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The monoid of endomorphisms of disconnected hypergraphs

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Abstract. We prove that the monoid of endomorphisms of an arbitrary disconnected hypergraph is isomorphic to a wreath product of a transformation semigroup with a certain small category. For disconnected hypergraphs we also study the structure of the monoid of strong endomorphisms and the group of automorphisms.

1. Introduction

For many types of geometrically-combinatorial objects in mathematics the concepts of a connectedness and a disconnectedness are defined naturally. In such cases each disconnected object is a union of its connected components and the reduction's problem of the description of some structures (e.g., the monoid of endomorphisms, the group of automorphisms etc.) of an arbitrary disconnected object to the description of its connected components is appeared. In the paper this problem is solved for monoids of endomorphisms of disconnected hypergraphs.

It is well known that the monoid of endomorphisms of any algebraic system carries a substantial information about a system and gives a new, convenient enough language with the help of which it is possible to study the structure of this system. Semigroups of endomorphisms of relational systems and their properties were studied by many authors.

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A survey of early works related to semigroups of endomorphisms of graphs and hypergraphs can be found in [1], [2] and more late works, for example, in [3], [4]. The specific attention was spared such problems as the definability of graphs by their semigroups of endomorphisms [5], [6], the description of abstract characteristics of semigroups of endomorphisms of graphs [7], the study of algebraic and combinatorial properties of endomorphisms of graphs [8–10], the description of representations of monoids of endomorphisms of graphs and hypergraphs [11–13] and etc.

Presently, in the semigroup theory the construction of a wreath product and its different modifications are actively used for the description of exact representations of monoids of endomorphisms. In [14] V. Fleischer introduced the construction of a wreath product of a monoid with a small category as a generalization of a wreath product of monoids and applied it in different cases. V. Fleischer and U. Knauer [15] used this construction for the description of an exact representation of the monoid of endomorphisms of an arbitrary act. In [16] the author proved that the semigroup of endomorphisms of a free product of semigroups from a given class is isomorphic to a wreath product of a transformation semigroup with some small category. U. Knauer and M. Nieporte in [17] proved that the monoid of strong endomorphisms of an arbitrary undirected finite graph without multiple edges is isomorphic to a wreath product of a canonical strong quotient graph with some category. Similar results for a suitable class of undirected infinite graphs without multiple edges and n-uniform hypergraphs were announced in [18]. In this direction the reduction's problem of the description of the semigroup of endomorphisms of an arbitrary disconnected hypergraph to the description of its connected components is opened. In terms of a wreath product construction we solve this problem for the monoid of endomorphisms of disconnected hypergraphs and some its submonoids.

The paper is organized in the following way. In Section 2, using a wreath product of a transformation semigroup with a small category, we prove the reduction's theorem for the semigroup of endomorphisms of a disconnected hypergraph. In Section 3 by means of different semigroup constructions we study the structure of the monoid of strong endomorphisms of disconnected hypergraphs. In Section 4 we show that the group of automorphisms of a disconnected hypergraph is isomorphic to a direct product of wreath products of groups. Finally, in Section 5 we study the monoid of endomorphisms of arbitrary relational structures.

2. The monoid of endomorphisms of an arbitrary disconnected hypergraph

It is known that there are different definitions of a hypergraph. In this paper we use the definition of a hypergraph which is given in [2].

Let V be an arbitrary non-empty set, E be a family of non-empty subsets of V. A pair (V, E) is called a (undirected) hypergraph with the set of vertices V and the set of edges E.

It is clear that any undirected graph is an example of a undirected hypergraph. Therefore all results of the paper are correct for arbitrary undirected graphs also.

Let $H = (V, E)$ be an arbitrary hypergraph. The set V and the set E of the hypergraph H are denoted also by $V(H)$ and $E(H)$ respectively.

By $\Im(X)$ we denote the symmetric semigroup on a set X. A transformation $f \in \Im(V)$ is called an endomorphism of a hypergraph $H = (V, E)$ if for all $A \subseteq V$ the condition $A \in E$ implies $Af \in E$.

The set of all endomorphisms of a hypergraph H is a semigroup with respect to the ordinary operation of the composition of transformations. This semigroup is called a semigroup of endomorphisms of a hypergraph H and is denoted by $End\, H$.

A sequence of edges $e_1, e_2, \ldots, e_n$ of a hypergraph H is called a chain if $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}$. Vertices a and b of a hypergraph $H = (V, E)$ are called connected [13] if there exists the chain $e_1, e_2, \ldots, e_n \in E$ such that $a \in e_1$ and $b \in e_n$.

A hypergraph H is called connected, if every pair of vertices in the hypergraph is connected, and disconnected, in other cases. A connected component is a maximal connected subhypergraph of H.

We denote by α the relation of a connectivity of vertices on an arbitrary hypergraph H. Obviously, α is an equivalence relation on the set V and connected components are then induced subhypergraphs formed by equivalence classes of this relation.

