (resp. V_k) appears in the direct sum decomposition of the module e_iR/e_iR^2 (resp. Re_i/R^2e_i) if and only if $e_iR^2e_k$ (resp. $e_kR^2e_i$) is strictly contained in e_iRe_k (resp. e_kRe_i).*

6. Quasi-Frobenius rings

Definition 6.1. *Let M be a right A-module. The socle of M ($soc\, M$) is the sum of all simple right submodules of M. If there are no such submodules, then $Soc\, M = 0$.*

If $M = A_A$, then $soc\, A_A$ is the sum of all minimal right ideals of A and it is a right ideal of A. If $\mathcal{J}$ a minimal right ideal in A, then for any $x \in A$ either $x\mathcal{J} = 0$ or $x\mathcal{J}$ is a minimal right ideal in A. Therefore, $soc\, A_A$ is a two-sided ideal in A. Analogously we can consider $soc_A A$. However $soc\, A_A \neq soc_A A$ in general. Let $A = T_2(k) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in k \right\}$, where k is a field. Obviously, $soc\, A_A = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & \gamma \end{pmatrix} \right\}$ and $soc\, {}_A A = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \right\}$. For semisimple module M we have $soc\, M = M$.

Let A be an Artinian ring and $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ (resp. ${}_A A = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$) be the canonical decomposition of A into a direct product of right (left) principal modules.

Let M be a right A-module and N be a left A-module. We set $top\, M = M/MR$ and $top\, N = N/RN$. Now we give a definition of a Nakayama permutation.

Definition 6.2. *We say that a two-sided Artinian ring A admits a Nakayama permutation $\nu(A) : i \to \nu(i)$ of $\{1, \ldots, s\}$ if the following conditions are satisfied*

(1) $soc\, P_k = top\, P_{\nu(k)}$

(2) $soc\, Q_{\nu(k)} = top\, Q_k$

Definition 6.3. *A two-sided Artinian ring A is called quasi-Frobenius, if A admits a Nakayama permutation $\nu(A)$ of $\{1, \ldots, s\}$. A quasi-Frobenius ring is called Frobenius if $n_{\nu(i)} = n_i$ for all $i = 1, \ldots, s$.*

The following theorem is a variant of the Morita theorem for semiperfect rings.

Theorem 6.4. *Let A be a semiperfect ring and $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of A_A into a direct sum of pairwise nonisomorphic sight ideals. Let $B = End\,(P_1 \oplus \ldots \oplus P_s)$ be the ring of endomorphisms of the module $P = P_1 + \ldots + P_s$. Then the categories of A-modules and B-modules are equivalent.*

Denote by $rad\, B$ the Jacobson radical of the ring B. Obviously, $B/rad\, B$ is a direct product of skew-fields. The ring B is called a basic ring of the ring A. If a ring A is quasi-Frobenius, B is Frobenius.

Definition 6.5. *An indecomposable projective right module over a semi-perfect ring A is called principal right module. Analogously, a principal left module can be defined.*

7. Examples

7.1.

Definition 7.1. *An A-lattice M is said to be relatively injective if $M \simeq_A P^*$, where ${}_AP$ is a finitely generated projective left A-module. An A-lattice M is called completely decomposable if it is a direct sum of irreducible A-lattices.*

Denote by Δ the completely decomposable left A-lattice A_A^*.

Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ be a reduced exponent matrix.

Let

$$\mathcal{E}(A) = \begin{pmatrix} 0 & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & 0 & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & 0 \end{pmatrix}.$$

Lemma 7.2. *A completely decomposable left A-lattice Δ is complete decomposable right A-lattice, and*

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -\alpha_{21} & \dots & -\alpha_{n1} \\ -\alpha_{12} & 0 & \dots & -\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \dots & 0 \end{pmatrix}.$$

Let us show that the k-th row $(-\alpha_{1k}, -\alpha_{2k}, \dots, -\alpha_{nk})$ of the matrix $\mathcal{E}(\Delta)$ defines an irreducible right A-lattice. Let $\beta_i = -\alpha_{ik}$. We can rewrite the inequality $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ in the form $-\alpha_{ik} + \alpha_{ij} \geqslant -\alpha_{jk}$, i.e., $\beta_i + \alpha_{ij} \geqslant \beta_j$, which implies the assertion of the lemma.

Corollary 7.3. *A fractional ideal Δ is a relatively injective right and a relatively injective left A-lattice.*

Let A be a reduced tiled order and $R = rad\, A$. Then

$$\mathcal{E}(R) = \begin{pmatrix} 1 & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & 1 & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & 1 \end{pmatrix}$$

and

$$\mathcal{E}(R^*_A) = \mathcal{E}({}_AR^*) = \begin{pmatrix} -1 & -\alpha_{12} & \dots & -\alpha_{1n} \\ -\alpha_{21} & -1 & -\dots & -\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & -\dots & 1 \end{pmatrix}$$

write $X =_A R^*$, $\Delta = (A_A)^*$.

It is true the following lemma.

