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RESEARCH ARTICLE

On the condensation property of the Lamplighter groups and groups of intermediate growth[1]

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Abstract. The aim of this short note is to revisit some old results about groups of intermediate growth and groups of the lamplighter type and to show that the Lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z}$ is a condensation group and has a minimal presentation by generators and relators. The condensation property is achieved by showing that L belongs to a Cantor subset of the space $\mathcal{M}_2$ of marked 2-generated groups consisting mostly of groups of intermediate growth.

1. Introduction

The modern development of group theory requires significant use of methods of geometry, topology, probability and measure theory, the theory of models etc. The space $\mathcal{M}_k$ of marked k-generated groups, introduced in [Gri84] plays an important role in this development. It is a compact totally disconnected metrizable space and it is important to know which groups belong to its perfect kernel (or condensation part), which is homeomorphic to a Cantor set. Groups in the perfect kernel are called *condensation groups*. The aim of this note is to revisit some results of [Gri84] and to use them to show that the so called Lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z}$, which is a popular object of study (see for example [GŻ01, GK12]) belongs to a Cantor set and hence is a condensation group.

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Let $\Omega = \{0,1,2\}^{\mathbb{N}}$ be the set of all infinite sequences over $\{0,1,2\}$ with the product (Tychonoff) topology.

Main Theorem. *There exists a subset* $\mathcal{L} = \{(L_\omega, T_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_2$ *with the following properties:*

a) *$\mathcal{L}$ is homeomorphic to Ω (and hence is a Cantor set),*

b) *If $\omega \in \Omega$ is not eventually constant, then L_ω has intermediate growth.*

c) *If $\omega \in \Omega$ is a constant sequence then $L_\omega \cong L$.*

d) *All groups in $\mathcal{L}$ are condensation groups.*

A simple argument shows that a group possessing an infinite minimal presentation is a condensation group. Surprisingly, it was observed in [BCGS14] that there are finitely generated groups which do not have a minimal presentation. It follows from [Bau61] that groups of the form $H \wr G$ where H and G are infinite and finitely generated are not finitely presented and it was observed in [Cor11] that such groups are condensation groups. It is probably well known (as indicated in [BCGS14]) that the standard presentation

$$L = \left\langle s, t \mid s^2, [s, s^{t^i}]\ i \geqslant 1 \right\rangle$$

is minimal. A proof of this fact using ideas of [Bau61] is presented for completeness. This provides an alternative proof of the fact that L is a condensation group.

An effective way to build large families of condensation groups is to construct closed subsets $X \subset \mathcal{M}_k, k \geqslant 2$ homeomorphic to a Cantor set. Such families were constructed in [Gri84, Gri85, Cha00, Nek07]. It will be interesting to produce such families based on new ideas.

2. Preliminaries

For a topological space X, let X' denote its set of accumulation points. For any ordinal α define the spaces $X^{(\alpha)}$ inductively as follows: $X^{(0)} = X, X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$ and $X^{(\lambda)} = \bigcap_{\beta<\lambda} X^{(\beta)}$ if λ is a limit ordinal. If X is a Polish space, (i.e., a completely metrizable, separable space) for some countable ordinal α_0 we will have $X^{(\alpha_0)} = X^{(\alpha)}$ for all $\alpha \geqslant \alpha_0$ (see [Kec95, Theorem 6.1]). The least ordinal with this property is called the

Cantor-Bendixon rank of X and will be denoted by $rk_{CB}(X)$. The set $X^{(\alpha_0)}$ is called the *perfect kernel* (or condensation part) of X which will be denoted by $\kappa(X)$. Note that if nonempty, $\kappa(X)$ is homeomorphic to a Cantor set and $\kappa(X)$ is empty if and only if X is countable. Points in $\kappa(X)$ are called condensation points and can be characterized as points for which every open neighborhood is uncountable (see [Kec95, I.6]).

