

# Infinitely iterated wreath products of metric spaces

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**ABSTRACT.** The construction of the finitary wreath product of metric spaces and its completion, the infinitely iterated wreath product of metric spaces are introduced. Their full isometry groups are described. Some properties and examples of these constructions are considered.

## Introduction

Let  $s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a strictly increasing continuous function with  $s(0) = 0$ , called a *scale*. A space  $s(X)$  which arises from a metric space  $(X, d_X)$  by replacing the metric  $d_X$  by  $s(d_X)$  is called a *metric transform* of  $(X, d_X)$ . This notion for metric spaces was introduced by Blumenthal in [1]. Metric transforms were studied in many papers, in particular, metric transforms of Euclidean spaces into subsets of Hilbert space have been investigated by Schoenberg and von Neumann ([2], [3]).

In general case a metric transform  $s(X)$  of a space  $(X, d_X)$  may not be a metric space. But if  $s(t)$  is differentiable scale and the derivative  $s'$  is non-increasing then  $s(d_X)$  is a metric. Metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are called *isomorphic* ([4]) if there exist a bijections  $g : X \rightarrow Y$  and a scale  $s$ , such that for arbitrary  $u, v \in X$

$$d_X(u, v) = s(d_Y(g(u), g(v))),$$

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i.e. the space  $X$  and the metric transform  $s(Y)$  are isometric. In this case the space  $(X, d_X)$  is denoted by  $s(Y)$ . If the space  $(X, d_X)$  is isomorphic to some subspace of the space  $(Y, d_Y)$ , then we say that  $(X, d_X)$  can be isomorphically embedded in the space  $(Y, d_Y)$ . Isomorphic spaces are topologically equivalent. Note, that isometry groups of isomorphic metric spaces are isomorphic.

In this paper we introduce two constructions of uniformly discrete metric spaces with finite diameters using the notion of isomorphism of metric spaces. The first one is the finitary wreath product of metric spaces. The second one is its completion and can be introduced independently from the first one. We call it the infinitely iterated wreath product of metric spaces. This construction can be regarded as a generalization of the boundary  $\partial T$  of the infinite spherically homogeneous rooted tree  $T$  (see [5]). We also describe the isometry group, some properties and some examples of finitary and infinitely iterated wreath products of metric spaces.

It is a well-known problem for a given permutation group  $(G, X)$  to find or just to prove the existence of a discrete structure  $X$  (e.g., graph, metric space, ordered set, etc.) such that its automorphism group  $Aut(X)$  is isomorphic to  $(G, X)$  as a transformation group, see [6]. Then a natural question arises. Assume that some transformation groups are realized as automorphism groups of certain structures. What can be said about realization of different constructions (e.g., direct or wreath products) of these groups? We partially answer this question. For arbitrary transformation groups  $(G_1, X_1), (G_2, X_2), \dots$ , which are isometry groups of uniformly discrete metric spaces of finite diameters, we construct a metric space of finite diameter such that the wreath product of the given transformation groups is isomorphic to the isometry group of this space as a transformation group.

Recall that a metric space  $(U, d_U)$  is called isomorphically universal for a collection  $M$  of metric spaces if any metric space  $(X, d_X)$  from  $M$  is isomorphically embeddable in  $(U, d_U)$ . The space  $l_2$  is isomorphically universal for finite metric spaces [4]. We define a continuum family of countable metric spaces isomorphically universal for finite metric spaces. The isometry groups of these spaces contain isomorphic copies of each countable residually finite group.

We show that for any finite group  $G$  there exists a self-similar metric space  $X$  such that the infinitely iterated wreath power  $\wr_{i=1}^{\infty} G$  of this group is isomorphic to the isometry group of  $X$ .

## 1. Preliminaries

Several constructions discussed below are based on the notion of the *wreath product of metric spaces* ([7]). To define it we need a few definitions.

Recall that a metric space  $(X, d_X)$  is said to be uniformly discrete if there exists a real number  $r > 0$  such that for any different points  $x, y \in X$  the inequality  $d_X(x, y) > r$  holds.

Let  $(X, d_X)$  be a uniformly discrete metric space and  $(Y, d_Y)$  be a metric space with finite diameter. Assume that for a positive number  $r$  and arbitrary points  $x_1, x_2 \in X$ ,  $x_1 \neq x_2$ , the inequality  $d_X(x_1, x_2) \geq r$  holds.

