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The algorithms that recognize Milnor laws and properties of these laws

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Abstract. We consider several equivalent definitions of the so-called Milnor laws (or Milnor identities) that is the laws which are not satisfied in $\mathfrak{A}_p\mathfrak{A}$ varieties. The purpose of this article is to provide algorithms that allow us to check whether a given identity $w(x, y)$ has one of the following properties:

- $w(x, y)$ is a Milnor law,
- every nilpotent group satisfying $w(x, y)$ is abelian,
- every finitely generated metabelian group satisfying $w(x, y)$ is finite-by-abelian.

1. Introduction

Many authors considered a special type of group laws using different names. All these laws have a common property which fully characterizes them. Namely, every solvable group satisfying such a law is nilpotent-by-(locally finite of finite exponent). We call these laws the Milnor laws after Point [31].

Let $F_2 = \langle x, y\rangle$ be a free group and $w := w(x, y)$ be a word in F_2. We speak of a group law w, meaning the law $w \equiv 1$.

The aim of this work is to provide algorithms that allow us to check whether a given word defines a Milnor law, and whether it satisfies the additional properties.

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If $\langle x\rangle^{F_2}$ denotes the normal closure of x and $[x,y] = x^{-1}y^{-1}xy = x^{-1}x^y = x^{-1+y}$, then every word in F_2' is conjugate to a word $x^{P(y)}\xi$ where $\xi \in (\langle x\rangle^F_2)'$, and $P(y)$ is a polynomial over integers.

All polynomials considered here have integer coefficients. Recall that such a polynomial is called **primitive** if it has coprime coefficients.

Point [31] introduced a Milnor identity $w(x,y)$ as the one which implies a law of the form $x^{P(y)}$, where the polynomial $P(y)$ is primitive. She used this name since groups satisfying such laws satisfy also the property considered by Milnor [21]. Later Endimioni [4] proved that such a property holds if and only if the law is not satisfied in a variety $\mathfrak{A}_p\mathfrak{A}$ for any prime p.

Unfortunately, for a given law the implication of a law $x^{P(y)}$ with primitive $P(y)$ is not constructive. For example, the laws $[x,y]^2 = x^{(-1+y)2}$ and $[x^2,y^2] = x^{-2+2y^2}$, written as $x^{P(y)}$, both have non-primitive $P(y)$ with even coefficients. However, only the second law is a Milnor law, because it is a positive law, so it is not satisfied in any of $\mathfrak{A}_p\mathfrak{A}$, since the wreath product $C_p \, wr \, C$, generating this variety contains a free subsemigroup [1] (the law $[x^2,y^2]$ is discussed in Example 6.4).

Results

Every word $w(x,y) \in F_2'$ can be written in the form $[x,y]^{P(x,y)}\xi$, where $P(x,y)$ is a polynomial with integer coefficients and $\xi \in F_2''$. Our main aim is to give algorithms which allow us to recognize properties of the law $w(x,y)$. We give:

- An algorithm for writing $w(x,y)$ in the form $[x,y]^{P(x,y)}\xi$ and a criterion for $P(x,y)$ which allows us to check whether the word defines a Milnor law.
- Three algorithms for writing $w(x,y)$ as a product $\prod_{kl}[x,_k y,_l x]^{b_{kl}}\zeta$ and a criterion for b_{kl}'s which allows us to check whether the word defines a Milnor law.
- A condition for $P(x,y)$ such that every nilpotent group satisfying the law $[x,y]^{P(x,y)}\xi$ is abelian.
- A condition for $P(x,y)$ such that every solvable group satisfying the law $[x,y]^{P(x,y)}\xi$ is (locally finite)-by-abelian.

Notations

All notations concerning varieties of groups can be found in [23]. For example $\mathfrak{A}$ is the variety of all abelian groups and $\mathfrak{A}_p$ is the variety of all abelian groups of exponent p, where p is a prime number. Then $\mathfrak{A}^n$ is a variety of solvable groups of derived length less or equal to n. If e is a positive integer, then $\mathfrak{B}_e$ is the variety of all groups of exponent dividing e, and for an integer $c \geq 1$, $\mathfrak{N}_c$ is the variety of nilpotent groups of class at most c.

If g, h, k are elements of a group G then we write $g^h = h^{-1}gh$ and $g^{h+k} = g^h g^k$. Under this convention we have $g^{hk} = (g^h)^k$. Multiplication and addition of exponents is left and right distributive but not commutative. However, if G is a metabelian group and g lies in the commutator subgroup of G then $g^{h+k} = g^{k+h}$ and $g^{hk} = g^{kh}$ for every $h, k \in G$.

As usual, $[g, h] = [g,_1 h] = g^{-1}h^{-1}gh = g^{-1+h}$ and for every integer $c > 0$ we define $[g,_{c+1} h] = [[g,_c h], h]$. It can be proved by induction on c that

$$[x,_{c+1} y] = [x, y]^{(-1+y)^c} = x^{(-1+y)^{c+1}}. \tag{1}$$

Commutator laws

We will use the commutator law (cf. [23] 33.34):

$$[xy, zt] = [x, t]^y [y, t][x, z]^{yt} [y, z]^t \tag{2}$$

and its particular forms (for $t = 1$ or $y = 1$):

$$[xy, z] = [x, z]^y [y, z], \quad [x, zt] = [x, t][x, z]^t. \tag{3}$$

We will also use the law:

$$[x, y^{-1}] = [x, y]^{-y^{-1}}. \tag{4}$$

Definitions of a Milnor law

We identify a word $w(x, y) \in F_2'$ with the group law $w(x, y) \equiv 1$. One can find in the literature the following properties of the laws $w(x, y)$:

(Mil 1) A variety of groups satisfying the law $w(x, y)$ does not contain a subvariety $\mathfrak{A}_p\mathfrak{A}$ for any prime p [7].

(Mil 2) Every metabelian group satisfying $w(x, y)$ is nilpotent-by-(locally finite of finite exponent) [7].

(Mil 3) $w(x,y)$ implies a law $x^{P(y)}\xi$, where $P(y)$ is a primitive polynomial over $\mathbb{Z}$, and $\xi \in (\langle x\rangle^{F_2})'$ [31]

Proposition 1.1. *Conditions* (Mil 1)–(Mil 3) *are equivalent.*

Proof. (Mil 1)⇒(Mil 2). Let $\mathfrak{W}$ be a variety of groups satisfying a law $w(x,y)$. Groves in [7] proved that if a solvable variety does not contain any $\mathfrak{A}_p\mathfrak{A}$ then every group in it is nilpotent-by-(finite exponent). Moreover, Endimioni showed in [4] (Theorem 2 (v)) that if $\mathfrak{W}$ satisfies (Mil 1), then there exist c, e such that every solvable group in $\mathfrak{W}$ belongs to $\mathfrak{N}_c\mathfrak{B}_e$. Now, let G be a finitely generated metabelian group in $\mathfrak{W}$. Then G has a normal nilpotent subgroup N such that G/N has a finite exponent. The group G/N is finitely generated and metabelian. So by [8] G/N is residually finite and by Zelmanov's positive solution to the restricted Burnside problem [38] G/N is finite. Hence G is nilpotent-by-(locally finite of finite exponent).

(Mil 2)⇒(Mil 3). Let G be a metabelian group satisfying a law $w(x,y)$. So by assumption it is nilpotent-by-(finite exponent). Hence G has a normal nilpotent subgroup N such that G/N has a finite exponent, e say. Thus $h^e \in N$ for every $h \in G$ and $[g,h^e] \in N$ for every $g \in G$. Since N is nilpotent there exists c such that $[[g,h^e],_c h^e] \equiv 1$. Hence $w(x,y)$ implies a law $[x,_{c+1} y^e]\xi$, where $\xi \in F_2''$. Thus $w(x,y)$ implies a law $x^{(-1+y^e)^{c+1}}\xi$, where $\xi \in F_2''$. The law $x^{(-1+y^e)^{c+1}}\xi$ satisfies condition (Mil 3) because the polynomial $P(y) = (-1+y^e)^{c+1} = (-1)^{c+1} + \ldots + y^{e(c+1)}$ is primitive and $\xi \in F_2'' \subseteq (\langle x\rangle^{F_2})'$.