Let $\mathcal{M}_k$ denote the space marked groups consisting of pairs (G, S) where G is a group and S is an ordered set of (not necessarily distinct) set of k generators. Two marked groups (G, S) and (H, T) in $\mathcal{M}_k$ are identified whenever the map $s_i \mapsto t_i, i = 1, \ldots, k$ extends to an isomorphism. Two points (G, S) and (H, T) are of distance $\leqslant 2^{-N}$ if the Cayley graphs of (G, S) and (H, T) have isomorphic balls of radius N. This (ultra) metric makes $\mathcal{M}_k$ into a compact, totally disconnected, separable space. It follows from the definition that a sequence $(G_n, S_n) \in \mathcal{M}_k$ converges to $(G, S) \in \mathcal{M}_k$, if and only if, for every element $w \in F_k$ (the free group of rank k), there exists $N = N_w \geqslant 0$, such that the the relation $w = 1$ holds in G if and only if it holds in G_n for $n \geqslant N$.

An important problem of geometric group theory (raised in [Gri05]) is the identification of $rk_{CB}(\mathcal{M}_k)$ for $k \geqslant 2$. It follows from [Cor11] that the lower bound $rk_{CB}(\mathcal{M}_k) > \omega^\omega, k \geqslant 2$ holds. By a classical result of B.H. Neumann [Neu37] there exists uncountably many non-isomorphic 2-generated groups. Therefore $\kappa(\mathcal{M}_k)$ is a Cantor set for all $k \geqslant 2$. A finitely generated group G is called a *condensation group*, if for some generating set S of size k the pair (G, S) belongs to $\kappa(\mathcal{M}_k)$. It follows that this property does not depend on the generating set (see [dCGP07, Lemma 1]).

In [Gri84] the second author constructed Cantor sets $\mathcal{G} \subset \mathcal{M}_k$ consisting essentially of groups of intermediate growth. Clearly, groups belonging to these families lie in the condensation part of $\mathcal{M}_k$. In general, it is a challenging problem to identify which groups are in the condensation part. It is expected that every group of intermediate growth is a condensation group. In contrast, it is easy to observe that virtually nilpotent groups are not condensation. In [Cor11, BCGS14] condensation properties of metabelian groups were considered and it was proven that restricted wreath products $H \wr G$ of two finitely generated infinite groups are condensation groups [Cor11, Proposition 8.1]. Also, by [Cha00] every non-elementary hyperbolic groups is a condensation group.

Let us briefly recall the groups constructed in [Gri84]. Although the original definition is in terms of measure preserving transformations of the unit interval, we will give here a definition in terms of automorphisms of rooted trees. Let Ω denote the set all infinite sequences over the alphabet

$\{0,1,2\}$. We identify Ω with the product $\{0,1,2\}^{\mathbb{N}}$ and endow it with the product topology. Let $\tau : \Omega \to \Omega$ be the shift transformation, i.e., $\tau(\omega)_n = \omega_{n+1}$. For each $\omega \in \Omega$ we will define a subgroup G_ω of $Aut(\mathcal{T}_2)$, where the latter denotes the automorphism group of the binary rooted tree $\mathcal{T}_2$ whose vertices are identified with the set of finite sequences $\{0,1\}^*$. Each group G_ω is the subgroup generated by the four automorphisms denoted by $a, b_\omega, c_\omega, d_\omega$ whose actions onto the tree is as follows.

For $v \in \{0,1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$\begin{array}{llllll} b_\omega(0v) = & 0\beta(\omega_1)(v) & c_\omega(0v) = & 0\zeta(\omega_1)(v) & d_\omega(0v) = & 0\delta(\omega_1)(v) \\ b_\omega(1v) = & 1b_{\tau(\omega)}(v) & c_\omega(1v) = & 1c_{\tau\omega}(v) & d_\omega(1v) = & 1d_{\tau\omega}(v), \end{array}$$

where

$$\begin{array}{lll} \beta(0) = a & \beta(1) = a & \beta(2) = e \\ \zeta(0) = a & \zeta(1) = e & \zeta(2) = a \\ \delta(0) = e & \delta(1) = a & \delta(2) = a \end{array}$$

and e denotes the identity.

