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RESEARCH ARTICLE

The diagonal limits of Hamming spaces

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Abstract. We consider a continuum family of subspaces of the Besicovitch–Hamming space on some alphabet B, naturally parametrized by supernatural numbers. Every subspace is defined as a diagonal limit of finite Hamming spaces on the alphabet B. We present a convenient representation of these subspaces. Using this representation we show that the completion of each of these subspace coincides with the completion of the space of all periodic sequences on the alphabet B. Then we give answers on two questions formulated in [1].

Introduction

Let $B = \{b_1, \ldots, b_q\}$ be an alphabet, $q \geq 2$. Denote by $H_n(q)$ the *Hamming space* of dimension n on the alphabet B. This space consists of all n-tuples $(a_1, \ldots, a_n)$, $a_i \in B$, $1 \leq i \leq n$, where the distance d_{H_n} between two n-tuples is equal to the number of coordinates where they differ. The *scaled Hamming space* $\hat{H}_n(q)$ have the same set of points, but the distance is defined as $\frac{1}{n} d_{H_n}$. A natural generalization of the scaled Hamming space is the Besicovitch space (in other terms, the Besicovitch–Hamming space), consisting of all infinite sequences on the alphabet B ([2], [3]). This space is used since the 1960s in symbolic dynamics and ergodic theory.

In this article we consider a family of subspaces of the Besicovitch–Hamming space on alphabet B, naturally parametrized by infinite supernatural numbers. The space corresponding to a supernatural number

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u is called the u-periodic Hamming space. For every u the u-periodic Hamming space is isometric to the direct limits of finite scaled Hamming spaces with respect to diagonal embeddings.

If the alphabet B consists of two elements, i.e. $B = \{0, 1\}$, then u-periodic Hamming spaces were characterized in [1] and [5]. In [1] the 2^∞-periodic Hamming space was considered as the space of finite unions of half-open subintervals of the interval $[0, 1)$ with binary-rational endpoints. In [5] u-periodic Hamming spaces were regarded as the spaces of clopen subsets of the boundaries of spherically homogeneous rooted trees. The completion of every space from these family is isometric to the completion of the space of 2^∞-periodic $(0, 1)$-sequences [1] or of all periodic $(0, 1)$-sequences [5].

In this paper we consider the case $q > 2$. In [1] P. J. Cameron and S. Tarzi formulated the questions:

(A) Is there a convenient representation of 2^∞-periodic Hamming space on alphabet B and its completion?

(B) Are completions of u-periodic Hamming spaces on alphabet B independent of choice of u?

We give answers to these questions. Namely, for every infinite supernatural numbers u we represent the u-periodic Hamming space on alphabet B and its completion as the spaces of functions defined in a special way on the boundaries of spherically homogeneous rooted trees. These functions define a measurable partition in sense of [4]. Using this representation we prove that the completion of every u-periodic Hamming space is independent of choice of u and coincides with the completion of the space of 2^∞-periodic sequences or of all periodic sequences.

1. Preliminaries

1. Let $\mathbb{P}$ be the set of all primes. A *supernatural number* (or Steinitz number) is an infinite formal product of the form

$$\prod_{p \in \mathbb{P}} p^{k_p}$$

where $k_p \in \mathbb{N} \cup \{0, \infty\}$. Denote by $\mathbb{SN}$ the set of all supernatural numbers. The set $\mathbb{N}$ is a natural subset of $\mathbb{SN}$. The elements of the set $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite supernatural* numbers. A supernatural number v divides a supernatural number u if there exists $t \in \mathbb{SN}$, such that $u = v \cdot t$. The divisibility relation $|$ transforms the set $\mathbb{SN}$ into a partially ordered set with the greatest element $I = \prod_{p \in P} p^\infty$ and the least element 1. Moreover, the poset $(\mathbb{SN}, |)$ is a complete lattice.