Let  $s(x)$  be a scale satisfying the inequality  $\text{diam}(s(Y)) < r$ . Such a function exists, for the diameter of the space  $Y$  is finite.

Define a metric  $\rho_s$  on the cartesian product  $X \times Y$  by the rule:

$$\rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2 \end{cases}.$$

This metric space is called the wreath product of metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  and denoted by  $XwrY$ . The metric space provided by this construction is unique up to isomorphism, that is it does not depend on the choice of the scale  $s(t)$ .

The following lemma is easily verified.

- Lemma 1.** 1) *The wreath product of metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  contains isomorphic copies of both spaces  $(X, d_X)$  and  $(Y, d_Y)$ .*
- 2) *Let  $(X, d_X)$ ,  $(Y, d_Y)$ ,  $(W, d_W)$  be metric spaces such that  $(X, d_X)$ ,  $(Y, d_Y)$  are uniformly discrete and  $(Y, d_Y)$ ,  $(W, d_W)$  have finite diameters. Then for any admissible scales  $s_1, s_2, s_3, s_4$  spaces*

$$(Xwr_{s_1}Y)wr_{s_2}W \quad \text{and} \quad Xwr_{s_3}(Ywr_{s_4}W)$$

*are isomorphic, i.e. the operation of wreath product of metric spaces is associative.*

## 2. Construction

Let  $(X_1, d_1), (X_2, d_2), \dots$  be an infinite sequence of uniformly discrete metric spaces of finite diameters. Assume that  $r_1, r_2, \dots$  is an infinite

sequence of positive numbers such that for arbitrary points  $a, b \in X_i$ ,  $a \neq b$ , the inequalities

$$d_i(a, b) \geq r_i, \quad i \geq 1 \quad (1)$$

hold.

Fix an infinite sequence of scales

$$\alpha = (s_2(x), s_3(x), s_4(x), \dots)$$

such that

$$\text{diam}(s_2(X_2)) < r_1, \quad \text{diam}(s_i(X_i)) < s_{i-1}(r_{i-1}), \quad i \geq 3. \quad (2)$$

By Lemma 1 we can consider the  $n$ -iterated wreath product of metric spaces using corresponding scales from sequence  $\alpha$ . Denote the  $n$ -iterated wreath product of metric spaces  $X_1, \dots, X_n$  by

$$wr_{i=1}^n(\alpha)X_i.$$

Fix a sequence of points  $x_i^0 \in X_i$ ,  $i \geq 1$ . One can define an isometric embedding

$$\eta_n : wr_{i=1}^n(\alpha)X_i \rightarrow wr_{i=1}^{n+1}(\alpha)X_i$$

given by the rule

$$\eta_n(x_1, \dots, x_n) = (x_1, \dots, x_n, x_{n+1}^0).$$

Then we have a directed system

$$\langle wr_{i=1}^n(\alpha)X_i, \eta_n \rangle, \quad n \in \mathbb{N}.$$

Denote the limit space of this system by  $\overline{wr}_{i=1}^\lambda(\alpha)X_i$  and by  $wr_{i=1}^\infty(\alpha)X_i$  its completion. We call the metric spaces  $\overline{wr}_{i=1}^\lambda(\alpha)X_i$  the *finitary wreath product* of metric spaces  $(X_1, d_1), (X_2, d_2), \dots$  with respect to the sequence of scales  $\alpha$  and the space  $wr_{i=1}^\infty(\alpha)X_i$  the *infinitely iterated wreath product* of the metric spaces  $(X_1, d_1), (X_2, d_2), \dots$  with respect to the sequence of scales  $\alpha$ .

Denote the set  $\prod_{i=1}^\infty X_i$  by  $X$ . Define a subset  $\tilde{X}$  of  $X$  as a set of all sequences  $(x_1, x_2, \dots)$  such that for some  $i \in \mathbb{N}$  the equalities

$$x_j = x_j^0, \quad j \geq i,$$

holds. Then the finitary wreath product and the infinitely iterated wreath product of metric spaces  $(X_1, d_1), (X_2, d_2), \dots$  with the sequence of scales