Note that from the definition, the following relations are immediate:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e$$

Denoting by $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$, we obtain a subset $\{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$. In [Gri84] it was observed that this subset is not closed. It was also shown in [Gri84] that modifying countably many groups in this family, one obtains a closed subset $\mathcal{G} = \{(G_\omega, S_\omega) \mid \omega \in \Omega\}$ with the following properties:

Theorem 1 ([Gri84]).

1) *$\mathcal{G}$ is homeomorphic to Ω via the map $\omega \mapsto (G_\omega, S_\omega)$,*

2) *If in $\omega \in \Omega$ all symbols $\{0,1,2\}$ appear infinitely often, then G_ω is a 2-group,*

3) *For $\omega \in \Omega$ which is not eventually constant (i.e., is not constant after some point), G_ω has intermediate growth,*

4) *If $\omega \in \Omega$ is eventually constant, then G_ω is virtually metabelian of exponential growth.*

3. Proof of the Main Theorem

We start with the following basic lemma:

Lemma 1. *Suppose that $\{(G_n, S_n)\}$ is a sequence in $\mathcal{M}_k$ converging to (G, S). Let F_k be the free group of rank k, with basis $\{x_1, \dots, x_k\}$ and let $\pi : F_k \to G$, $\pi_n : F_k \to G_n$ be the canonical maps. Given $w_1, \dots, w_m \in F_k$, let $T = \{\pi(w_1), \dots, \pi(w_m)\}$, $T_n = \{\pi_n(w_1), \dots, \pi_n(w_m)\}$ and $H = \langle T \rangle \leqslant G, H_n = \langle T_n \rangle \leqslant G_n$. Then the sequence $\{(H_n, T_n)\}$ converges to (H, T) in $\mathcal{M}_m$.*

Proof. Let F_m be the free group of rank m with basis $\{y_1, \dots, y_m\}$ and let $\gamma : F_m \to H$ and $\gamma_n : F_k \to H_n$ be the canonical maps. Also, let $p : F_m \to F_k$ be the group homomorphism defined by $p(y_i) = w_i$, $i = 1, \dots, m$. Note that we have the following:

$$\gamma_n = \pi_n \circ p \quad \text{for every} \quad n$$

and

$$\gamma = \pi \circ p$$

It follows that, given $w \in F_m$, $w = 1$ in H if and only if $p(w) = 1$ in G. This shows that the sequence $\{(H_n, T_n)\}$ converges to (H, T) in $\mathcal{M}_m$. $\square$

The following is a description of the structure of the group $G_{000\dots}$.

Theorem 2. *The group $G_{000\dots}$ is isomorphic to the group $L \rtimes \mathbb{Z}_2$ where $L = \mathbb{Z}_2 \wr \mathbb{Z}$ is the Lamplighter group given by presentation $\left\langle s, t \mid s^2, [s, s^{t^i}], i \geqslant 1 \right\rangle$ and $\mathbb{Z}_2$ acts on L by the automorphism*

$$s \mapsto s^t$$

$$t \mapsto t^{-1}$$

Proof. Let us denote $G_{000\dots}$ by G and denote its canonical generators by a, b, c, d. Let H be the subgroup of G generated by the elements b, c, d, b^a, c^a, d^a. There exists an embedding (see [Gri84])

$$\begin{array}{ccc}
\psi : H & \to & G \times G \\
b & \mapsto & (a, b) \\
c & \mapsto & (a, c) \\
d & \mapsto & (1, d) \\
b^a & \mapsto & (b, a) \\
c^a & \mapsto & (c, a) \\
d^a & \mapsto & (d, 1)
\end{array}$$

Let $D = \langle\langle d \rangle\rangle$ be the normal closure of d in G. By induction on word length one can see that D is an abelian group (see [Gri84, Lemma 6.1]).