A sequence of positive integers $\tau = (m_1, m_2, \ldots)$ is called *divisible* if $k_i | k_{i+1}$ for all $i \in \mathbb{N}$. For divisible sequence $\tau = (m_1, m_2, \ldots)$ the supernatural number

$$m_1 \cdot \frac{m_2}{m_1} \cdot \frac{m_3}{m_2} \ldots$$

is called the *characteristic of the sequence* τ and denoted by $char(\tau)$. It is easy to see that the sequence τ is a strictly increasing divisible sequence iff $char(\tau)$ is an infinite supernatural number.

2. Let X be a nonempty set, $\Sigma_1 = \{X_i, 1 \leq i \leq q\}$ and $\Sigma_2 = \{Y_i, 1 \leq i \leq q\}$ be ordered partitions of the set X. These partitions decompose the set X into q (possibly empty) blocks. Introduce the *symmetric difference of partitions* Σ_1 and Σ_2 as the set $\Sigma_1 \triangle \Sigma_2$ defined by the equation

$$\Sigma_1 \triangle \Sigma_2 = \bigcup_{i \neq j} (X_i \cap Y_j). \tag{1}$$

Then $\Sigma_1 \triangle \Sigma_2 \subset X$. We can formulate some properties of the symmetric difference of partitions which are not difficult to verify.

Lemma 1. *Let $\Sigma_1 = \{X_i, 1 \leq i \leq q\}$, $\Sigma_2 = \{Y_i, 1 \leq i \leq q\}$ and $\Sigma_3 = \{Z_i, 1 \leq i \leq q\}$ be ordered partitions of the set X. Then the following properties hold.*

1) $\Sigma_1 \triangle \Sigma_2 = \varnothing$ *iff* $\Sigma_1 = \Sigma_2$.
2) $\Sigma_1 \triangle \Sigma_2 = \Sigma_2 \triangle \Sigma_1$.
3) $(\Sigma_1 \triangle \Sigma_2) \triangle \Sigma_3 = \Sigma_1 \triangle (\Sigma_2 \triangle \Sigma_3)$.
4) $\Sigma_1 \triangle \Sigma_2 \subseteq (\Sigma_1 \triangle \Sigma_3) \cup (\Sigma_2 \triangle \Sigma_3)$.

2. The periodic Hamming space

Let $\tau = (m_1, m_2, \ldots)$ be an increasing divisible sequence, $\hat{H}_{m_1}(q)$, $\hat{H}_{m_2}(q), \ldots$ be the corresponding infinite sequence of scaled Hamming spaces on alphabet B. Denote by $(s_1, s_2, \ldots)$ the sequence of ratios of the sequence τ, i. e.

$$s_1 = m_1, \qquad s_{i+1} = \frac{m_{i+1}}{m_i}, \ i \geq 1. \tag{2}$$

For any $i \geq 1$ define an isometric embedding $\psi_{s_i} : \hat{H}_{m_i}(q) \to \hat{H}_{m_{i+1}}(q)$ by the rule:

$$\psi_{s_i}(x_1, \ldots, x_{m_i}) = (\underbrace{x_1, \ldots, x_{m_i} | x_1, \ldots, x_{m_i} | \ldots | x_1, \ldots, x_{m_i}}_{s_i \cdot m_i}). \tag{3}$$

Then the sequence τ determines the directed system of scaled Hamming spaces on the alphabet B

$$\langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle_{i \in \mathbb{N}} \tag{4}$$

with the diagonal embeddings ψ_{s_i}, $i \geq 1$, defined by (3).

The limit space of the directed system (4)

$$\mathcal{H}(\tau, q) = \varinjlim \langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle$$

is called a *diagonal limit* of spaces $\hat{H}_{m_i}$.

Proposition 1. *Let τ_1, τ_2 be increasing divisible sequences. Then the spaces $\mathcal{H}(\tau_1, q)$ and $\mathcal{H}(\tau_2, q)$ are isometric iff $char(\tau_1) = char(\tau_2)$.*

The diagonal limit $\mathcal{H}(\tau, q)$ admits a natural description using supernatural numbers.