(Mil 3)⇒(Mil 1). Consider a group law $x^{P(y)}\xi$, where $P(y) = n_0 + n_1 y + \ldots + n_t y^t$ is primitive and $\xi \in (\langle x\rangle^{F_2})'$. It is sufficient to show that a wreath product $C_p \, wr \, C \in \mathfrak{A}_p\mathfrak{A}$ of a cyclic group of prime order p and an infinite cyclic group does not satisfy the law $x^{P(y)}\xi$. If $C_p = \langle d\rangle$ and $C = \langle c\rangle$ then every element of $C_p \, wr \, C$ has a unique representation of the form $d^{\cdots + m_{-1}c^{k_{-1}} + m_0 c^{k_0} + m_1 c^{k_1} + \cdots} c^{\ell}$. If we substitute $x \to d, y \to c$ then $\xi(d,c) = 1$ because $\xi \in (\langle x\rangle^{F_2})'$ and d^{c^i}, d^{c^j} commute for every i, j. Moreover, $d^{P(c)} = d^{n_0 + n_1 c + \ldots + n_t c^t} \neq 1$ since $\gcd(n_0, \ldots, n_t) = 1$, which implies that at least one n_i is not zero modulo p. □

Definition 1.1. We say that $w(x,y)$ is **a Milnor law** if it satisfies one of the conditions (Mil 1)–(Mil 3).

Other conditions defining a Milnor law can be found in the literature, see e.g. [2, 4, 5, 16–18]. A condition similar to the one of Point (Mil 3) was used by Black [2]. However, Black wrote a word $w(x,y)$ in two forms

$x^{P(y)}$ and $y^{Q(x)}$, where P and Q are not necessary polynomials as they may contain negative powers of y (and x respectively). It can be deduced from ([34], Lemma 4.8) that $w(x,y)$ is a Milnor law if and only if every metabelian group satisfying it does not contain a free monoid on two generators. A.I. Mal'tsev in [19] and independently Neumann and Taylor [24] showed that every nilpotent group satisfies a positive law. Thus $w(x,y)$ is a Milnor law if and only if every solvable group satisfying it satisfies a positive law.

The definition says that in the class of solvable groups every group satisfying a Milnor law is nilpotent-by-(locally finite). There are some papers in which authors describe larger classes of groups with this property. For example Kim and Rhemtulla in [15] show that every residually finite group satisfying a Milnor law is nilpotent-by-(finite exponent). Burns and Medvedev extended this result to the class $\mathcal{S}$ [3]. The class $\mathcal{S}$ consists of all soluble-by-(locally finite of finite exponent) groups and is closed under the operators L and R, where for any group-theoretic class $\mathcal{X}$, $L\mathcal{X}$ denotes the class of all groups locally in $\mathcal{X}$ and $R\mathcal{X}$ the class of groups residually in $\mathcal{X}$. In particular, the class $\mathcal{S}$ contains all locally and residually finite groups. The following Theorem is the consequence of Dichotomy Theorem in [3].

Theorem 1.1 (Burns and Medvedev, Dichotomy Theorem, cf. [3])**.** *If $w(x,y)$ is a Milnor law then every group in the class $\mathcal{S}$ satisfying the law $w(x,y)$ is nilpotent-by-(finite exponent).*

2. A constructive condition for a Milnor law

In this section we give one more condition for a word in F_2' to be a Milnor law. This condition allows us to check algorithmically whether a given word is a Milnor law or not. Namely:

Theorem 2.1. *Let $w \in F_2'$. Then there exist integers m, n and a polynomial $P(x,y)$ over integers such that the word w equals to $[x,y]^{P(x,y)x^{-m}y^{-n}}$ modulo F_2''. Therefore w is a Milnor law if and only if $P(x,y)$ is primitive.*

The proof of this theorem follows from two lemmas.

Lemma 2.1. *Every word $w \in F_2'$ is equal to $[x,y]^{P(x,y)x^{-n}y^{-m}}\xi$ where n, m are nonnegative integers, $P(x,y)$ is a polynomial over $\mathbb{Z}$ and $\xi \in F_2''$.*

Proof. Every commutator word in F_2 is a product of commutators of the form $[x^k, y^l]$. So it is enough to prove the Lemma for commutators $[x^k, y^l]$.

We use commutator laws $[a^k, b] = [a, b]^{a^{k-1}+\ldots+1}$ for $k > 1$, $[a^{-1}, b] = [a, b]^{-a^{-1}}$ and the fact that modulo F_2'' addition and multiplication of exponents of $[x, y]$ are commutative. Hence the commutator $[x^k, y^l]$ is equal to $[x, y]^{Q(x,y)}\xi$ where $Q(x, y)$ belongs to $\mathbb{Z}[x, y, x^{-1}, y^{-1}]$ and $\xi \in F_2''$. Moreover, there exist nonnegative integers n, m such that $Q(x, y) = P(x, y)x^{-n}y^{-m}$ and $P(x, y)$ is a polynomial over integers. Thus $[x^k, y^l] = [x, y]^{P(x,y)x^{-n}y^{-m}}\xi$, as required. □

Lemma 2.2. *A word* $[x, y]^{P(x,y)x^{-m}y^{-n}}\xi$ *is a Milnor law if and only if* $P(x, y)$ *is primitive.*

Proof. As we showed in Lemma 2.1 every word $w \in F_2'$ is equal to $[x, y]^{P(x,y)x^{-m}y^{-n}}\xi$ where ξ lies in F_2'' and $P(x, y)$ is a polynomial with integer coefficients. It is clear that the law $w(x, y)$ is equivalent modulo F_2'' to $[x, y]^{P(x,y)}$ and the law $w(x, y)$ is a Milnor law if and only if $[x, y]^{P(x,y)}$ is a Milnor law.

First we prove that if $P(x, y)$ is non-primitive then w is not a Milnor law. It suffices to show that there exists a prime p such that every group in $\mathfrak{A}_p\mathfrak{A}$ satisfies the law $[x, y]^{P(x,y)}$. If $P(x, y)$ is non-primitive then there exist p and a polynomial $R(x, y)$ such that $P(x, y) = pR(x, y)$. Hence modulo F_2'':

$$[x, y]^{P(x,y)} = [x, y]^{pR(x,y)} = ([x, y]^{R(x,y)})^p \in (F_2')^p.$$

So $[x, y]^{P(x,y)}$ is a law in the variety $\mathfrak{A}_p\mathfrak{A}$, whence it is not a Milnor law (by (Mil 1)).

Conversely, let $P(x, y) = m_1x^{k_1}y^{l_1} + m_2x^{k_2}y^{l_2} + \ldots + m_sx^{k_s}y^{l_s}$ be primitive. We can assume that $l_i \neq l_j$ for $i \neq j$. If not, choose n which does not divide all $l_i - l_j > 0$ and substitute $x \to y^nx$, $y \to y$. Then we get $[y^nx, y]^{P(y^nx,y)} = [x, y]^{P(y^nx,y)}$ and

$$Q(x, y) := P(y^nx, y) = m_1x^{k_1}y^{l_1+nk_1} + m_2x^{k_2}y^{l_2+nk_2} + \ldots + m_sx^{k_s}y^{l_s+nk_s}.$$

Polynomials $P(x, y)$ and $Q(x, y)$ have the same coefficients and now the exponents $l_i' = l_i + nk_i$ in the polynomial $Q(x, y)$ are different. Moreover laws $[x, y]^{P(x,y)}$ and $[x, y]^{Q(x,y)}$ are equivalent.

If we substitute $x \to [x, y]$, we get the law $[x,_2 y]^{Q([x,y],y)} = [x,_2 y]^{Q(1,y)}$ and the polynomial $Q(1, y)$ has the same coefficients as $Q(x, y)$ and $P(x, y)$. Moreover, $[x,_2 y]^{Q(1,y)} = x^{(-1+x)^2Q(1,y)}$ and if $P(x, y)$ is primitive then $(-1 + x)^2Q(1, y)$ is also primitive. So by condition (Mil 3) $w(x, y)$ is a Milnor law. □

3. The condition for Engel commutators

In this section we give one more condition for a commutator word to be a Milnor law. Namely:

Theorem 3.1. *Every word $w(x,y)$ in F_2' can be expressed in the form $w(x,y) = \left(\prod [x,_l\, y,_k\, x]^{b_{lk}}\right)^{x^{-n}y^{-m}} \xi$ where n, m are nonnegative integers, b_{lk} are integers and $\xi \in F_2''$.*

Moreover, $w(x,y) = \left(\prod [x,_l\, y,_k\, x]^{b_{lk}}\right)^{x^{-n}y^{-m}} \xi$ is a Milnor law if and only if exponents b_{lk} are coprime.

We divide the proof into several consecutive steps.