We claim that $D = \langle d^g \mid g \in \langle a, b \rangle\rangle$. Let us denote the right hand side by T. Clearly T is contained in D. It suffices to show T is normal. Since $bcd = 1$ it is enough to show that $(d^g)^c \in T$ for all $g \in \langle a, b\rangle$. By induction on k one can see that the following equality holds:

$$\psi(d^{(ab)^n}) = \begin{cases} (1, d^{(ab)^k}) & , n = 2k \\ (d^{(ab)^k a}, 1) & , n = 2k+1 \end{cases}$$

We will show by induction on $|g|$ that $(d^g)^c = (d^g)^b$. Suppose $|g| = 1$, the case $g = b$ is obvious since $bcd = 1$. If $g = a$, we have $\psi((d^a)^b) = \psi((d^a)^c) = (d^a, 1)$ and hence $(d^g)^c = (d^g)^b$. Now assume $|g| > 1$. Since $d^b = d$ we can assume that g starts with a. There are two cases, either $g = (ab)^n$ or $g = (ab)^n a$ for some n. In the first case (using induction assumption)

$$\psi((d^{(ab)^n})^c) = \begin{cases} (1, (d^{(ab)^k})^c) = (1, (d^{(ab)^k})^b) & , n = 2k \\ ((d^{(ab)^k a})^a, 1) = (d^{(ab)^k}, 1) & , n = 2k+1 \end{cases}$$

and in the second case

$$\psi((d^{(ab)^n a})^c) = \begin{cases} ((d^{(ab)^k})^a, 1) & , n = 2k \\ (1, (d^{(ab)^k a})^c) = (1, (d^{(ab)^k})^b) & , n = 2k+1 \end{cases}$$

In any case $\psi((d^g)^c) = \psi((d^g)^b)$. This shows T is normal and hence $D = T$.

Now letting

$$t_n = \begin{cases} d^{(ab)^n} & n \geq 0 \\ d^{(ab)^{-n-1} a} & n < 0 \end{cases}$$

we see that $T = \langle t_n \mid n \in \mathbb{Z}\rangle$. Looking at $\psi(t_n)$ we see that the t_n are mutually distinct, therefore $T \cong \prod_{\mathbb{Z}} \mathbb{Z}_2$.

Since $\psi((ab)^2) = (ba, ab)$, it follows that the element ab is of infinite order in G.

We will show that the subgroups D and $\langle ab \rangle$ intersect trivially. Suppose not, then $d^g = (ab)^n$ for some $g \in \langle a, b\rangle$ and $n \in \mathbb{Z}$. Necessarily n has to be even since left hand side of $d^g = (ab)^n$ has even number of a's. If $n = 2k$ then $\psi((ab)^{2k}) = ((ba)^k, (ab)^k)$ whereas $\psi(d^g) = (d^h, 1)$ or $(1, d^h)$ for some element $h \in G$. It follows that $(ab)^k = 1$ which is a contradiction since ab has infinite order.

Now the subgroup $K = D \rtimes \langle ab \rangle$ is isomorphic to $(\mathbb{Z}_2^{\infty}) \rtimes \mathbb{Z} \cong \mathbb{Z}_2 \wr \mathbb{Z}$ which is the Lamplighter group. This is true since we have

$$t_n^{ab} = t_{n+1}, \quad n \in \mathbb{Z}$$

and hence the generator $\langle ab \rangle$ acts on D by shifting its generators.