$\alpha$  can be described as metric spaces defined on the sets  $\widetilde{X}$  and  $X$  correspondingly, where the metric is defined by the rule:

$$\rho_\alpha((a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots)) = \begin{cases} d_1(a_1, b_1), & \text{if } a_1 \neq b_1; \\ s_2(d_2(a_2, b_2)), & \text{if } a_1 = b_1 \text{ and } a_2 \neq b_2; \\ s_3(d_3(a_3, b_3)), & \text{if } a_1 = b_1, a_2 = b_2, a_3 \neq b_3; \\ \dots & \dots \end{cases} \quad (3)$$

The infinitely iterated wreath product of metric spaces  $(X_1, d_1), (X_2, d_2), \dots$  is homeomorphic to the projective limit of finitely iterated wreath products of metric spaces  $wr_{i=1}^n X_i, n \geq 1$ , with natural projections, where  $wr_{i=1}^1 X_i = X_1$ .

**Proposition 1.** *Let  $(X_1, d_1), (X_2, d_2), \dots$ , and  $(Y_1, b_1), (Y_2, b_2), \dots$  be sequences of uniformly discrete metric spaces of finite diameters,  $\alpha_1$  and  $\alpha_2$  be sequences of scales such that the inequalities (2) hold for both of them. If for each  $i, i \geq 1$ , spaces  $X_i$  and  $Y_i$  are isomorphic then spaces  $wr_{i=1}^\infty(\alpha_1)X_i$  and  $wr_{i=1}^\infty(\alpha_2)Y_i$  ( $\overline{wr}_{i=1}^\infty(\alpha_1)X_i$  and  $\overline{wr}_{i=1}^\infty(\alpha_2)Y_i$ ) are isomorphic as well.*

*Proof.* Let  $h_1(x), h_2(x), \dots$  be a sequence of scales such that for each  $i, i \geq 1, Y_i = h_i(X_i)$ . Assume that

$$\alpha_1 = (s_2(x), s_3(x), \dots), \quad \alpha_2 = (g_2(x), g_3(x), \dots).$$

Define an infinite sequence of numbers  $q_2, q_3, \dots$  such that

$$q_n = \sup_{u, v \in Y_n} \{g_n(b_n(u, v))\}, \quad n \geq 2.$$

As  $\alpha_2$  satisfies inequalities (2), the following inequalities hold

$$q_2 < r_1, \quad q_n < g_{n-1}(r_{n-1}), \quad n \geq 3.$$

Define a new function  $\widetilde{S}(x)$  on the  $R^+$  by the rule:

$$\widetilde{S}(x) = \begin{cases} x, & \text{if } x > r_1; \\ r_1 + \frac{s_2(h_2(g_2^{-1}(q_2))) - r_1}{q_2 - r_1}(x - r_1), & \text{if } q_2 \leq x \leq r_1; \\ s_2(h_2(g_2^{-1}(x))), & \text{if } g_2(r_2) < x < q_2; \\ \dots \quad \dots \quad \dots & \\ s_n(h_n(g_n^{-1}(q_n))) + \beta_n(x - q_n), & \text{if } q_n \leq x \leq g_{n-1}(r_{n-1}) \\ s_n(h_n(g_n^{-1}(x))), & \text{if } g_n(r_n) < x < q_n; \\ \dots \quad \dots \quad \dots, & \end{cases}$$

where

$$\beta_n = \frac{s_{n-1}(h_{n-1}(g_{n-1}^{-1}(r_{n-1}))) - s_n(h_n(g_n^{-1}(q_n)))}{g_{n-1}(r_{n-1}) - q_n}, \text{ for all } n \geq 3,$$

i.e. if  $q_2 \leq x \leq r_1$  or  $q_n \leq x \leq g_{n-1}(r_{n-1})$ , then the graph of this function is the line segment joining points  $(q_2, s_2(g_2^{-1}(q_2)))$  and  $(r_1, r_1)$ , or  $(q_n, s_n(g_n^{-1}(q_n)))$  and  $(g_{n-1}(r_{n-1}), s_{n-1}(g_{n-1}^{-1}(r_{n-1})))$ ,  $n \geq 3$ , respectively.