Lemma 3.1. *Every word $w(x,y)$ in F_2' can be expressed in the form $w(x,y) = \left(\prod [x,_l\, y,_k\, x]^{b_{lk}}\right)^{x^{-n}y^{-m}} \xi$ where n, m are nonnegative integers, b_{lk} are integers and $\xi \in F_2''$.*

Proof. By Lemma 2.1 every word $w \in F_2'$ is equal to $[x,y]^{P(x,y)x^{-n}y^{-m}}\xi$. We show that $[x,y]^{P(x,y)}$ can be expressed modulo F_2'' in the form $\prod [x,_l\, y,_k\, x]^{b_{lk}}$. It is enough to prove that $[x,y]^{x^k y^l}$ has the required form. Indeed, modulo F_2'' we have $[x,y]^{x^k y^l} = [x,y]^{y^l x^k}$ and since $[x,_r\, y]^y = [x,_r\, y][x,_{r+1}\, y]$ the word $[x,y]^{y^l}$ is the product of words of the form $[x,_t\, y]$. Similarly, every word $[x,_t\, y]^{x^k}$ is the product of words $[x,_t\, y,_s\, x]$. □

Lemma 3.2. *The following equalities holds modulo F_2'':*

1) $[x,_{l_1}\, y,_{k_1}\, x,_{l_2}\, y,_{k_2}\, x, \ldots,_{l_s}\, y,_{k_s}\, x] = [x,_{l_1+\ldots+l_s}\, y,_{k_1+\ldots+k_s}\, x]$,

2) $\prod_{k,l} [x,_l\, y,_k\, x]^{b_{lk}} = [x,y]^{\sum_{k,l} b_{lk}(x-1)^k(y-1)^{l-1}}$.

Proof. Indeed, modulo F_2'' we have

$$\begin{aligned}
&[x,_{l_1}\, y,_{k_1}\, x,_{l_2}\, y,_{k_2}\, x, \ldots,_{l_s}\, y,_{k_s}\, x] \\
&\quad = [x,y]^{(y-1)^{l_1-1}(x-1)^{k_1}(y-1)^{l_2}(x-1)^{k_2}\ldots(y-1)^{l_s}(x-1)^{k_s}} \\
&\quad = [x,y]^{(y-1)^{l_1+\ldots+l_s-1}(x-1)^{k_1+\ldots+k_s}} \\
&\quad = [x,_{l_1+\ldots+l_s}\, y,_{k_1+\ldots+k_s}\, x].
\end{aligned}$$

The second equality is clear. □

Now we show how to rewrite the word $[x,y]^{P(x,y)}$ as a product of commutators of the form $[x,_l y,_k x]$. There are three ways. The first one follows from the proof of Lemma 3.1. The second one uses the Taylor's formula and is explained in Section 6. The third one is the matrix method. We explain it now since it is useful to prove the main theorem of this section.

By equality 2) of Lemma 3.2, in order to write $[x,y]^{P(x,y)}$ on the form $\prod[x,_l y,_k x]^{b_{lk}}$ we have to express the polynomial $P(x,y)$ in the form $P(x,y)=\sum b_{lk}(x-1)^k(y-1)^{l-1}$.

Let us consider the following (upper unitriangular) matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \binom{2}{1} & \binom{3}{1} & \dots & \binom{n-1}{1} & \binom{n}{1} \\ 0 & 0 & 1 & \binom{3}{2} & \dots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & \binom{n}{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

that is $A=[a_{ij}]_{(n+1)\times(n+1)}$, such that $a_{ij}=\binom{i-1}{j-1}$ for $i,j\in\{1,\dots,n+1\}$. If $f(x)=a_0+a_1x+\ldots+a_nx^n=b_0+b_1(x-1)+\ldots+b_n(x-1)^n$, then we have

$$[b_0,\dots,b_n]^t = A[a_0,\dots,a_n]^t.$$

Let $\mathbb{Z}[x]_n$, $\mathbb{Z}[y]_n$ denote the free $\mathbb{Z}$-module of polynomials of one variable of degree less than or equal to n and let $\mathbb{Z}[x,y]_n$ be the $\mathbb{Z}$-module of polynomials $G(x,y)=a_1x^{k_1}y^{l_1}+a_2x^{k_2}y^{l_2}+\ldots+a_sx^{k_s}y^{l_s}$ such that $k_i\le n$ and $l_i\le n$ for $i=1,\dots,s$.

Lemma 3.3. *Every polynomial*

$$G(x,y)=a_{00}+\ldots+a_{kl}x^ky^l+\ldots+a_{nn}x^ny^n\in\mathbb{Z}[x,y]_n$$

can be written in the form

$$G(x,y)=b_{00}+\ldots+b_{kl}(x-1)^k(y-1)^l+\ldots+b_{nn}(x-1)^n(y-1)^n,$$

where

$$[b_{00},\dots,b_{kl},\dots,b_{nn}]^t=(A\otimes A)[a_{00},\dots,a_{kl},\dots,a_{nn}]^t,$$

and $A\otimes A$ is the tensor product of matrix A by A.

Proof. It is clear that $\mathbb{Z}[x,y]_n$ as the $\mathbb{Z}$-module is isomorphic to the tensor product $\mathbb{Z}[x]_n \otimes_{\mathbb{Z}} \mathbb{Z}[y]_n$. Hence the matrix which changes the basis $\{1,\dots,x^k y^l,\dots,x^n y^n\}$ into the basis $\{1,\dots,(x-1)^k(y-1)^l,\dots,(x-1)^n(y-1)^n\}$ is the tensor product of the matrix which changes the basis $\{1,\dots,x^n\}$ into the basis $\{1,\dots,(x-1)^n\}$ (that is A) by the matrix which changes the basis $\{1,\dots,y^n\}$ into the basis $\{1,\dots,(y-1)^n\}$ (that is also A). □

Lemma 3.4. *Let $[x,y]^{P(x,y)} = \prod_{k,l}[x,_l y,_k x]^{b_{lk}}$ modulo F_2''. Then the polynomial $P(x,y)$ is primitive if and only if exponents b_{lk} are coprime.*

Proof. We have by Lemma 3.3:

$$[b_{00},\dots,b_{kl},\dots,b_{nn}]^t = (A\otimes A)[a_{00},\dots,a_{kl},\dots,a_{nn}]^t.$$

Moreover, A is the upper unitriangular matrix, so the matrix $A\otimes A$ is also upper unitriangular. Hence $A\otimes A$ has the determinant equal to 1 and defines the automorphism of the free abelian group $\mathbb{Z}^{n^2}$.

By [25] (Theorem II.1, p.13) a vector $[a_{00},\dots,a_{kl},\dots,a_{nn}]$ has coprime coefficients if and only if it can be extended to a basis of the free abelian group $\mathbb{Z}^{n^2}$. The lemma follows since the automorphism maps every basis onto another basis. □

Proof of Theorem 3.1.

By Theorem 2.1 $w(x,y) = [x,y]^{P(x,y)}\xi = \prod_{k,l}[x,_l y,_k x]^{b_{lk}}\zeta$ is a Milnor law if and only if $P(x,y)$ is primitive and by Lemma 3.4 $P(x,y)$ is primitive if and only if exponents b_{lk} are coprime. □

4. Varieties without non-abelian nilpotent groups

As an application of Theorem 2.1 we give a criterion for a word $w(x,y)$ to satisfy the condition that every nilpotent group satisfying $w(x,y)$ is abelian.

Proposition 4.1. *Let w be a word in F_2. If every nilpotent group satisfying the law $w(x,y)$ is abelian then $w(x,y)$ is a Milnor law.*

Proof. Every variety $\mathfrak{A}_p\mathfrak{A}$ contains a finite non-abelian nilpotent group of order p^3. So the variety $\mathfrak{W}$ defined by the law w satisfying the hypothesis does not contain any $\mathfrak{A}_p\mathfrak{A}$. Thus $w(x,y)$ is a Milnor law. □

Corollary 4.1. *If every nilpotent group satisfying a law $w(x, y)$ is abelian then every metabelian group satisfying the law is abelian-by-(locally finite).*

Proof. By Proposition 4.1 if every nilpotent group satisfying $w(x, y)$ is abelian then $w(x, y)$ is a Milnor law. Hence by the condition (Mil 2) every metabelian group in $\mathfrak{W}$ is nilpotent-by-(finite exponent) and by the hypothesis it is abelian-by-(finite exponent). Let G be a finitely generated, metabelian group satisfying $w(x, y)$ and let N be an abelian, normal subgroup of G such that G/N has finite exponent. The group G/N is a finitely generated metabelian group. By [8] every finitely generated solvable group is residually finite. So G/N is residually finite of finite exponent and by Zelmanov's positive solution of the restricted Burnside problem [38] it is finite. □

Proposition 4.2. *A word $w(x, y) = [x, y]^{f(x,y)x^{-n}y^{-m}}\xi$, where $\xi \in F_2''$ belongs to $\gamma_3(F_2)$ if and only if $f(1, 1) = 0$.*

Proof. Every element of $\gamma_3(F_2)$ is a product of commutators $[a, b, c]$, where a, b, c are arbitrary words in F_2. We have an equality $[a, b, c] = [a, b]^{-1+c}$ and for $g(x, y) = 1 - c(x, y)$ we have $g(1, 1) = 1 - c(1, 1) = 0$.