Conjugating the generators of K by the generators of G we see that K is a normal subgroup. The quotient G/D is isomorphic to the infinite dihedral group D_∞ (see [Gri84, Lemma 6.1]) and maps onto the quotient G/K. The kernel of this homomorphism contains the image of ab in G/D. From this It follows that K has index 2 in G. Hence we have $G = K \rtimes \langle a \rangle \cong L \rtimes \mathbb{Z}_2$. Identifying $s = d, t = ab$ we see that conjugation by a gives the asserted automorphism of K. □

For $\omega \in \Omega$ let $L_\omega = \langle d_\omega, ab_\omega \rangle \leqslant G_\omega$. By virtue of the relations $a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = 1$ we see that L_ω is a normal subgroup of index 2 in G_ω and hence share many properties with G_ω. Let us denote by $T_\omega = \{d_\omega, ab_\omega\}$ and $\mathcal{L}_\omega = \{(L_\omega, T_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_2$.

Proof of the main Theorem:

a) Consider the map $\phi : \Omega \to \mathcal{L}$ given by $\omega \mapsto (L_\omega, T_\omega)$. ϕ is continuous since, if w_n converges to w, then by Theorem 1 $(G_{\omega_n}, S_{\omega_n})$ converges to (G_ω, S_ω) and hence by Lemma 1 $(L_{\omega_n}, T_{\omega_n})$ converges to (L_ω, T_ω). To see that ϕ is injective: By [Gri84, Section 5], the following is true: Given $\omega_1 \neq \omega_2$ in Ω, there exists $u \in F_4$ (depending on ω_1 and ω_2), such that u is trivial in G_{ω_1} and nontrivial in G_{ω_2} (this amounts to saying that the map $a \mapsto a, b_{\omega_1} \mapsto b_{\omega_2}, c_{\omega_1} \mapsto c_{\omega_2}, d_{\omega_1} \mapsto d_{\omega_2}$ does not extend to an isomorphism from G_{ω_1} to G_{ω_2} i.e., $(G_{\omega_1}, S_{\omega_1})$ and $(G_{\omega_2}, S_{\omega_2})$ are distinct points in $\mathcal{M}_4$). One observes that such u is a 2 power and (since L_ω has index 2 in G_ω) its image in G_ω lies in L_ω. Therefore the image of u in L_{ω_1} is trivial but its image in L_{ω_2} is nontrivial which implies that $(L_{\omega_1}, T_{\omega_1})$ and $(L_{\omega_2}, T_{\omega_2})$ are distinct. This shows that ϕ is injective and by compactness we have that ϕ is a homeomorphism.

b) This follows from Theorem 1 and the fact that L_ω has finite index in G_ω.

c) By Theorem 2 $L_{000\ldots}$ is isomorphic to L and it is immediate from the definition of the groups that $G_{000\ldots} \cong G_{111\ldots} \cong G_{222\ldots}$ and $L_{000\ldots} \cong L_{111\ldots} \cong L_{222\ldots}$.

d) This follows from part a).

As a corollary we obtain the following:

Corollary 1. *The Lamplighter group L is a condensation group.*

4. Minimal presentations of the Lamplighter groups

For a subset $A \subset G$ of a group let $\langle\langle A\rangle\rangle$ denote the normal subgroup generated by A. A presentation $\langle X \mid R\rangle$ is called *minimal* if for every $r \in R$ we have $r \notin \langle\langle R \setminus \{r\}\rangle\rangle$. The following is well known.

Proposition 1. *Let $\langle X \mid r_1, r_2, \ldots\rangle$ be an infinite minimal presentation where $|X| = k$. Then the marked group (G, X) lies in the condensation part of $\mathcal{M}_k$.*