Let H/α be a set of all connected components $K_i, i \in I$, of a hypergraph H, that is

$$H/\alpha = \{K_i \mid i \in I\}.$$

We will also designate by $H = \bigcup_{i \in I} K_i$ an arbitrary hypergraph H. The set of all homomorphisms from the connected component K_i to the component K_j is denoted by $Hom(K_i; K_j)$.

If $\varphi : A \to B$ is an arbitrary mapping and $\emptyset \neq Y \subseteq A$, then by $\varphi|_Y$ we designate the restriction φ on the subset Y.

The description of endomorphisms of a disconnected hypergraph gives the following

Lemma 1. *A transformation φ of an arbitrary disconnected hypergraph $H = \bigcup_{i\in I} K_i$ is an endomorphism if and only if for any $i \in I$ there exists $j \in I$ such that $\varphi\,|_{K_i} \in Hom(K_i; K_j)$.*

Proof. Let $\varphi \in EndH$ and K_i be an arbitrary connected component of H. If $|K_i| = 1$, then it is clear that $\varphi\,|_{K_i} \in Hom(K_i; K_j)$ for some $j \in I$. Suppose that $|K_i| \geq 2$ and $a, b \in K_i, a \neq b$. Then there exists the chain $e_1, e_2, \ldots, e_n \in E$ such that $a \in e_1$ and $b \in e_n$. As φ is an endomorphism of H, then $e_i\varphi \in E$ for all $i \in \{1, 2, \ldots, n\}$. Therefore there exists the chain $e_1\varphi, e_2\varphi, \ldots, e_n\varphi \in E$ which connects $a\varphi$ and $b\varphi$, that is $a\varphi, b\varphi \in K_j$ for some $j \in I$.

The converse statement is obvious. □

Let $\mathscr{K}$ be a small category, S be a monoid with an identity 1 which acts on the left on a set of objects $X = Ob\mathscr{K}$ of the category $\mathscr{K}$. We put

$$M = \bigcup_{a,b\in X} Mor_{\mathscr{K}}(a; b)$$

and denote by $Map(X; M)$ the set of all mappings from X to M.

Further let

$$W = \{(s; f) | s \in S, f \in Map(X; M), xf \in Mor_{\mathscr{K}}(x; sx) \;\; for \;\; x \in X\}.$$

For all $(r; f), (p; g) \in W$ we define the operation:

$$(r; f)(p; g) = (rp; f_p g),$$

where $x(f_p g) = (px)f\,xg$ for all $x \in X$ and $(px)f\,xg$ is a composition of morphisms $(px)f$, xg in the category $\mathscr{K}$.

The set W with a such multiplication is a monoid with the identity $(1; e)$, where mapping $e \in Map(X; M)$ is such that $xe \in Mor(x; x)$ is an identical morphism id_x for every object x in $\mathscr{K}$.

The monoid W is called the wreath product of the monoid S with the category K and is denoted by $S\,wr\,\mathscr{K}$ [14].

Observe that if $W = S\,wr\,\mathscr{K}$ is a finite wreath product, then

$$|W| = \sum_{s\in S} \prod_{x\in X} |Mor_{\mathscr{K}}(x; sx)|.$$

Let $H = \bigcup_{i\in I} K_i$ be an arbitrary hypergraph, α be the relation of a connectivity of vertices on H and $\mathscr{C}$ be the small category such that

$$Ob\mathscr{C} = H/\alpha, \;\; Mor_{\mathscr{C}}(K_i;K_j) = Hom(K_i;K_j),$$

$$Mor\mathscr{C} = \bigcup_{i,j\in I} Mor_{\mathscr{C}}(K_i;K_j).$$

We denote by $T(I)$ the set of all transformations ζ of the set I such that $Mor_{\mathscr{C}}(K_i;K_{i\zeta}) \neq \varnothing$ for all $i \in I$. It is clear that $T(I)$ is a subsemigroup of the symmetric semigroup $\Im(I)$ and objects of the category $\mathscr{C}$ naturally set left $T(I)$-act:

$$\zeta K_i = K_{i\zeta}, \;\; \zeta \in T(I).$$

Thus, we obtain the construction of the wreath product $T(I)\, wr\, \mathscr{C}$ of the transformation monoid $T(I)$ with the small category $\mathscr{C}$.

Let $(g;f) \in T(I)\, wr\, \mathscr{C}$, where $g \in T(I)$, $f \in Map(Ob\mathscr{C};Mor\mathscr{C})$. Then $t(K_i f) \in K_{ig}$ for all $t \in K_i$.

Note that the composition of mappings in the next theorem only is defined from the right to the left.

The main result of this paper is the following theorem:

Theorem 1. *The monoid of endomorphisms $EndH$ of an arbitrary disconnected hypergraph $H = \bigcup_{i\in I} K_i$ is isomorphic to the wreath product $T(I)\, wr\, \mathscr{C}$ of the transformation monoid $T(I)$ with the small category $\mathscr{C}$.*

Proof. Let $\eta \in End\, H$. By η^* we designate the transformation of I which in according to Lemma 1 is induced by the endomorphism η:

$$\eta^* : I \to I : i \mapsto i\eta^* = j, \;\; \textit{if}\;\; K_i\eta \subseteq K_j.$$

Define the mapping ξ from the semigroup $End\, H$ into the wreath product $T(I)\, wr\, \mathscr{C}$ by the rule:

$$\xi : \eta \mapsto (\eta^*;h), \textit{where}\;\; K_i h = \eta|_{K_i} \;\; \textit{for all}\;\; i \in I.$$

This mapping is well-defined as $K_i h \in Hom(K_i;K_{i\eta^*})$ for all $i \in I$.