Conversely, if $f(1, 1) = 0$, then $f(x, y) = (-1 + x)a(x, y) + (-1 + y)b(x, y)$ for some polynomials $a(x, y), b(x, y) \in \mathbb{Z}[x, y]$ (it can be deduced from Lemma 3.3). Hence

$$\begin{aligned}
&[x, y]^{f(x,y)x^{-n}y^{-m}}\\
&\quad = [x, y]^{(-1+x)a(x,y)x^{-n}y^{-m}}[x, y]^{(-1+y)b(x,y)x^{-n}y^{-m}}\\
&\quad = [x, y, x]^{a(x,y)x^{-n}y^{-m}}[x, y, y]^{b(x,y)}x^{-n}y^{-m}\eta,
\end{aligned}$$

where $\eta \in F_2''$. Since $F_2'' \subseteq \gamma_3(F_2)$ we have $[x, y]^{f(x,y)x^{-n}y^{-m}} \in \gamma_3(F_2)$. □

Lemma 4.1. *Let $w(x, y) = [x, y]^{f(x,y)x^{-n}y^{-m}}\xi$ where $f(x, y) \in \mathbb{Z}$, n, m are nonnegative integers and $\xi \in F_2'$. Then every nilpotent group satisfying the law $w(x, y)$ is abelian if and only if every nilpotent group satisfying $[x, y]^{f(x,y)}$ is abelian.*

Proof. First assume that every nilpotent group satisfying $[x, y]^{f(x,y)x^{-n}y^{-m}}\xi$ is abelian. Let G be a nilpotent group satisfying $[x, y]^{f(x,y)}$. The group G is nilpotent so it is solvable. We show that G must be metabelian.

Let us assume that G is not metabelian. There exists n such that $G^{(n+1)} = 1$ and $G^{(n)} \neq 1$. Thus $G^{(n-1)}$ is metabelian and it satisfies laws $[x, y]^{f(x,y)}$ and ξ for every $\xi \in F_2''$. Hence, it satisfies also the law

$[x,y]^{f(x,y)x^{-n}y^{-m}}\xi$. So $G^{(n-1)}$ is abelian and $G^{(n)} = 1$, which is a contradiction.

Thus G is metabelian and it satisfies the law $[x,y]^{f(x,y)x^{-n}y^{-m}}\xi$, so G is abelian.

The proof of the converse implication is similar. □

Theorem 4.1. *Let $w(x,y) = [x,y]^{f(x,y)x^{-n}x^{-m}}\xi$ where $f(x,y) \in \mathbb{Z}[x,y]$, n, m are nonnegative integers and $\xi \in F_2'$. Then every nilpotent group satisfying the law $w(x,y)$ is abelian if and only if $f(1,1) = \pm 1$.*

Proof. By Lemma 4.1 it is sufficient to consider the law $[x,y]^{f(x,y)}$.

Let $f(1,1) = d$. Then $[x,y]^{f(x,y)}$ can be written in the form $[x,y]^d v(x,y)$ where $v(x,y) = [x,y]^{g(x,y)}$ and $g(x,y)$ is a polynomial with $g(1,1) = 0$. Thus by Proposition 4.2 $v(x,y)$ lies in $\gamma_3(F_2)$.

We have three cases.

Case 1. $d = \pm 1$. Then the law $[x,y]^{f(x,y)}$ has the form $[x,y]^{\pm 1}v(x,y)$ for some $v(x,y) \in \gamma_3(F_2)$. Let G be a nilpotent group satisfying the law $[x,y]^{\pm 1}v(x,y)$. We show by induction on the class of nilpotency of G that G is abelian. Let G be a nilpotent group of class 2 satisfying the law $[x,y]^{\pm 1}v(x,y)$, then G satisfies the law $v(x,y)$ and it satisfies the law $[x,y]^{\pm 1}$. Thus G is abelian.

Now, let us assume that every nilpotent group of nilpotency class less than $c > 2$ satisfying the law $[x,y]^{f(x,y)}$ is abelian and let G be a nilpotent group of class c satisfying $[x,y]^{f(x,y)}$. Then $\gamma_{c+1}(G) = 1$ and $G/\gamma_c(G)$ is nilpotent of class $c-1$. So $G/\gamma_c(G)$ is abelian. Thus $\gamma_c(G) \supseteq G'$ and because the opposite inclusion is also valid we have $\gamma_c(G) = G' = \gamma_2(G)$. Hence $\gamma_{c+1}(G) = [\gamma_c(G), G] = [\gamma_2(G), G] = \gamma_3(G) = 1$ and G is of class 2. So G is abelian.

Case 2. $d = 0$. Then $[x,y]^{f(x,y)} = v(x,y) \in \gamma_3(F_2)$ and every nilpotent group of class 2 satisfies the law $v(x,y)$. So a nilpotent group satisfying the law $[x,y]^{f(x,y)}$ need not be abelian.

Case 3. $|d| > 1$. Let p be any prime divisor of d. Let G be a group of 3×3 (upper) unitriangular matrices $UT(3,p)$ over a prime field of order p. Then G is nonabelian, nilpotent group , satisfying the law $[x,y]^p$. Hence G also satisfies $[x,y]^{f(x,y)}$.

Case 1 shows that if $f(1,1) = \pm 1$ then every nilpotent group satisfying the law $[x,y]^{f(x,y)}$ is abelian. Cases 2 and 3 show that if $f(1,1) \neq 1$ then there are nilpotent non-abelian groups satisfying the law $[x,y]^{f(x,y)}$. □

Now, we show a few examples of varieties each nilpotent group in which is abelian.

4.1. A variety of A-groups

A finite group is called an A**-group** if all its Sylow p-subgroups are abelian (cf. [35]). Ol'shanskii in [26] studied varieties in which every group is residually finite. He proved the following theorem:

Theorem 4.2 (A. Yu. Ol'shanskii, cf. [26]). *Let $\mathfrak{V}$ be a variety of groups. Then every group in $\mathfrak{V}$ is residually finite if and only if $\mathfrak{V}$ is generated by a finite A-group.*

The question is what can be said about a variety of groups in which every finite group is an A-group. And the answer is:

Proposition 4.3. *Let $\mathfrak{V}$ be a variety of groups. Then every finite group in $\mathfrak{V}$ is an A-group if and only if every nilpotent group in $\mathfrak{V}$ is abelian.*

Proof. It is clear that if every nilpotent group in $\mathfrak{V}$ is abelian then every finite group in $\mathfrak{V}$ is an A-group.

Now, let every finite group in the variety $\mathfrak{V}$ be an A-group. Then every finite nilpotent group in $\mathfrak{V}$ is abelian.

Let G be a non-abelian nilpotent group in $\mathfrak{V}$. Then there exists a two generated nonabelian nilpotent subgroup H of G. By Hirsch theorem ([37], 2.13), H is polycyclic. By another Hirsch theorem ([33], 5.4.18) H has a non-abelian finite (nilpotent) image, which is a contradiction. □

Ol'shanskii posed the question whether a variety in which every finite group is an A-group has a finite basis of identities ([14], 4.48).

4.2. Pseudo-abelian laws

We say that a law $w(x, y)$ is **a pseudo-abelian law** if every metabelian group satisfying it is abelian but there are non-abelian groups satisfying this law. Oates showed that $w(x, y)$ is pseudo-abelian if and only if every finite group satisfying $w(x, y)$ is abelian ([23], 21.4). Neumann posed the question whether there exist pseudo-abelian laws ([23], Problem 5). O'lshanskii in ([27], section 9.29) gave the positive answer to this question.