The infinite sequence $\mathbf{a} = (a_1, a_2, \ldots)$, $a_i \in B$ is said to be *periodic* if there exists a natural number k such that the equality $a_i = a_{i+k}$ holds for all $i \in \mathbb{N}$. In this case the number k is called a *period* of the sequence $\mathbf{a}$. A periodic sequence $\mathbf{a}$ is called *u-periodic* for some supernatural number u if its minimal period divides u.

Let u be some infinite supernatural number and $\mathcal{H}(u, q)$ be the set of all u-periodic sequences on B. In particular, the space $\mathcal{H}(I, q)$ is the space of all periodic sequences on B. We can introduce a natural metric on $\mathcal{H}(u, q)$ putting

$$d_{\mathcal{H}(u,q)}((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \frac{1}{l} d_{H_l}((x_1, \ldots, x_l), (y_1, \ldots, y_l)), \tag{5}$$

where l is a common period of sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ from $\mathcal{H}(u, q)$. It is clear that definition (5) is independent of choice of a common period. We call the metric space $(\mathcal{H}(u, q), d_{\mathcal{H}(u,q)})$ the *u-periodic Hamming space* over the alphabet B. It is not difficult to verify

Proposition 2. *Let τ be an increasing divisible sequence, u be some infinite supernatural number. Then the spaces $\mathcal{H}(\tau, q)$ and $\mathcal{H}(u, q)$ are isometric iff $char(\tau) = u$.*

3. Representations on boundaries of rooted trees

Let T be an infinite locally finite rooted tree with the root v_0, n be some nonnegative integer. The n-th level of the tree T is the set L_n of all vertices v of T such that the length of the unique simple path connecting v and v_0 in T equals n.

A rooted tree (T, v_0) is called *spherically homogeneous* if for every nonnegative integer n the degrees of all vertices from L_n are equal. A spherically homogeneous rooted tree T is uniquely defined by its *spherical index*, i.e. by an infinite sequence of positive integers $[s_1; s_2; \ldots,]$ such that s_i is the number of edges joining a vertex of the $(i-1)$th level with vertices of the ith level, $i \geq 1$. If the tree (T, v_0) has the spherical index $[s_1; s_2; \ldots,]$ then the sequence $m_i = s_1 \cdot s_2 \cdot \ldots \cdot s_i$, $i \geq 1$, is divisible and additionally $|L_i| = m_i$, $i \geq 1$.

The boundary ∂T of a tree T is the set of infinite rooted paths, i.e. the set of infinite sequences of pairwise distinct vertices $(v_0, v_1, v_2, \ldots)$ such that the vertices v_i, v_{i+1} are connected by an edge for every i, $i \geq 0$. These paths are also called the ends of T. Define a distance ρ on the set ∂T as

$$\rho(\gamma_1, \gamma_2) = \begin{cases} \frac{1}{k+1}, & \text{if } \gamma_1 \neq \gamma_2 \\ 0, & \text{if } \gamma_1 = \gamma_2 \end{cases}, \qquad (6)$$

where k is the length of the common beginning of rooted paths γ_1 and γ_2. The space $(\partial T, \rho)$ is an ultrametric totally disconnected compact space with diameter 1.

Let as before $\tau = (m_1, m_2, \ldots)$ be an increasing divisible sequence with the sequence of its ratios $(s_1, s_2, \ldots)$ defined by (2). Assume that T_τ is a spherically homogeneous rooted tree with spherical index $[s_1; s_2; \ldots]$ and ρ_τ is a metric defined by (6) on ∂T_τ. The set of all rooted paths from ∂T_τ passing through a vertex v is denoted

$$C_v = \{\gamma \in \partial T_\tau \mid v \in \gamma\}$$

and called the *cylindrical set* C_v corresponding to v.