For any elements $\varphi, \psi \in End\, H$ we have

$$(\varphi\psi)\xi = ((\varphi\psi)^*;\mu), \textit{where}\;\; K_i\mu = (\varphi\psi)|_{K_i}, i \in I,$$

$$\varphi\xi = (\varphi^*;f) \;\; \textit{and} \;\; \psi\xi = (\psi^*;g), \;\; \textit{where}\;\; K_i f = \varphi|_{K_i}, \;\; K_i g = \psi|_{K_i}, i \in I.$$

Then

$$\varphi\xi\, \psi\xi = (\varphi^*;f)(\psi^*;g) = (\varphi^*\psi^*; f_{\psi^*} g).$$

It is clear that $(\varphi\psi)^* = \varphi^*\psi^*$. Moreover, for all $K_i \in Ob\mathscr{C}$

$$K_i\,\mu = (\varphi\psi)|_{K_i} = \varphi|_{K_i\psi}\psi|_{K_i} = \varphi|_{K_{i\psi^*}}\psi|_{K_i} =$$

$$= K_{i\psi^*}f \;\; K_i g = (\psi^* K_i)f \;\; K_i g = K_i(f_{\psi^*}g).$$

Thus, $f_{\psi^*}g = \mu$ and so, ξ is a homomorphism.

If $\varphi \neq \psi$, then $x\varphi \neq x\psi$ for some $x \in K_i$. It means that $x(K_i f) \neq x(K_i g)$, hence $\varphi\xi \neq \psi\xi$.

Assume $(\alpha;\beta) \in T(I)\,wr\,\mathscr{C}$ and define a transformation λ of the hypergraph H by

$$x\lambda = x(K_i\beta), \;\; if\; x \in K_i$$

for all $x \in H$. Easily to see that $\lambda \in End\,H$ and $\lambda\xi = (\lambda^*;\beta) = (\alpha;\beta)$. Therefore, $End\,H \cong T(I)\,wr\,\mathscr{C}$. □

Let φ be an arbitrary endomorphism of a finite disconnected hypergraph $H = \bigcup_{i\in I} K_i$. By Theorem 1 φ can be represented as $\varphi = (\varphi^*; f)$. Since the mapping f is defined by the rule: $K_i f = \varphi|_{K_i}$, $i \in I$, then it can be chosen by

$$\prod_{i\in I} |\; Hom(K_i; K_{i\,\varphi^*}) \;|$$

ways. Thus, from Theorem 1 we obtain the following

Corollary 1. *Let $H = \bigcup_{i\in I} K_i$ be a disconnected finite hypergraph. Then*

$$|End\,H| = \sum_{\zeta\in T(I)} \prod_{i\in I} |\; Hom(K_i; K_{i\zeta}) \;| \,.$$

Consider as an example the hypergraph $H = (V, E)$, where

$$V = \{a, b, c, d, e, f, g\}, \;\; E = \{\{a\}, \{b, c\}, \{c, d\}, \{c\}, \{e, f, g\}\}.$$

Connected components of H are K_1, K_2, K_3 such that

$$V(K_1) = \{a\}, E(K_1) = \{\{a\}\},$$

$$V(K_2) = \{b, c, d\}, E(K_2) = \{\{b, c\}, \{c, d\}, \{c\}\},$$

$$V(K_3) = \{e, f, g\}, E(K_3) = \{\{e, f, g\}\}.$$

In this case, $Ob\mathscr{C} = \{K_1, K_2, K_3\}$ and morphisms of $\mathscr{C}$ are the following: sets $Hom(K_1; K_1)$, $Hom(K_1; K_2)$, $Hom(K_2; K_1)$, $Hom(K_3; K_1)$ are single-element, $Hom(K_3; K_3)$ is the symmetric group on K_3 and

$$Hom(K_2; K_2) = \{\varphi \in \Im(K_2) | \varphi(c) = c\},$$

$$Hom(K_3; K_2) = \{\varphi \in Map(K_3; K_2) | im(\varphi) \in E(K_2)\}.$$

From here, putting $I = \{1, 2, 3\}$, we obtain

$$T(I) = \{\varphi \in \Im(I) | \varphi|_{\{1,2\}} \in \Im(\{1, 2\})\}$$

and by Theorem 1 the monoid $EndH$ is isomorphic to $T(I)\, wr\, \mathscr{C}$.

Besides, by Corollary 1 we have

$$|End\, H| = 2 \cdot (1 + 9 + 13 + 117 + 6 + 54) = 400.$$

Further we consider another construction for the description of an exact representation of the semigroup of endomorphisms of any disconnected hypergraph.