Proposition 4.4. *If $w(x, y)$ is a pseudo-abelian law then every nilpotent group satisfying it is abelian.*

Proof. Every nilpotent group is solvable. So if G is a nilpotent group satisfying $w(x, y)$ then there exists n such that $G^{(n)} = 1$ and $G^{(n-1)} \neq 1$. Let us assume that G is not abelian. Then $G^{(n-2)}$ is metabelian and it satisfies $w(x, y)$. Hence $G^{(n-2)}$ is abelian and $G^{(n-1)} = 1$, a contradiction. □

4.3. Laws of the form $u(x,y)v(x,y) \equiv v(x,y)u(x,y)$

Let u, v be words in F_2. We consider laws of the form $u(x,y)v(x,y) \equiv v(x,y)u(x,y)$. The question is: for which words u, v the law $u(x,y)v(x,y) \equiv v(x,y)u(x,y)$ is equivalent to the abelian law, that is every group satisfying $u(x,y)v(x,y) \equiv v(x,y)u(x,y)$ is abelian. For example Gupta [9] showed that groups satisfying the law $[x,y] \equiv [x,_n y]$ for $n = 2,3$ are abelian and asked whether the laws $[x,y] \equiv [x,_n y]$ for $n \geq 4$ are equivalent to the abelian law. This question is still open. Next Ghandi, Moravec and Tomkinson and Kappe investigated laws of the form $[x,y] \equiv [a,b,c]$ or $[x,y] \equiv [a,b,c,d]$ where $a,b,c,d \in \{x^{\pm 1}, y^{\pm 1}\}$ [6,13,22]. They proved that all these laws are equivalent to the abelian law. All laws considered in their papers are of the form $[u, x^{\pm 1}] \equiv [x,y]$ or $[u, y^{\pm 1}] \equiv [x,y]$. If a law has the form $[u,y] \equiv [x,y]$, then we can transform it as follows:

$$\begin{aligned}
&[u,y] \equiv [x,y] \\
&\quad \longleftrightarrow u^{-1}y^{-1}uy \equiv x^{-1}y^{-1}xy \longleftrightarrow u^{-1}y^{-1}u \equiv x^{-1}y^{-1}x \\
&\quad \longleftrightarrow xu^{-1}y^{-1} \equiv y^{-1}xu^{-1} \longleftrightarrow (xu^{-1})y \equiv y(xu^{-1}).
\end{aligned}$$

So it can be written in the form $ab \equiv ba$ where $a = y$ and $b = xu^{-1}$.

Psomopoulos in [32] investigated laws of the form $x^t[x^n, y] \equiv [x, y^m]x^s$ in rings. He showed that in many cases rings satisfying such laws are commutative. For example, if m and n are coprime then every ring satisfying the law $x^t[x^n, y] \equiv [x, y^m]x^s$ is commutative ([32], Theorem 2). He also noticed that the last statement for groups may not be true. For example, the symmetric group on three symbols, which is non-abelian satisfies the law $x^6[x^7, y] \equiv [x,y]x^6$. However, he showed that in some cases groups satisfying such laws are abelian. The following proposition describes such case.

Proposition 4.5 ([32], Theorem 3)**.** *Every group satisfying a law $[x^n, y] \equiv [x, y^{n+1}]$ is abelian.*

Proof. First we substitute yx for x and obtain $[(yx)^n, y] \equiv [yx, y^{n+1}] = [x, y^{n+1}] \equiv [x^n, y]$. Hence we get the law $[(yx)^n, y] \equiv [x^n, y]$. Now we replace y by yx^{-1} and get $[y^n, yx^{-1}] \equiv [x^n, yx^{-1}]$. Thus $[y^n, x^{-1}] \equiv [x^n, y]^{x^{-1}}$ and $[y^n, x]^{-x^{-1}} \equiv [x^n, y]^{x^{-1}}$. We conjugate the last law by x and get $[y^n, x]^{-1} \equiv [x^n, y]$. Thus we have

$$[x, y^{n+1}] \equiv [x^n, y] \equiv [y^n, x]^{-1} = [x, y^n]$$

and it is easy to check that the law $[x, y^{n+1}] \equiv [x, y^n]$ is equivalent to the abelian law. □

We can see that every law of the form $[x^k, y^n] \equiv [x^m, y^l]$ is equivalent to a law of the form $uv \equiv vu$, which is actually positive. Namely

Proposition 4.6. *Laws* $[x^k, y^n] \equiv [x^m, y^l]$ *and* $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ *are equivalent.*

Proof. We transform the second law as follows:

$$\begin{aligned}&(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)\\ &\quad\longleftrightarrow x^{-m} y^l x^m y^{-l} \equiv x^{-k} y^n x^k y^{-n} \longleftrightarrow [x^m, y^{-l}] \equiv [x^k, y^{-n}].\end{aligned}$$

After the substitution $y \to y^{-1}$ we get $[x^k, y^n] \equiv [x^m, y^l]$. □

Now, we give conditions for words u, v providing that every group satisfying the law $uv \equiv vu$ is abelian. Note that words $u(x, y)$ and $v(x, y)$ can be written as $u(x, y) = x^k y^l c$, $v(x, y) = x^m y^n d$ where m, n are integers and c, d are commutator words.

Proposition 4.7. *Let* $u(x, y) = x^k y^l c$, $v(x, y) = x^m y^n d$ *where* c, d *are integers and* c, d *are commutator words. If* $W = \begin{vmatrix} k & l \\ m & n \end{vmatrix} = \pm 1$ *then a law* $u(x, y)v(x, y) \equiv v(x, y)u(x, y)$ *is a Milnor law. Moreover, every nilpotent group satisfying such a law is abelian.*

Proof. We can write the law $x^k y^l c x^m y^n d \equiv x^m y^n d x^k y^l c$ in the form:

$$[x^m y^n, x^k y^l] \equiv c[c, x^m y^n] d c^{-1} [d, x^k y^l] d^{-1}. \tag{5}$$

The word on the right side of (5) belongs to $\gamma_3(F_2)$. Indeed, $[c, x^m y^n]$ and $[d, x^k y^l]$ are in $\gamma_3(F_2)$ since c, d lie in F_2'. Hence

$$c[c, x^m y^n] d c^{-1} [d, x^k y^l] d^{-1} = (c[c, x^m y^n] c^{-1}) c d c^{-1} d^{-1} (d[d, x^k y^l] d^{-1})$$

belongs to $\gamma_3(F_2)$ since $cdc^{-1}d^{-1} \in F_2'' \subseteq \gamma_3(F_2)$. Thus by Proposition 4.2 the right side of (5) can be written in the form $[x, y]^{f(x,y)}$ where $f(1, 1) = 0$. The word $[x^m y^n, x^k y^l]$ modulo F_2'' equals

$$\begin{aligned}[x^m y^n, x^k y^l] &= [x^m, y^l]^{y^n} [y^n, x^k]^{y^l}\\ &= [x, y]^{(1+\ldots+x^{m-1})(1+\ldots+y^{l-1})y^n + (1+\ldots+x^{k-1})(1+\ldots+y^{n-1})y^l}\end{aligned}$$

Thus the law (5) can be written in the form $[x, y]^{g(x,y) - f(x,y)} \equiv 1$ where $g(x, y) = (1 + \ldots + x^{m-1})(1 + \ldots + y^{l-1}) y^n + (1 + \ldots + y^{n-1})(1 + \ldots + y) y^l$. Consequently, $g(1, 1) - f(1, 1) = g(1, 1) = ml - kl = -W = \pm 1$. So $g(x, y) - f(x, y)$ is primitive and by Theorem 2.1 $uv \equiv vu$ is a Milnor law. Moreover, by Theorem 4.1, every nilpotent group satisfying this law is abelian. □

The next two propositions show that condition $W = \pm 1$ is necessary but not sufficient for the identity $uv \equiv vu$ to be equivalent to the abelian identity.

Proposition 4.8. *Let $u = x^k y^l c$, $v = x^m y^n d$ be words in F_2 where $c, d \in F_2'$. If the law $uv \equiv vu$ is equivalent to the abelian law then* $\begin{vmatrix} k & l \\ m & n \end{vmatrix} = \pm 1$.