The metric ρ_τ induces a topology on ∂T_τ. The clopen subsets are finite unions of cylindrical sets. Denote by ΩT_τ the set of all clopen subsets of ∂T_τ. Define the Bernoulli measure μ on the Borel σ-algebra of ∂T_τ by the rule:

$$\mu(C_v) = \frac{1}{n_v},$$

where n_v is the number of vertices of T_τ on the level containing the vertex v. The space $(\partial T_\tau, \mu)$ is isomorphic as a measure space to the space $([0,1], l)$, where l is the Lebesgue measure (see [6] for instance).

Introduce the discrete metric ϱ on the set B, i.e.

$$\varrho(b_i, b_j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases},$$

for all $1 \leq i, j \leq q$. The metric ϱ induces the discrete topology on B. Denote by $C(\partial T_\tau, B)$ the set of all continuous functions from the space ∂T_τ to the space B. Note that for every $f \in C(\partial T_\tau, B)$ the subsets $\{f^{-1}(b_i), 1 \leq i \leq q\}$ form an ordered partition of the set ∂T_τ. Define a mapping $d_\mu : C(\partial T_\tau, B) \times C(\partial T_\tau, B) \to \mathbb{R}^+$ by putting

$$d_\mu(f, g) = \mu(\Sigma_f \bigtriangleup \Sigma_g), \tag{7}$$

where $f, g \in C(\partial T_\tau, B)$ and the symmetric difference of $\Sigma_f = \{f^{-1}(b_i), 1 \leq i \leq q\}$ and $\Sigma_g = \{g^{-1}(b_i), 1 \leq i \leq q\}$ is determined by (1). The proof of the next proposition follows from Lemma 1.

Lemma 2. *The function d_μ is a metric on the set $C(\partial T_\tau, B)$.*

Theorem 1. *Let $B = \{b_1, \ldots, b_q\}$ be some alphabet, $q > 2$, $\tau = (m_1, m_2, \ldots)$ be an increasing divisible sequence with the sequence of its ratios $(s_1, s_2, \ldots)$. Assume that T_τ is a spherically homogeneous rooted tree with spherical index $[s_1; s_2; \ldots]$ and u is a supernatural number with $char(\tau) = u$. Then the u-periodic Hamming space $\mathcal{H}(u, q)$ on the alphabet B is isometric to the space of all continuous functions $C(\partial T_\tau, B)$ with the metric d_μ defined by (7).*

Proof. Let n be a positive integer. A function $f : \partial T_\tau \to B$ is called *n-determined* if for every $v \in L_n$ there exists i, $1 \leq i \leq q$, such that the equality $f(x) = b_i$ holds for any $x \in C_v$. We write $f(C_v) = b_i$ in this case. Note that every n-determined function is continuous. Conversely, for any continuous function $f \in C(\partial T_\tau, B)$ there exists a level l such that for every $n \geq l$ the function f is n-determined.

Denote by $Fun(n)$ the set of all n-determined function. Enumerate all vertices in L_n and assume that $L_n = \{v_1, \ldots, v_{m_n}\}$. Define a mapping $\varphi_n : Fun(n) \to H_{m_n}(q)$ by the rule:

$$\varphi_n(f) = (f(C_{v_1}), \ldots, f(C_{v_{m_n}})).$$

The mapping φ_n is bijective. We are going to show that φ_n preserves distances between points. Let $f, g \in Fun(n)$. The sets $f^{-1}(b_i)$ and $g^{-1}(b_i)$ are unions of cylindrical sets for all $1 \leq i \leq q$. Hence, from the definitions of the measure μ and the metric d_μ we obtain

$$d_\mu(f, g) = \mu(\Sigma_f \bigtriangleup \Sigma_g) = \frac{1}{m_n} \sum_{i=1}^{m_n} (f(C_{v_i}) \oplus g(C_{v_i})), \tag{8}$$

where

$$f(C_{v_i}) \oplus g(C_{v_i}) = \begin{cases} 1, & \text{if } f(C_{v_i}) \neq g(C_{v_i}) \\ 0, & \text{if } f(C_{v_i}) = g(C_{v_i}) \end{cases}.$$