Let $H = \bigcup_{i \in I} K_i$ be a disconnected hypergraph and $\mathscr{C}$ be the small category defined above. We put $Mor^0\mathscr{C} = Mor\mathscr{C} \cup \{0\}$, where $0 \notin Mor\mathscr{C}$, and define on this set a such operation:

$$\varphi\psi = \begin{cases} \varphi \circ \psi, & \varphi \neq 0 \neq \psi \text{ and composition } \varphi, \psi \text{ is defined,} \\ 0 & \text{in other cases,} \end{cases}$$

where $\varphi \circ \psi$ is a composition of morphisms.

It is clear that the set $Mor^0\mathscr{C}$ is a semigroup with respect to the above defined operation. This semigroup up to an isomorphism is contained in the semigroup B_H of all binary relations on H.

Let T be an arbitrary semigroup, $G(T)$ be a set of all its subsets. Putting $AB = \{ab | a \in A, b \in B\}$ for all $A, B \subseteq T$, we obtain a semigroup on $G(T)$ which is called a global supersemigroup of a semigroup T.

The set of all elements which are taken one from each block of a partition $D_X = \{X_\lambda | \lambda \in \Lambda\}$ of some set X is called a cross-section of this partition.

For the partition

$$L_{Mor^0\mathscr{C}} = \{Hom(K_i; *) | i \in I\} \cup \{\{0\}\},$$

where $Hom(K_i; *) = \bigcup_{j \in I} Hom(K_i; K_j)$, of the semigroup of morphisms $Mor^0\mathscr{C}$ we denote the set of all its cross-sections by $T(L_{Mor^0\mathscr{C}})$.

Lemma 2. *The set $T(L_{Mor^0\mathscr{C}})$ is a subsemigroup of the global supersemigroup $Gl(Mor^0\mathscr{C})$ of the semigroup $Mor^0\mathscr{C}$.*

Proof. Let $A, B \in T(L_{Mor^0\mathscr{C}})$. Since

$$\{dom(\eta)|\eta \in B, \eta \neq 0\} = Ob\mathscr{C},$$

then for each $\varphi \in A, \varphi \neq 0$ there exists a unique element $\psi \in B$, $\psi \neq 0$ such that $im(\varphi) \subseteq dom(\psi)$. From here $\varphi\psi \neq 0$. Besides, in those cases when φ is the same and $f \in B$ such that $f \neq \psi$ we have $\varphi f = 0$.

Thus, taking into account that $0 \in AB$ and

$$\{dom(\eta)|\eta \in A, \eta \neq 0\} = Ob\mathscr{C},$$

$$dom(\varphi) = dom(\varphi\psi), when \ \varphi\psi \neq 0,$$

we obtain $AB \in T(L_{Mor^0\mathscr{C}})$. □

The representation of the monoid of endomorphisms of a disconnected hypergraph by unary relations gives

Theorem 2. *For any disconnected hypergraph H the following isomorphism holds:*

$$End\,H \cong T(L_{Mor^0\mathscr{C}}).$$

Proof. The isomorphism ξ from $End\,H$ to $T(L_{Mor^0\mathscr{C}})$ is defined by the next rule:

$$f\xi = \{f|_A \colon A \in Ob\mathscr{C}\} \cup \{0\}$$

for all $f \in End\ H$. □

3. The monoid of strong endomorphisms of an arbitrary disconnected hypergraph

Let $H = (V, E)$ and $H' = (V', E')$ be arbitrary hypergraphs. A homomorphism $\varphi : H \to H'$ is called a strong homomorphism if $A\varphi \in E'$ implies $A \in E$ for all $A \subseteq V$ such that there exists an edge $e \in E$ with $|e| = |A|$. The set of all strong homomorphisms from H into H' is denoted by $SHom(H; H')$. It is clear that $SEndH$ is a submonoid of the monoid $EndH$.

Recall that a subhypergraph of H is called a correct hypergraph if it contains all edges of H.

Lemma 3. *An endomorphism φ of an arbitrary disconnected hypergraph $H = \bigcup_{i\in I} K_i$ is a strong endomorphism if and only if*

(i) for any $i \in I$ there exists $j \in I$ such that $\varphi\,|_{K_i} \in SHom(K_i; K_j)$;

(ii) any edge of the correct subhypergraph $H\varphi$ can not be represented as a union of two or more non-empty subsets from the corresponding images $K_i\varphi, i \in I$.

Proof. Let $\varphi \in SEndH$. Taking into account Lemma 1 we immediately obtain the condition (i). Assume that there exists $e \in E(H\varphi)$ such that $e = \bigcup_{j\in J} A_j$, $\emptyset \neq A_j \subseteq K_j\varphi$ for some subset $J \subseteq I, |J| \geq 2$. Then there exists a subset $A \subseteq e\varphi^{-1}$ such that $|A| = |e|, A\varphi = e$ and $A \nsubseteq K_i$ for all $i \in I$. On the other hand, since the endomorphism φ is strong, then $A \in E(H)$ and so $A \subseteq K_\alpha$ for some $\alpha \in I$.