Lemma 7.4. *For $i = 1, \dots, n$ we have that $e_{ii}X(Xe_{ii})$ is the unique minimal overmodule of $e_{ii}A(\Delta e_{ii})$ and $e_{ii}X/e_{ii}\Delta = U_i$, $Xe_{ii}/\Delta e_{ii} = V_i$, where U_i is simple right A-module and V_i is a simple left A-module.*

Lemma 7.5. *Every relatively injective irreducible A-lattice Q has only one minimal overmodule. Let Q_1 and Q_2 be relatively injective irreducible A-lattices and let $X_1 \supset Q$ and $X_2 \supset Q_2$ be the unique minimal overmodules of Q_1 and Q_2 respectively. Then the simple A-modules X_1/Q_1 and X_2/Q_2 are isomorphic if and only if $Q_1 \simeq Q_2$.*

Proposition 7.6. *An irreducible A-lattice is relatively injective if and only if it has exactly one minimal overmodule.*

Consider the tiled order

$$H_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi\mathcal{O} & \pi\mathcal{O} & \dots & \mathcal{O} \end{pmatrix},$$

i.e., $H = H_n(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(H_n(\mathcal{O}))\}$, where

$$\mathcal{E}(H_n(\mathcal{O})) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 0 \end{pmatrix},$$

i.e., $\alpha_{ij} = 0$ for $i \leqslant j$ and $\alpha_{ij} = 1$ for $i > j$; $i, j = 1, \ldots, n$.

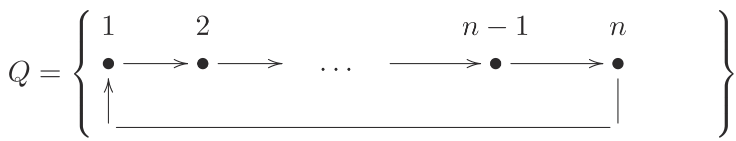
Let $\Delta = H_H^*$. Obviously,

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -1 & \ldots & -1 \\ 0 & 0 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Let $M_n(\mathbb{Z})$ be the ring of $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $U_n = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} \in M_n(\mathbb{Z})$ and all elements of U_n are $1 \in \mathbb{Z}$. Consider

$$\mathcal{E}(\Delta) + U_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Obviously, $\mathcal{E}(\Delta) + U_n$ is the serial ring with the quiver



The quiver Q is the simple cycle $(1\,2\,\ldots\,n)$ and the adjacency matrix $[Q]$ is the permutation matrix which corresponds to the simple cycle $(1\,2\,\ldots\,n)$. Obviously, $(1\,2\,\ldots\,n) = \sigma^{-2}$, where σ is the permutation for Gorenstein matrix $\mathcal{E}(H_n(\mathcal{O}))$. Moreover, the quotient ring H_n/R^2 is the Frobenius ring with Nakayama permutation $\nu = (1\,2\,\ldots\,n)$ (the simple cycle σ^{-1}).

7.2.

Let $\mathcal{O}$ be a discrete valuation ring with the unique maximal ideal $\mathfrak{M} = \pi\mathcal{O} = \mathcal{O}\pi$ and let

$$K_n = K_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathfrak{M} & \ldots & \mathfrak{M} & \mathfrak{M} \\ \mathfrak{M} & \mathcal{O} & \ldots & \mathfrak{M} & \mathfrak{M} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \mathcal{O} & \mathfrak{M} \\ \mathfrak{M} & \mathfrak{M} & \ldots & \mathfrak{M} & \mathcal{O} \end{pmatrix}$$

be a tiled order. Obviously,

$$\mathcal{E}_n = \mathcal{E}({}_{K_n}K_n^*) = \mathcal{E}(K_n{}^*_{K_n}) = \begin{pmatrix} 0 & -1 & \dots & -1 & -1 \\ -1 & 0 & \dots & -1 & -1 \\ \dots & \dots & \dots & \dots & \\ -1 & -1 & \dots & 0 & -1 \\ -1 & -1 & \dots & -1 & 0 \end{pmatrix}$$

and

$$U_n + \mathcal{E}_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Denote $e_i = (0, \dots 0, 1, 0, \dots 0)$ (1 is on the i-th place). By lemmas 7.4 and 7.5 irreducible A-lattices Q_i with $\mathcal{O}$-basis $(\mathcal{O}, \mathcal{O}, \dots, \mathcal{O}, \pi\mathcal{O}, \mathcal{O}, \dots, \mathcal{O})$ ($\pi\mathcal{O}$ is on the i-th place) are relatively injective and has exactly one minimal overmodule $(\mathcal{O}, \mathcal{O}, \dots, \mathcal{O}, \mathcal{O}, \mathcal{O}, \dots, \mathcal{O})$. Let $\sigma : i \to \sigma(i)$ be a permutation of $\{1, 2, \dots, n\}$. Denote by $I_{n,m} = (\mathfrak{M}^{m_{ij}})$ the two-sided ideal of $K_n(\mathcal{O})$, where $w_{i\sigma(i)} = m + 1$, $w_{ij} = m$, $j \neq \sigma(i)$, $i, j = 1, \dots, n$.

Obviously, $I_{m,n}$ is relatively injective right and left A-lattice and Nakayama permutation of Frobenius quotient ring $K_n(\mathcal{O})/I_{m,n}$ coincides with σ.

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