It is clear that  $\widetilde{S}(x)$  is a scale and  $\widetilde{S}(\rho_{\alpha_1}) = \rho_{\alpha_2}$ . Hence, spaces  $wr_{i=1}^{\infty}(\alpha_1)X_i$  and  $wr_{i=1}^{\infty}(\alpha_2)Y_i$  ( $\overrightarrow{wr}_{i=1}^{\infty}(\alpha_1)X_i$  and  $\overrightarrow{wr}_{i=1}^{\infty}(\alpha_2)Y_i$ ) are isomorphic. □

Therefore, to consider metric spaces  $\overrightarrow{wr}_{i=1}^{\infty}(\alpha)X_i$  or  $wr_{i=1}^{\infty}(\alpha)X_i$  up to isomorphism we can assume that the corresponding sequence of scales  $\alpha$  is fixed. In this case we denote the infinitely iterated wreath product and finitary wreath product of metric spaces  $X_1, X_2, \dots$  by  $wr_{i=1}^{\infty}X_i$  and  $\overrightarrow{wr}_{i=1}^{\infty}X_i$  correspondingly.

As  $(X_i, d_i)$ ,  $i \geq 1$ , are uniformly discrete metric spaces, the metric  $d_i$  induces the discrete topology on  $X_i$ , for all  $i \geq 1$ . The metric  $\rho_{\alpha}$  induces the topology  $\sigma$  on  $\prod_{i=1}^{\infty} X_i$ . The topology  $\sigma$  coincides with the Tykhonov's product topology on the product  $\prod_{i=1}^{\infty} X_i$  of discrete spaces. If  $|X_i| < \infty$ ,  $i \geq 1$ , then the space  $wr_{i=1}^{\infty}X_i$  and the Cantor space are homeomorphic. In this case the space  $wr_{i=1}^{\infty}X_i$  is a compact totally disconnected metric space, while the space  $\overrightarrow{wr}_{i=1}^{\infty}X_i$  is a countable everywhere dense subspace in  $wr_{i=1}^{\infty}X_i$ .

### 3. Characterization

Let  $T$  be an infinite spherically homogeneous rooted tree with the root  $v_0$ . Recall the definition of the space  $\partial T$  of paths in  $T$ , i.e. the boundary of  $T$  (see, e.g., [5]). For every nonnegative integer  $l$  the  $l$ -th level is the set

$V_l$  of all vertices  $v \in V(T)$  such that the length of the unique simple path connecting  $v$  and  $v_0$  in  $T$  equals  $l$ . The tree  $T$  is uniquely defined by its *spherical index*, i.e. by an finite sequence of cardinal numbers  $[k_1; k_2; \dots, ]$ , where  $k_i$  is the number of edges joining a vertex of the  $i - 1$ -th level with vertices of the  $i$ -th level. A *rooted path* is an infinite sequence of vertices  $(v_0, v_1, \dots, v_n, \dots)$  such that the vertices  $v_i, v_{i+1}$  are connected by an edge for every  $i, i \geq 0$ . The metric space  $\partial T$  is the set of all infinite rooted paths of  $T$  with the ultrametric  $\rho$ , defined by the rule:

$$\rho(\gamma_1, \gamma_2) = 1/(m + 1),$$

where  $m$  is the length of the common beginning of rooted paths  $\gamma_1$  and  $\gamma_2$ .

Recall, that a metric space is called discrete if all non-zero distances in this space equal 1.

**Proposition 2.** *Let  $T$  be an infinite spherically homogenous rooted tree with spherical index  $[k_1; k_2; \dots, ]$ . Assume that  $(X_1, d_1), (X_2, d_2), \dots$  are discrete metric spaces, such that  $|X_i| = k_i, i \geq 1$ . Then there exists a sequences of scales  $\alpha$  such that spaces  $\partial T$  and  $wr_{i=1}^\infty(\alpha)X_i$  are isometric.*

To prove this statement it is sufficient to pick  $\alpha = (\frac{1}{2}x, \frac{1}{3}x, \frac{1}{4}x, \dots)$ .

Let now  $(X_1, d_1), (X_2, d_2), \dots$  be uniformly discrete metric spaces of finite diameters. Consider an infinite spherically homogenous rooted tree  $T$  with spherical index  $[|X_1|; |X_2|; \dots, ]$ . Fix an infinite sequence of scales

$$\alpha = (s_2(x), s_3(x), s_4(x), \dots)$$

such that inequalities (2) hold. Let  $s_1(x) = x, x \in \mathbb{R}^+$ . We can introduce a natural metric on the set  $\partial T$  of all rooted path of tree  $T$ . For arbitrary paths  $\gamma_1 = (v_0, u_1, u_2, \dots), \gamma_2 = (v_0, v_1, v_2, \dots)$  we put

$$\sigma(\gamma_1, \gamma_2) = \begin{cases} s_{n+1}(d_{n+1}(v_n, u_n)), & \text{if } \gamma_1 \neq \gamma_2; \\ 0, & \text{if } \gamma_1 = \gamma_2, \end{cases}$$

where  $n$  is the length of the common beginning of rooted paths  $\gamma_1$  and  $\gamma_2$ .