The sufficiency of the assertion is obvious. □

Let $H = \bigcup_{i\in I} K_i$ be an arbitrary hypergraph. Denote by $\mathscr{C}_S$ the small category such that

$$Ob\mathscr{C}_S = H/\alpha, \;\; Mor_{\mathscr{C}_S}(K_i; K_j) = SHom(K_i; K_j).$$

We denote by $T_S(I)$ the monoid of all transformations of the set I which in according to Lemma 3 are induced by all strong endomorphisms of the hypergraph H.

Theorem 3. *The monoid of strong endomorphisms $SEndH$ of any disconnected hypergraph $H = \bigcup_{i\in I} K_i$ is isomorphic to the wreath product $T_S(I)\, wr\, \mathscr{C}_S$ of the transformation monoid $T_S(I)$ with the category $\mathscr{C}_S$.*

Proof. The proof of the theorem is the same as the proof of Theorem 1. □

Let H be an arbitrary disconnected finite hypergraph and $\varphi = (\varphi^*; f)$ $\in SEnd\,H$ (see Sect. 2). In this case the mapping f for the given transformation φ^* can be chosen by

$$\prod_{i\in I} |\; SHom(K_i; K_{i\,\varphi^*}) \;|$$

ways. So, we have the following assertion:

Corollary 2. *Let $H = \bigcup_{i\in I} K_i$ be a disconnected finite hypergraph. Then*

$$|SEnd\,H| = \sum_{\zeta\in T_S(I)} \prod_{i\in I} |\; SHom(K_i; K_{i\zeta}) \;| \, .$$

Assume that $T(L_{Mor^0\mathscr{C}_S})$ be a semigroup which is defined on the set of morphisms of the category $\mathscr{C}_S$ in the same way as $T(L_{Mor^0\mathscr{C}})$ is defined on $Mor^0\mathscr{C}$ (see Sect. 2). Denote by $T^*(L_{Mor^0\mathscr{C}_S})$ the subsemigroup of $T(L_{Mor^0\mathscr{C}_S})$ which consists of all elements A such that any edge of the correct subhypergraph defined on the set $\bigcup_{\varphi\in A,\varphi\neq 0} im(\varphi)$ can not be represented as a union of two or more non-empty subsets from the corresponding images $im(\varphi), 0 \neq \varphi \in A$. Thus, for any disconnected hypergraph H we have

$$SEndH \cong T^*(L_{Mor^0\mathscr{C}_S}).$$

Further we describe an exact representation of the monoid of strong endomorphisms of undirected finite graphs in terms of the wreath product of a group of automorphisms with a suitable category.

Let $G = (V, E)$ be a undirected finite graph without multiple edges. For every $x \in V$ by $N(x)$ we denote the neighborhood of x, that is a set of all vertices $y \in V$ such that $\{x, y\} \in E$.

A binary relation ν on the set of vertices of G defined by

$$x\nu y \Leftrightarrow N(x) = N(y)$$

is an equivalence relation. The equivalence class of ν which contains x is denoted by x_ν.

Define on the quotient set V/ν a new graph G/ν putting $V(G/\nu) = V/\nu$ and $\{x_\nu, y_\nu\} \in E(G/\nu)$ if and only if $\{x, y\} \in E(G)$. In particular, a loop $\{z_\nu\} \in E(G/\nu)$ if and only if a loop $\{z\} \in E(G)$. The defined graph is called a canonical strong quotient graph of G.

Recall that a generalized lexicographic product of the graph U with graphs $Y_u, u \in U$, is a graph $U[(Y_u)_{u\in U}]$ such that

$$V(U[(Y_u)_{u\in U}]) = \{(u; y_u) | u \in U, y_u \in Y_u\}$$

and $\{(u; y_u), (v; z_v)\} \in E(U[(Y_u)_{u\in U}])$ if and only if one of the following conditions holds:

(i) $\{u, v\} \in E(U)$,
(ii) $u = v, \{u\} \in E(U)$,
(iii) $u = v, \{u\} \notin E(U), \{y_u, z_u\} \in E(Y_u)$.

It is well known that any undirected graph G without multiple edges can be exactly represented as a generalized lexicographic product of the quotient graph $U = G/\nu$ with graphs $Y_u, u \in U$, such that $| Y_u |=| u |$ for all $u \in U$.

By $AutG$ we denote the group of all automorphisms of G.

Theorem 4. [17] *Let $G = U[(Y_u)_{u\in U}]$ be a undirected finite graph without multiple edges, $U = G/\nu$ and $\mathscr{K}$ be a small category such that $Ob\mathscr{K} = U$, $Mor_{\mathscr{K}}(Y_u; Y_v) = Map(Y_u; Y_v)$. Then*

$$SEndG \cong AutUwr\mathscr{K}.$$

The isomorphic mapping ξ from $SEnd\,G$ to $AutUwr\mathscr{K}$ is defined by the condition:

$$f\xi = (f^*; f'), \ \ where\ \ f^* : x_\nu \mapsto (xf)_\nu, \ \ f' : A \mapsto f|_A$$

for all $f \in SEnd\,G$.