Proof. Let $\begin{vmatrix} k & l \\ m & n \end{vmatrix} = t \neq \pm 1$. Then there exists a prime number p dividing t. Let G be the group of 3×3 (upper) unitriangular matrices $UT(3, p)$. The group G is nilpotent of class 2 and satisfies the following identities:

$$[xy, z] \equiv [x, z][y, z], \quad [x, yz] \equiv [x, y][x, z], \quad [x, y]^p \equiv 1, [xc, yd] \equiv [x, y],$$

for every $c, d \in F_2'$. Thus:

$$[u, v] = [x^k y^l c, x^m y^n d] \equiv [x, y]^{kn-ml} = [x, y]^t \equiv 1.$$

So G is a non-abelian nilpotent group satisfying $uv \equiv vu$, a contradiction. □

Proposition 4.9. *Let $u_n = x^{-1}[x,_n y]$ and $v_n = y^{-1}[y,_n x]$ where n is a positive integer. Then the symmetric group S_3 satisfies laws $u_n v_n \equiv v_n u_n$ for $n \geq 1$ and $W = \pm 1$.*

Proof. The symmetric group S_3 is metabelian and satisfies the following laws:

$$[x, y]^3 \equiv 1,\ [x, y]^x \equiv [x, y]^{x^{-1}},\ [x, y]^{xy+x+y+1} \equiv 1,\ [x,_2 y] \equiv [x,_3 y]. \quad (6)$$

Thus the group S_3 also satisfies laws

$$[x,_2 y] \equiv [x,_n y], \quad (7)$$

for every integer $n > 1$. By the law (2) we get

$$[u_n, v_n] = [x^{-1}[x,_n y], y^{-1}[y,_n x]]$$
$$= [x^{-1}, [y,_n x]]^{[x,_n y]}[[x,_n y], [y,_n x]][x^{-1}, y^{-1}]^{[x,_n y][y,_n x]}[[x,_n y], y^{-1}]^{[y,_n x]}.$$

Since S_3 is metabelian it satisfies the laws $[x, y]^c \equiv [x, y]$ where $c \in F_2'$. So, we get $[u_n, v_n] \equiv [x^{-1}, [y,_n x]][x^{-1}, y^{-1}][[x,_n y], y^{-1}]$. Next, we use (4) and obtain $[u_n, v_n] \equiv [y,_{n+1} x]^{x^{-1}}[x, y]^{x^{-1}y^{-1}}[x,_{n+1} y]^{-y^{-1}}$. Now by the law (7) we get $[u_n, v_n] \equiv [y,_2 x]^{x^{-1}}[x, y]^{x^{-1}y^{-1}}[x,_2 y]^{-y^{-1}}$, and using the law (1) for $c = 1$ we have $[u_n, v_n] \equiv [x, y]^{x^{-1}y^{-1}-x^{-1}(-1+x)-y^{-1}(-1+y)}$. Hence, by laws (6):

$$[u_n, v_n] \equiv [x, y]^{xy+x+y-2} \equiv [x, y]^{xy+x+y+1} \equiv 1.$$

So S_3 satisfies all laws $[u_n, v_n] \equiv 1$. Moreover $W = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$. □

4.4. SM-varieties

Let $w(x_1, \ldots, x_n)$ be a word in n variables and let G be any group. A word w defines an n- ary operation f in G in the following way:

$$f : G^n \to G,\ f(g_1, \ldots, g_n) = w(g_1, \ldots, g_n).$$

We define $f^\sigma(g_1, \ldots, g_1) = (g_{1^\sigma}, \ldots, g_{n^\sigma})$ for $\sigma \in S_n$. We say that f is n-symmetric in G if $f^\sigma = f$ for every $\sigma \in S_n$. In other words, the operation f is symmetric if and only if $w \equiv w^\sigma$ is the identity in the group G for every $\sigma \in S_n$. Then w is called a symmetric word in G.

We say that G is an SM-group if the group operation xy in G is a composition of symmetric operations (not necessarily with the same number of arguments), that is there are symmetric words $w, w_1, \ldots, w_n$ such that $x \cdot y \equiv w(w_1, \ldots, w_n)$ is a law in G. It is clear that all abelian groups are SM-groups. We can also show the following fact:

Proposition 4.10. *If G is an SM-group then every group in var(G) is an SM-group.*

Proof. It is clear because both $x \cdot y \equiv w(w_1, \ldots, w_n)$ and symmetric words are identities in G. □

So we can define SM-variety as the variety of groups in which every group is an SM-group.

Marczewski asked whether only abelian groups are SM-groups (see [20]). Płonka gave in [28] an example of a non-abelian SM-group. But which groups are SM-groups is still the question (see [29]). It follows from [28] that

Proposition 4.11. *The variety $\mathfrak{A}_3\mathfrak{A}_2$ is an SM-variety.*

Proof. Płonka showed that S_3 is an SM-group. The variety var(S_3) is $\mathfrak{A}_3\mathfrak{A}_2$, so by Proposition 4.10 it is an SM-variety. □

In [30] Płonka described symmetric words in nilpotent groups of class at most 3 and from his description we get:

Proposition 4.12. *Let $\mathfrak{V}$ be a variety of groups. If $\mathfrak{V}$ is an SM-variety then every nilpotent group in $\mathfrak{V}$ is abelian.*

One can find a description of symmetric words in several groups in [10–12, 36].

5. (Locally finite)-by-abelian groups

In this section we give a condition for a Milnor law $w(x,y)$ such that every metabelian group satisfying the law $w(x,y)$ is (locally finite)-by-abelian. We prove

Theorem 5.1. *Let $w(x,y) = [x,y]^{P(x,y)}$ be a law which modulo F_2'' implies two laws $[x,_2 y]^{P(y)}$ and $[x,_2 y]^{Q(y)}$. If $P(y)$ and $Q(y)$ are coprime and $P(1,1) \neq 0$ then $w(x,y)$ implies modulo F_2'' a law $[x,y]^n$ for some n. In particular, every metabelian group satisfying $w(x,y)$ is (locally finite)-by-abelian.*

We will first prove some auxiliary statements.

Lemma 5.1. *If a metabelian group G satisfies laws $[x,y]^{P(x,y)}$ and $[x,y]^{Q(x,y)}$ then it also satisfies a law $[x,y]^{A(x,y)P(x,y)+B(x,y)Q(x,y)}$, where $A(x,y)$ and $B(x,y)$ are polynomials over integers.*

Proof. It suffices to observe that a law $[x,y]^{P(x,y)}$ implies $[x,y]^{mx^ky^lP(x,y)}$ for all integers m,k,l and if $[x,y]^{H(x,y)}$ and $[x,y]^{K(x,y)}$ are laws in G then $[x,y]^{H(x,y)+K(x,y)}$ is also the law in G. □

Lemma 5.2. *Let G be a metabelian group satisfying laws $[x,y]^{P(x,y)}$ and $[x,_2 y]^n$ then G also satisfies the law $[x,y]^{nP(1,1)}$.*

Proof. It follows from [23], 34.33 that if $[x,_2 y]^k \equiv 1$ is a law then $[x,y,x]^k \equiv 1$ also is a law. Therefore in metabelian groups for $a \in \{x,y\}$ we have $[[x,y]^n,a] = [x,y,a]^n = 1$ and $[x,y]^{na} = [x,y]^n$. Now we raise $[x,y]^{P(x,y)}$ to the power n and obtain the law

$$\begin{aligned}[x,y]^{nP(x,y)} &= [x,y]^{n(m_1x^{k_1}y^{l_1}+m_2x^{k_2}y^{l_2}+\ldots+m_sx^{k_s}y^{l_s})}\\ &= [x,y]^{n(m_1+m_2+\ldots+m_s)},\end{aligned}$$

and since $m_1+\ldots+m_s = P(1,1)$, the statement follows. □

Proof of Theorem 5.1. If $P(y)$ and $Q(y)$ are coprime then there exist polynomials $A(y), B(y)$ with rational coefficients such that

$$A(y)Q(y) + B(y)P(y) = 1.$$

Thus there exists an integer n such that both $nA(y)$ and $nB(y)$ have integer coefficients and

$$(nA(y))Q(y) + (nB(y))P(y) = n.$$

By Lemma 5.1 a law $w(x, y)$ implies a law

$$[x,_2 y]^{(nA(y))Q(y)+(nB(y))P(y))} = [x,_2 y]^n,$$

and by Lemma 5.2 it implies a law $[x, y]^{nP(1,1)}$ which is nontrivial since by assumption $P(1, 1) \neq 0$. □

Corollary 5.1. *Let $w(x, y) = [x, y]^{P(x,y)}$ be a law which modulo F_2'' implies two laws $[x,_2 y]^{P(y)}$ and $[x,_2 y]^{Q(y)}$ such that $P(y)$, $Q(y)$ are coprime and $P(1, 1) \neq 1$. If $w(x, y)$ is a Milnor law and G is a finitely generated metabelian group satisfying $w(x, y)$ then G is finite-by-abelian. Moreover, there exists a positive integer d such that $w(x, y)$ implies modulo F_2'' laws $[x^d, y] \equiv 1$ and $[x, y]^d \equiv 1$.*

Proof. Let G be a finitely generated free group in a variety of all groups satisfying a law $w(x, y) \equiv 1$. It is proved in [16] (Theorem 3) that a finitely generated metabelian group satisfying a Milnor law has a finitely generated commutator subgroup G'. So it follows from Theorem 5.1 that G' is finite and abelian and thus it satisfies a law $[x, y]^n$, where n is an exponent of G'.