Proof. It is enough to show that any open ball around (G, X) is uncountable. Let $B = B((G, X), 2^{-N})$ be a ball of radius 2^{-N} around (G, X). A marked group $(H, T) \in \mathcal{M}_k$ lies in B if and only if for all $w \in F_k$ such that $|w| \leqslant 2N + 1$, we have $w = 1$ in $G \iff w = 1$ in H. Let $A = \{w \in F_k \mid |w| \leqslant 2N + 1 \text{ and } w = 1 \text{ in } G\}$. Choose $M = M(N) \in \mathbb{N}$ large enough so that $A \subset \langle\langle r_1, r_2, \ldots, r_M\rangle\rangle$. For any subset $U \subset \mathbb{N}$ such that $\{1, 2, \ldots, M\} \subset U$, let (G_U, X) be the group $\langle X \mid r_i, i \in U\rangle$. Clearly all $(G_U, X) \in B$ and since the initial presentation is minimal all of them are distinct marked groups. Hence B is uncountable. □

We will give an alternative proof of Corollary 1 by showing that the standard presentation of L is minimal.

For a group G and a subset $S \subset G$ let

$$T_S = \{(s_1 g, s_2 g) \mid s_1, s_2 \in S, g \in G\} \subset G \times G$$

Theorem 3 ([Bau61]). *Let G and H be two groups and $S \subset G$ be a subset. Then there exists a group $W = W(H, G, S)$ (called the circle product of G and H with respect to S) with the following properties:*

- *W contains subgroups $H_g, g \in G$ all isomorphic to H,*
- *W is generated by G and H_1,*
- *The subgroup $K = \langle H_g \mid g \in G\rangle$ is normal in W and $W = K \rtimes G$,*
- *For $h_{g_1} \in H_{g_1}$ and $g_2 \in G$ we have $h_{g_1}^{g_2} \in H_{g_1 g_2}$,*
- *$[H_{g_1}, H_{g_2}] = 1$ if and only if $(g_1, g_2) \in T_S$.*

Note that W can also be realized by using graph products: Let Γ be the graph with vertex set G and edges T_S, and let K be the graph product where each vertex group is H. Clearly G acts on K and one can see that $W \cong K \rtimes G$.

Proposition 2. *For every $n \geqslant 2$, the presentation*

$$\left\langle s, t \mid s^n, [s, s^{t^i}]\ i \geqslant 1 \right\rangle$$

is a minimal presentation of $\mathbb{Z}_n \wr \mathbb{Z}$.

Proof. Clearly the relation s^n is not redundant. For $i \geqslant 1$ let $r_i = [s, s^{t^i}]$ and suppose that for some $m \geqslant 1$ r_m is redundant. Let

$$a_0 = 0 \ , \ a_{2j} = j(m+1) + (1 + \ldots + j) \quad j \geqslant 1$$

and

$$a_{2j+1} = \begin{cases} a_{2j} + j + 1 & \text{if} \quad j < m-1 \\ a_{2j} + j + 2 & \text{if} \quad j \geqslant m-1 \end{cases} \quad , \ j \geqslant 0 \ .$$

Note that

$$a_{2j+1} - a_{2j} = \begin{cases} j + 1 & \text{if} \quad j < m-1 \\ j + 2 & \text{if} \quad j \geqslant m-1 \end{cases} \quad , \ j \geqslant 0$$

and

$$|a_k - a_\ell| > m \ \text{ if } \ |k - \ell| \geqslant 2.$$

Finally let $S = \{a_0, a_1, a_2, \ldots\} \subset \mathbb{Z}$ and observe that the set $S - S = \mathbb{Z} \setminus \{-m, m\}$. Form the circle product $W = W(\mathbb{Z}_n, \mathbb{Z}, S)$ with generators x, y. By the properties of W we have for $i \geqslant 1$

$$[x, x^{y^i}] = 1 \iff (0, i) \in T_S \iff i \in S - S \iff i \neq m.$$

Therefore, under the assumption that r_m is redundant in $\mathbb{Z}_n \wr \mathbb{Z}$, the map $s \mapsto x, t \mapsto y$ defines a homomorphism from $\mathbb{Z}_n \wr \mathbb{Z}$ to W which contradicts the fact that $r_m = 1$ in $\mathbb{Z}_n \wr \mathbb{Z}$ but $[x, x^{y^m}] \neq 1$ in W. □

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