Observe that Theorem 4 does not hold for undirected infinite graphs without multiple edges. Indeed, let N be the set of all positive integers and let

$$E = \{\{n, n+1\}, \{n+1, n+2\}, \{n+2, n+3\} | n \in 4N\} \cup \{\{1, 2\}, \{2, 3\}\}.$$

Then the transformation $f : n \mapsto n + 4$ of the graph (N, E) is a strong endomorphism such that for the two-element class $\{1, 3\}$ of $(N, E)/\nu$ we have $1_\nu f^* = 5_\nu$, $3_\nu f^* = 7_\nu$ and $5_\nu \neq 7_\nu$.

Besides, in [17] it was proved that the monoid of all strong endomorphisms of an arbitrary undirected finite graph without multiple edges is a regular monoid. For arbitrary undirected graphs conditions of a regularity of the monoid of strong endomorphisms were studied in [19], [20].

4. The group of automorphisms of an arbitrary disconnected hypergraph

A bijective homomorphism $\varphi : H \to H'$ from a hypergraph H to a hypergraph H' is called an isomorphism if f^{-1} is a homomorphism also. The set of all isomorphisms from H to H' is denoted by $ISo(H; H')$.

Let $H = \bigcup_{i\in J} K_i$ be an arbitrary hypergraph. Denote by $\mathscr{C}_{IS}$ the small category such that

$$Ob\mathscr{C}_{IS} = H/\alpha, \ \ Mor_{\mathscr{C}_{IS}}(K_i; K_j) = ISo(K_i; K_j).$$

We denote the group of all bijective transformations ζ of the set J such that $ISo(K_i; K_{i\zeta}) \neq \varnothing$ for all $i \in J$ by $T_{IS}(J)$.

Theorem 5. *The group of automorphisms $AutH$ of an arbitrary disconnected hypergraph $H = \bigcup_{i\in J} K_i$ is isomorphic to the wreath product*

$T_{IS}(J)\, wr\, \mathscr{C}_{IS}$ of the permutation group $T_{IS}(J)$ with the category $\mathscr{C}_{IS}$. Moreover, if H is a finite hypergraph, then

$$|Aut\, H| = \sum_{\zeta\in T_{IS}(J)} \prod_{i\in J} \mid ISo(K_i; K_{i\zeta}) \mid .$$

Proof. The proof of this theorem is the same as the proof of Theorem 1. □

Let $T(L_{Mor^0\mathscr{C}_{IS}})$ be a semigroup which is defined on the set of morphisms of the category $\mathscr{C}_{IS}$ in the same way as $T(L_{Mor^0\mathscr{C}})$ is defined on $Mor^0\mathscr{C}$ (see Sect. 2). Denote by $T^*(L_{Mor^0\mathscr{C}_{IS}})$ the subgroup of $T(L_{Mor^0\mathscr{C}_{IS}})$ which consists of all elements A such that the following conditions hold:

(i) $im(\varphi) \neq im(\psi)$ for all non-zero different $\varphi, \psi \in A$;

(ii) for all $K_i \in H/\alpha$ there exists a non-zero $\varphi \in A$ with $im(\varphi) = K_i$.

Therefore, for any disconnected hypergraph H we have

$$Aut\, H \cong T^*(L_{Mor^0\mathscr{C}_{IS}}).$$

Now we consider other representation of the group of automorphisms of a disconnected hypergraph which is similar to the description of the group of automorphisms of a disconnected graph (see, for instance, [21]).

Let α be the relation of a connectivity of vertices on a hypergraph H, P be a set of all representatives taken on one from every class of an isomorphism's relation on H/α and let $Y = \{i \mid K_i \in P\}$.

For every $i \in Y$ we put

$$\overline{i} = \{j \in J \mid K_i \cong K_j\} \ \ and \ \ F_i = \bigcup_{j\in\overline{i}} K_j.$$

Thus, H is a disjoint union of hypergraphs $F_i, i \in Y$, and we obtain

Lemma 4. *The group of automorphisms $Aut\, H$ of an arbitrary disconnected hypergraph H is isomorphic to the direct product $\prod_{i\in Y} Aut\, F_i$ of groups of automorphisms $Aut\, F_i$ of hypergraphs $F_i, i \in Y$.*

Proof. Let φ be an automorphism of the hypergraph H. It is clear that $\varphi\mid_{F_i}\in Aut\, F_i$ for all $i \in Y$. Conversely, if some transformation φ of H isomorphically maps each hypergraph $F_i, i \in Y$, on itself then, obviously, φ is an automorphism of H. Thus, the mapping

$$\xi : Aut\, H \to \prod_{i\in Y} AutF_i : \phi \mapsto \phi\xi = (\phi_i), \ \ \phi_i = \phi|_{F_i},$$

is an isomorphism of groups. □

Let G be an arbitrary group, X be a non-empty set. By $\Im[X]$ we denote the symmetric group on X and by $Fun(X;G)$ the direct product of isomorphic copies G_x of the group G which are indexed by elements of X. Thus, $Fun(X;G)$ is a group of all functions $X \to G$ with the ordinary composition of functions.

Define on the set $\Im[X] \times Fun(X;G)$ the operation by

$$(g;f)(g';f') = (gg';f^{g'}f'),$$

where $xf^g = (xg^{-1})f$ for all $x \in X$.