Since G' is finite, then there exist integers k, l such that $0 \leq k < l$ and $[x^k, y] = [x^l, y]$. Hence we get $[x, y]^{1+x+\ldots+x^{k-1}} = [x, y]^{1+x+\ldots+x^{l-1}}$ and $[x, y]^{x^k+\ldots+x^{l-1}} = [x, y]^{x^k(1+x+\ldots+x^{l-k-1})}$. Consequently $[x, y]^{1+x+\ldots+x^{l-k-1}} = [x^{l-k}, y]$. If $d = l - k$ then from a law $[x^d, y] = [x, y]^{1+x+\ldots+x^{d-1}}$, by substituting $x \to [x, y]$, we get a law $[x,_2 y]^d$. Thus $n|d$ and $[x, y]^d$ is a law in G. □

In practise, there are many ways for obtaining polynomials $P(y)$ and $Q(y)$ satisfying hypotheses of Theorem 5.1. One of the methods is described in Propositions 5.1 and 5.2 below.

Proposition 5.1. *Let $P(x, y)$ be any polynomial over integers and let $P(y) = P(1, y)$, $Q(y) = y^t P(1/y)$, where t is a degree of $P(y)$. Then a law $[x, y]^{P(x,y)}$ implies (modulo F_2'') laws $[x,_2 y]^{P(y)}$ and $[x,_2 y]^{Q(y)}$.*

Proof. We substitute $x \to [x,y]$ in $[x,y]^{P(x,y)}$ and we get modulo F_2'':

$$[[x,y],y]^{P([x,y],y)} = [x,_2 y]^{P(1,y)} = [x,_2 y]^{P(y)}.$$

Now we substitute y^{-1} for y and we obtain:

$$[x,_2 y^{-1}]^{P(y^{-1})}. \tag{8}$$

So

$$\begin{aligned}[x,_2 y^{-1}] &= [[x,y^{-1}],y^{-1}] \\ &= [[x,y]^{-y^{-1}},y]^{-y^{-1}} = [[x,y]^{-1},y]^{-y^{-2}} = [x,y,y]^{[x,y]^{-1}y^{-2}}.\end{aligned}$$

Thus modulo F_2'' we have $[x,_2 y^{-1}] = [x,y,y]^{y^{-2}}$. So the word (8) is equal to $[x,_2 y]^{y^{-2}P(1/y)}$. If t is a degree of $P(y)$ then we conjugate $[x,_2 y]^{y^{-2}P(1/y)}$ by y^{t+2} getting $[x,_2 y]^{y^t P(1/y)} = [x,_2 y]^{Q(y)}$, which is the second law. □

Proposition 5.2. *Let $P(x,y)$ be a polynomial such that $P(y) = P(1,y) = n_0 + n_1 y + \ldots + n_t y^t$ is not divisible over complex numbers by a polynomial of the form $y^2 + zy + 1$. Then polynomials $P(y)$ and $Q(y) = y^t P(1/y)$ are coprime. If, moreover, $P(1) \neq 1$ then the law $[x,y]^{P(x,y)}$ implies a law $[x,y]^n$ for some integer n.*

Proof. Suppose that $P(y)$ and $Q(y)$ are not coprime. Then $P(y)$ and $Q(y)$ have a common complex root α. We can assume that $\alpha \neq 0$. But if α is a root of $Q(y) = y^t P(1/y)$ then α^{-1} is a root of $P(y)$. Thus α and α^{-1} are roots of $P(y)$ and $(y-\alpha)(y-\alpha^{-1}) | P(y)$. But $(y-\alpha)(y-\alpha^{-1}) = y^2 + zy + 1$, where $z = \alpha + \alpha^{-1}$ and by assumption it does not divide $P(y)$, a contradiction. □

Example 5.1. Let $w(x,y) = x^{-2}y^{-3}xy^2xy$. Then modulo F_2''

$$\begin{aligned}w(x,y) &= [x^2,y^3][y^3,x][x,y] \\ &= [x,y]^{(1+x)(1+y+y^2)-(1+y+y^2)+1} = [x,y]^{1+x+xy+xy^2}.\end{aligned}$$

Thus $P(y) = 2 + y + y^2$ and $Q(y) = 1 + y + 2y^2$. Using the Euclidean algorithm we get:

$$(5-2y)P(y) + (y-2)Q(y) = 8.$$

Hence by Theorem 5.1 we get the law $[x,_2 y]^8 \equiv 1$ and since $P(1,1) = 4$ we get by Lemma 5.2 the law $[x,y]^{32} \equiv 1$ modulo F_2''.

We can improve the above result. First, we substitute xy for y in $w(x, y)$ and we get $w(x, xy) = [x, y]^{1+x+x^2y+x^3y^2}$. Next, we substitute $[y, x]$ for y and we obtain $[x, [y, x]]^{1+x+x^2+x^3} = [y,_2 x]^{-(1+x+x^2+x^3)}$. We interchange x and y, invert and we get $[x,_2 y]^{1+y+y^2+y^3}$. If use the Euclidean algorithm to $1 + y + y^2 + y^3$ and $2 + y + y^2$ then we get $\alpha(x)(1 + y + y^2 + y^3) + \beta(x)(2 + y + y^2) = 4$. Hence $[x,_2 y]^4 \equiv 1$ and finally $[x, y]^{16} \equiv 1$.

6. Algorithms and examples

Now we present two algorithms which summarize the steps from sections 2 and 3. We call the first one **a Milnor algorithm**.

Algorithm I

Data: a binary word $w(x, y) = x^{k_1}y^{l_1}x^{k_2}y^{l_2} \dots x^{k_s}y^{l_s}$ in F_2'.
Question: is $w(x, y)$ a Milnor law?

Step 1. Write the word $w(x, y)$ as the product of commutators:

$$w(x,y) = [x^{q_2}, y^{p_1}]^{-1}[x^{q_2}, y^{p_2}][x^{q_3}, y^{p_2}]^{-1} \dots [x^{q_s}, y^{p_{s-1}}]^{-1}[x^{q_s}, y^{p_s}], \quad (9)$$

where $q_i = k_i + \dots + k_s$, $p_i = l_i + \dots + l_s$ for $i = 1, \dots, s$.

Step 2. Write every commutator $[x^m, y^n]$ in (9) in the form $[x, y]^{q(x,y)}$ (modulo F_2''), where

— if $m, n > 0$ then $q(x, y) = (1 + \dots + x^{m-1})(1 + \dots + y^{n-1})$,

— if $m>0, n<0$ then $q(x, y) = -(1+\dots+x^{m-1})(1+\dots+y^{|n|-1})y^n$,

— if $m<0, n>0$ then $q(x, y) = -(1+\dots+x^{|m|-1})(1+\dots+y^{n-1})x^m$,

— if $m, n < 0$ then $q(x, y) = (1 + \dots + x^{|m|-1})(1 + \dots + y^{|n|-1})x^m y^n$.

Step 3. Calculate $Q(x, y)$ which is the sum of all functions obtained in Step 2.

Step 4. Express $Q(x, y)$ in the form $Q(x, y) = P(x, y)x^{-m}y^{-n}$ where m, n are nonnegative integers and

$$P(x, y) = m_1x^{s_1}y^{t_1} + m_2x^{s_2}y^{t_2} + \dots + m_rx^{s_r}y^{t_r}$$

is a polynomial over integers.

Step 5. $w(x, y)$ is a Milnor law if and only if $\gcd(m_1, \dots, m_r) = 1$.

Step 6. Every nilpotent group satisfying the law $w(x, y)$ is abelian if and only if $P(1, 1) = m_1 + \dots + m_r = 1$.

Example 6.1. $w(x,y) = xy^2x^2yx^{-1}y^{-2}x^{-2}y^{-1}$.