Since

$$\frac{1}{m_n}\sum_{i=1}^{m_n} f(C_{v_i}) \oplus g(C_{v_i}) =$$
$$= d_{H_{m_n}}((f(C_{v_1}),\dots,f(C_{v_{m_n}})),(g(C_{v_1}),\dots,g(C_{v_{m_n}}))),$$

using (8) we have

$$d_\mu(f,g) = d_{H_{m_n}}((f(C_{v_1}),\dots,f(C_{v_{m_n}})),(g(C_{v_1}),\dots,g(C_{v_{m_n}}))).$$

Hence the mapping φ_n is an isometry between $(Fun(n), d_\mu)$ and $H_{m_n}(q)$.

Every cylindrical set corresponding to a vertex of the n-th level splits into the union of s_{n+1} cylindrical subsets corresponding to vertices of the $(n+1)$th level in T_τ. Every n-determined function is $(n+1)$-determined. Thus we can define an injection $\psi_n : Fun(n) \to Fun(n+1)$. Let us define an isometric embedding $\chi_n : H_{m_n}(q) \hookrightarrow H_{m_{n+1}}(q)$ by the rule:

$$\chi_n(x_1,\dots,x_{m_n}) = (\underbrace{x_1,\dots,x_1}_{s_{n+1}},\dots,\underbrace{x_{m_n},\dots,x_{m_n}}_{s_{n+1}}).$$

Then for any positive integer n the diagram

$$\begin{tikzcd} Fun(n) \arrow[r, "\psi_n"] \arrow[d, "\varphi_n"'] & Fun(n+1) \arrow[d, "\varphi_{n+1}"] \\ \hat{H}_{m_n}(q) \arrow[r, "\chi_n"] & \hat{H}_{m_{n+1}}(q) \end{tikzcd}$$

is commutative. Therefore, the spaces

$$\bigcup_{n=1}^{\infty} Fun(n) = C(\partial T_\tau, B) \quad \text{and} \quad \varinjlim \langle \hat{H}_{m_i}(q), \psi_{s_i}\rangle = \mathcal{H}(\tau, q)$$

are isometric. The proof of the theorem is complete. □

Define an equivalence $\sim$ on the set of all measurable functions $Measurable(\partial T_\tau, B)$. Let $f \sim g$ iff for every $Y \subseteq B$ the sets $f^{-1}(Y)$ and $g^{-1}(Y)$ coincide up to measure zero sets.

The following statement is the answer to the question (A).

Corollary 1. *Let $B = \{b_1,\dots,b_q\}$ be an alphabet, $q > 2$, $\tau = (m_1, m_2, \dots)$ be an increasing divisible sequence with the sequence of its ratios $(s_1, s_2, \dots)$. Assume that T_τ is a spherically homogeneous rooted tree of the spherical index $[s_1; s_2; \dots]$ and u is a supernatural number with $char(\tau) = u$. Then the completion of the u-periodic Hamming space $\mathcal{H}(u,q)$ on the alphabet B*

is isometric to the space of all measurable functions $Measurable(\partial T_\tau, B)$ *(well-defined up to measure zero sets) with the metric* d_μ *defined by* (7).

As the space $(\partial T_\tau, \mu)$ is a standard probability space and independent of the choice of a divisible sequence τ, the space of all measurable functions $Measurable(\partial T_\tau, B)$ (well-defined up to measure zero sets) with metric d_μ independent of the choice of a divisible sequence too. Thus we get an answer to the question (B) formulated in [1].

Corollary 2. *For every infinite strictly increasing divisible sequence* $\tau = (m_1, m_2, \ldots)$ *the completion of the space* $\mathcal{H}(\tau, q)$ *is isometric to the completion of the space* $\mathcal{H}(2^\infty, q)$ *or completion of the space* $\mathcal{H}(I, q)$ *of all periodic sequences over the alphabet* B.

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