Concerning the defined operation $\Im[X] \times Fun(X;G)$ is a group which is called the wreath product of the group G with the symmetric group $\Im[X]$ (see, e.g., [22]). This group is denoted by $G\,Wr\,\Im[X]$.

In terms of the wreath product of groups we obtain the description of groups of automorphisms of disconnected hypergraphs $F_i, i \in Y$.

Lemma 5. *For any $i \in Y$ the group of automorphisms $AutF_i$ of a disconnected hypergraph F_i is isomorphic to the wreath product $Aut\,K_i\,Wr\,\Im[\overline{i}]$ of the group of automorphisms $Aut\,K_i$ with the symmetric group $\Im[\overline{i}]$.*

Proof. Let φ be an automorphism of the hypergraph $F_i, i \in Y$. Denote by $\delta_\varphi : \overline{i} \to \overline{i}$ the bijection which is induced by φ and for all $\lambda, \mu \in \overline{i}$ fix isomorphisms $f^{(\lambda;\mu)} : K_\lambda \to K_\mu$ such that $f^{(\lambda;\lambda)}$, $\lambda \in \overline{i}$, are identical automorphisms and the equality $f^{(\lambda;\mu)}f^{(\mu;\lambda)} = f^{(\lambda;\lambda)}$ holds.

Further we define the mapping

$$\xi : AutF_i \to Aut\,K_i\ \ Wr\ \ \Im[\overline{i}] : \varphi \mapsto \varphi\xi = (\delta_\varphi;\widetilde{\varphi}),$$

putting $j\widetilde{\varphi} = f^{(i;j\delta_\varphi^{-1})}\varphi|_{K_{j\delta_\varphi^{-1}}} f^{(j;i)}$ for all $j \in \overline{i}$.

Let $\varphi, \psi \in AutF_i$, then

$$\varphi\xi = (\delta_\varphi;\widetilde{\varphi}),\ \ \psi\xi = (\delta_\psi;\widetilde{\psi})\ \ and\ \ (\varphi\psi)\xi = (\delta_{\varphi\psi};\widetilde{\varphi\psi}).$$

It is clear that $\delta_{\varphi\psi} = \delta_\varphi\delta_\psi$. Besides, $\varphi\xi\psi\xi = (\delta_\varphi\delta_\psi;\widetilde{\varphi}^{\delta_\psi}\widetilde{\psi})$, *where*

$$j(\widetilde{\varphi}^{\delta_\psi}\widetilde{\psi}) = j\widetilde{\varphi}^{\delta_\psi}j\widetilde{\psi} = (j\delta_\psi^{-1})\widetilde{\varphi}\ \ j\widetilde{\psi} =$$

$$= (f^{(i;(j\delta_\psi^{-1})\delta_\varphi^{-1})}\varphi|_{K_{(j\delta_\psi^{-1})\delta_\varphi^{-1}}} f^{(j\delta_\psi^{-1};i)})\ (f^{(i;j\delta_\psi^{-1})}\psi|_{K_{j\delta_\psi^{-1}}} f^{(j;i)}) =$$

$$= f^{(i;j(\delta_\varphi\delta_\psi)^{-1})}\varphi|_{K_{j(\delta_\varphi\delta_\psi)^{-1}}} f^{(j\delta_\psi^{-1};i)} f^{(i;j\delta_\psi^{-1})}\psi|_{K_{j\delta_\psi^{-1}}} f^{(j;i)} =$$

$$= f^{(i;j\delta_{\varphi\psi}^{-1})}\varphi|_{K_{j\delta_{\varphi\psi}^{-1}}}\psi|_{K_{j\delta_{\psi}^{-1}}}f^{(j;i)} = f^{(i;j\delta_{\varphi\psi}^{-1})}(\varphi\psi)|_{K_{j\delta_{\varphi\psi}^{-1}}}f^{(j;i)} = j\widetilde{\varphi\psi}$$

for all $j \in \overline{i}$. Thus, ξ is a homomorphism.

Finally, note that the bijectivity of ξ is checked directly. □

From Lemmas 4 and 5 we obtain

Theorem 6. *The group of automorphisms $Aut\, H$ of any disconnected hypergraph H is isomorphic to the direct product $\prod_{i\in Y} Aut\, K_i\, Wr\, \Im[\overline{i}]$ of wreath products of groups of automorphisms $Aut\, K_i$ with symmetric groups $\Im[\overline{i}]$, $i \in Y$.*

From here it follows that it is enough to consider groups of automorphisms of connected hypergraphs.

5. The monoid of endomorphisms of relational structures

In this section we show that obtained results for undirect hypergraphs can be used to describe the monoid of endomorphisms of arbitrary relational structures and its submonoids.

Recall that a pair $(X, \mathscr{R})$, where $\mathscr{R}$ is some set of relations on a set X, is called a relational structure over X. Examples of relational structures are arbitrary relational clones, ordered sets, quasi-ordered sets, graphs, hypergraphs, different algebras of relations etc. The most important relational structures are those in which each relation from $\mathscr{R}$ is a binary relation. For instance, such structures are so-called coherent configurations and, in particular, associative schemes, Heming's schemes, Johnson's schemes (see, e.g., [21]).