- $w(x,y) = [x^{-1},y^{-2}][x^{-3},y^{-2}]^{-1}[x^{-3},y^{-3}][x^{-2},y^{-3}]^{-1}[x^{-2},y^{-1}]$
- $[x^{-1},y^{-2}] = [x,y]^{(1+y)x^{-1}y^{-2}}$
  $[x^{-3},y^{-2}]^{-1} = [x,y]^{-(1+x+x^2)(1+y)x^{-3}y^{-2}}$
  $[x^{-3},y^{-3}] = [x,y]^{(1+x+x^2)(1+y+y^2)x^{-3}y^{-3}}$
  $[x^{-2},y^{-3}]^{-1} = [x,y]^{-(1+x)(1+y+y^2)x^{-2}y^{-3}}$
  $[x^{-2},y^{-1}] = [x,y]^{(1+x)x^{-2}y^{-1}}$,
- $Q(x,y) = x^{-1}y^{-1} - x^{-2}y^{-2} + x^{-3}y^{-3} = (1-xy+x^2y^2)x^{-3}y^{-3}$, $P(x,y) = 1-xy+x^2y^2$,
- $w(x,y)$ is a Milnor law since all the coefficients of $P(x,y)$ are equal to ± 1. Moreover $P(1,1) = 1$, so every nilpotent group satisfying $w(x,y)$ is abelian.

Example 6.2. While Example 6.1 was done by hand, the second one is generated by a computer program (written in C language).

$$w(x,y) = xyx^{-1}y^{-1}xy^2xy^{-1}x^{-3}y^{-1}xyx^{-1}y^{-1}xyx^2yx^{-1}y^{-1}x^{-1}y^{-1}.$$

- $w(x,y) = [x^{-1},y^{-1}][x^{-1},y^{-2}][x^{-2},y^{-2}]^{-1}[x^{-2},y^{-1}][x,y^{-1}]^{-2}$
  $[x^{-2},y^{-1}]^{-1}[x^{-2},y^{-2}][x^{-1},y^{-2}]^{-1}[x^{-1},y^{-1}]$,
- $Q(x,y) = 2(x^{-1}y^{-1}+y^{-1}) = 2(1+x)x^{-1}y^{-1}$, $P(x,y) = 2(1+x)$.
- $w(x,y)$ is **not** a Milnor law since $\gcd(2,2) \neq 1$.

Example 6.3. Let w be an Engel word $w = [x,_c y]$ then $w = [x,y]^{(-1+y)^{c-1}}$ and since coefficients of $(-1+y)^{c-1}$ are coprime, w is a Milnor law.

Example 6.4. The aim of this example is to compare the condition from the Theorem 2.1 with the condition (Mil 3). We show that the method described in the statement is more effective. Let us take an element $w = x^{-2}y^{-2}x^2y^2$. We have $w = [x^2,y^2] = [x,y]^{(x+1)(1+y)} = [x,y]^{x+1+xy+y}$, so by the Theorem 2.1 w is a Milnor law.

However, if we write w as a power of x we get $w = x^{-2+2y^2} = x^{-2(1-y)(1+y)}$ with a non-primitive polynomial $-2+2y^2$ in exponent. By definition of Point (Mil 3) w must imply a law $x^{P(y)}\xi$ with primitive $P(y)$. Lemma 2.2 gives an algorithm how we can find $P(y)$. All calculations are done modulo F_2'' (and also modulo $(\langle x\rangle^{F_2})' \supseteq F_2''$). We substitute y^nx for x in $[x,y]^{1+y+x+yx}$ and get $[y^nx,y]^{1+y+y^nx+y^{n+1}x} = [x,y]^{1+y+y^nx+y^{n+1}x}$ which implies a law $[x,_2 y]^{1+y+y^n[x,y]+y^{n+1}[x,y]}$. This last word modulo F_2'' is equal to $[x,_2 y]^{1+y+y^n+y^{n+1}}$. Since $[x,_2 y] = x^{(-1+y)^2}$ we get

$$[x,_2 y]^{1+y+y^n+y^{n+1}} = x^{(-1+y)^2(1+y+y^n+y^{n+1})}.$$

Thus w implies modulo $(\langle x\rangle^{F_2})'$ a law $x^{(-1+y)^2(1+y+y^n+y^{n+1})}$, where the polynomial $P(y) = (-1+y)^2(1+y+y^n+y^{n+1})$ is primitive.

The second algorithm presents how to express a word $w(x,y) \in F_2''$ as the product of commutators of the form $[x,_k y,_l y]$. In the algorithm we use the Taylor's formula twice.

Algorithm II

Data: a binary word $w(x,y) = x^{k_1}y^{l_1}x^{k_2}y^{l_2}\dots x^{k_s}y^{l_s}$ in F_2'.
Aim: express $w(x,y)$ as the product of commutators of the form $[x,_l y,_k x]$ and check properties of the law $w(x,y)$.

Step 1. Write $w(x,y)$ in the form $w(x,y) = [x,y]^{P(x,y)x^{-m}y^{-n}}$, where $P(x,y)$ is a polynomial over integers.

Step 2. Write $P(x,y)$ in the form $P(x,y) = P_y(x) = f_n(y)x^n + \dots + f_1(y)x + f_0(y)$.

Step 3. Express $P(x,y)$ in the form

$$P(x,y) = g_n(y)(x-1)^n + \dots + g_1(y)(x-1) + g_0(y),$$

where $g_i(y) = \frac{P_y^{(i)}(1)}{i!}$ for $i = 0,\dots,n$ and $P_y^{(i)}(1)$ is the i-th derivative of $P_y(x)$ at the point x.

Step 4. Express every function $g_i(y)$ for $i = 0,\dots,n$ in the the form

$$g_i(y) = b_{im}(y-1)^m + \dots + b_{i1}(y-1)y + b_{i0},$$

where $b_{ij} = \frac{g_i^{(j)}(1)}{j!}$ for $j = 0,\dots,m$ and $g_i^{(j)}(y)$ is the j-th derivative of $g_i(y)$ at the point y.

Step 5. Multiply and order components of $P(x,y)$ to obtain

$$P(x,y) = \sum b_{kl}(x-1)^k(y-1)^{l-1}.$$

Step 6. $w(x,y) = [x,y]^{P(x,y)x^{-n}y^{-m}} = \left(\prod [x,_l y,_k x]^{b_{kl}}\right)^{x^{-n}y^{-m}}.$

Step 7. $w(x,y)$ is a Milnor law if and only if b_{kl} are coprime.

Step 8. Every nilpotent group satisfying the law $w(x,y)$ is abelian if and only if $b_{10} = \pm 1$.

Example 6.5. Let $w(x,y)=[x,y]^{xy^3+x^2y+x^2y^2+2x+3}$.

- $f_y(x)=P(x,y)=xy^3+x^2y+x^2y^2+2x+3$
  $\quad=3+(y^3+2)x+(y+y^2)x^2.$
- $f'_y(x)=y^3+2+2(y+y^2)x,\ f''_y(x)=2(y+y^2).$
- $P(x,y)=\frac{f(y)(1)}{0!}+\frac{f'_y(1)}{1!}(x-1)+\frac{f''_y(1)}{2!}(x-1)^2$
  $\quad=5+y+y^2+y^3+(2+2y+2y^2+y^3)(x-1)+(y+y^2)(x-1)^2.$

Now, using the Taylor's formula we write polynomials $g_0(y)=5+y+y^2+y^3$, $g_1(y)=2+2y+2y^2+y^3$, $g_2(y)=y+y^2$ as linear combinations of powers of $y-1$. Thus we get

$$\begin{aligned}g_0(y)&=(y-1)^3+4(y-1)^2+6(y-1)+8,\\ g_1(y)&=(y-1)^3+5(y-1)^2+9(y-1)+7,\\ g_2(y)&=(y-1)^2+3(y-1)+2.\end{aligned}$$

Hence we get

$$\begin{aligned}P(x,y)=8&+7(x-1)+2(x-1)^2+6(y-1)+9(y-1)(x-1)\\ &+3(y-1)(x-1)^2+4(y-1)^2+(y-1)^3+5(y-1)^2(x-1)\\ &+(y-1)^3(x-1)+(y-1)^2(x-1)^2,\end{aligned}$$

and $w(x,y)$ equals modulo F''_2:

$$[x,y]^8[x,y,x]^7[x,y,_2x]^2[x,_2y]^6[x,_2y,x]^9[x,_2y,_2x]^3[x,_3y]^4[x,_3y,x]^5$$
$$[x,_3y,_2x][x,_4y][x,_4y,x].$$

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