Let $(X, \mathscr{R})$ be an arbitrary relational structure. A transformation φ of the set X is called an endomorphism of the structure $(X, \mathscr{R})$ if φ is an endomorphism of each relation from $\mathscr{R}$. The set of all endomorphisms of $(X, \mathscr{R})$ is a semigroup with respect to the ordinary operation of the composition of transformations. This semigroup is called a monoid of endomorphisms of the relational structure $(X, \mathscr{R})$ and is denoted by $End\,(X, \mathscr{R})$.

The monoid of strong endomorphisms and the group of automorphisms of an arbitrary relational structure $(X, \mathscr{R})$ are defined by the similar way. These monoids are denoted by $SEnd\,(X, \mathscr{R})$ and $Aut\,(X, \mathscr{R})$ respectively.

Now it remains to define the concept of a connectivity in the relational structure with a single relation. Take an arbitrary structure $(X, \mathscr{R})$ and assume that $\rho \in \mathscr{R}$ is a relation of an arity k. If $x = (x_1, x_2, \ldots, x_k) \in \rho$

then by x^* we denote the set $\{x_1, x_2, \ldots, x_k\}$. A sequence of elements $e_1, e_2, \ldots, e_n$ of the relation ρ we call a chain if $e_i^* \cap e_{i+1}^* \neq \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}$. Elements a and b of the relational structure (X, ρ) we call connected if there exists the chain $e_1, e_2, \ldots, e_n \in \rho$ such that $a \in e_1^*$ and $b \in e_n^*$. A relational structure (X, ρ) we will call connected, if every pair of elements in the structure is connected, and disconnected, in other cases. In addition, a connected component is a maximal connected substructure of (X, ρ).

It is clear that the relation of a connectivity in (X, ρ) is an equivalence on the set X and connected components are induced substructures formed by equivalence classes of this relation.

Further, in the same way as for hypergraphs the reduction's problem of the description of the semigroup of endomorphisms of an arbitrary disconnected relational structure (X, ρ) to the description of its connected components is solved.

Thus, knowing how to construct the monoid $End\,(X, \rho)$ for all relations $\rho \in \mathscr{R}$ we obtain

$$End\,(X, \mathscr{R}) = \bigcap_{\rho \in \mathscr{R}} End\,(X, \rho).$$

The structure of the monoid $SEnd\,(X, \mathscr{R})$ and the group $Aut\,(X, \mathscr{R})$ is described analogously.

Finally, we consider the following example. Let $(X, \mathscr{R})$ be a relational structure with $X = \{1, 2, \ldots, 7\}$ and $\mathscr{R} = \{\rho_1, \rho_2, \rho_3, \}$, where $\rho_i, 1 \leq i \leq 3$, is a relation of the arity $i+1$ and

$$\rho_1 = \{\{3\}, \{4, 5\}, \{4, 6\}, \{4, 7\}\}, \rho_2 = \{\{1, 2, 3\}, \{3, 4, 5\}, \{7\}\}, \rho_3 = \{X\}.$$

Here a set from a relation of an arity k means all k-permutations of elements of the given set. We find elements of the monoid $SEnd\,(X, \mathscr{R})$.

Obviously, $SEnd\,(X, \rho_3) = \Im(X)$. Using Theorem 4, we obtain

$$SEnd\,(X, \rho_1) = \{\varphi \in \Im(X) | \{1, 2\}\varphi \subseteq \{1, 2\}, 3\varphi = 3 \text{ and}$$

$$4\varphi = 4, \{5, 6, 7\}\varphi \subseteq \{5, 6, 7\} \text{ or } 4\varphi \in \{5, 6, 7\}, \{5, 6, 7\}\varphi = \{4\}\}.$$

Similarly as in Theorem 3 we establish that

$$SEnd\,(X, \rho_2) = \{\varphi \in \Im(X) | \varphi|_{\{3,6,7\}} = i_{\{3,6,7\}} \text{ and } \varphi|_{\{1,2\}} \in \Im[\{1, 2\}],$$

$$\varphi|_{\{4,5\}} \in \Im[\{4, 5\}] \text{ or } \varphi|_{\{1,2\}} \in \Im[\{4, 5\}], \varphi|_{\{4,5\}} \in \Im[\{1, 2\}]\},$$

where i_A is an identity transformation of a set A. Thus,

$$\bigcap_{1\leq i\leq 3} SEnd\,(X,\rho_i) = SEnd\,(X,\rho_1) \cap SEnd\,(X,\rho_2) \text{ and so,}$$

$$SEnd\,(X,\mathscr{R}) = \{\varphi \in \Im(X) | \varphi|_{\{1,2\}} \in \Im[\{1,2\}], \varphi|_{X\setminus\{1,2\}} = i_{X\setminus\{1,2\}}\}.$$

Therefore, $SEnd\,(X,\mathscr{R}) \cong \Im[Y]$, where $|Y| = 2$.

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