**Proposition 3.** *The infinitely iterated wreath product of metric spaces  $(X_1, d_1), (X_2, d_2), \dots$  with the sequence of scales  $\alpha$  is isometric to the space  $(\partial T, \sigma)$ .*

Note that for arbitrary  $i, i \geq 1$  the space  $(X_i, d_i)$  is isomorphically embeddable in the space  $(\partial T, \sigma)$ . Indeed, fix a point  $a_j$  from the space  $X_j, j \geq 1, j \neq i$ . Then the subspace of paths

$$(v_0, a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots), \quad x_i \in X_i$$

is isomorphic to the space  $(X_i, d_i)$ . Such a subspace of  $(\partial T, \sigma)$  will be called *naturally* isomorphic to the metric space  $(X_i, d_i)$ .

**Lemma 2.** *Let  $u = (u_1, u_2, \dots), v = (v_1, v_2, \dots), w = (w_1, w_2, \dots)$  be different points of the space  $wr_{i=1}^{\infty} X_i$ . The points  $u, v, w$  are vertices of a scalene triangle iff there exists  $k$  such that  $u_1 = v_1 = w_1, \dots, u_{k-1} = v_{k-1} = w_{k-1}, u_k \neq v_k, w_k \neq v_k, u_k \neq w_k$  and the triangle  $u_k, v_k, w_k$  is scalene in the space  $X_k$ .*

*Proof.* Let  $u, v, w \in wr_{i=1}^{\infty} X_i$  be vertices of a scalene triangle. Assume that there exist  $k$  and  $l, k \neq l$ , such that  $u_1 = v_1 = w_1, \dots, u_{k-1} = v_{k-1} = w_{k-1}, u_k \neq v_k, w_k = v_k, \dots, w_{l-1} = v_{l-1}, w_l \neq v_l$ . Using (3) we obtain

$$\rho_{\alpha}(u, v) = \rho_{\alpha}((u_1, u_2, \dots), (v_1, v_2, \dots)) = s_k(d_k(u_k, v_k)),$$

$$\rho_{\alpha}(u, w) = \rho_{\alpha}((u_1, u_2, \dots), (w_1, w_2, \dots)) = s_k(d_k(u_k, w_k)).$$

Therefore,  $\rho_{\alpha}(u, v) = \rho_{\alpha}(u, w)$ .

The converse statement directly follows from the definition of the space  $wr_{i=1}^{\infty} X_i$ .  $\square$

**Proposition 4.** (A) *The space  $wr_{i=1}^{\infty} X_i$  is totally disconnected.*

(B) *The space  $wr_{i=1}^{\infty} X_i$  is compact iff for each  $i \geq 1$  the space  $X_i$  is finite.*

(C) *The space  $wr_{i=1}^{\infty} X_i$  is separable iff for each  $i \geq 1$  the space  $X_i$  is countable or finite.*

(D) *The space  $wr_{i=1}^{\infty} X_i$  is ultrametric iff for each  $i \geq 1$  the space  $X_i$  is ultrametric.*

*Proof.* (A) The space  $wr_{i=1}^{\infty} X_i$  is a product of totally disconnected spaces. Hence  $wr_{i=1}^{\infty} X_i$  is totally disconnected.

(B) It follows from the Tykhonov's compactness theorem that the product  $\prod_{i=1}^{\infty} X_i$  is compact iff for all  $i \geq 1$  the space  $X_i$  is compact. As  $(X_1, d_1), (X_2, d_2), \dots$ , is an infinite sequence of uniformly discrete metric spaces of finite diameters,  $X_i$  is compact iff  $X_i$  is finite.

(C) For each  $j \geq 1$  fix a point  $a_j$  from  $X_j$ . Consider the subspace of all sequences  $(x_1, x_2, \dots)$ ,  $x_i \in X_i$ , such that for some number  $m$  equalities  $x_i = a_i$  hold,  $i \geq m$ . Then this subspace is a countable everywhere dense subset of  $wr_{i=1}^\infty X_i$ .

Conversely, assume that the space  $X_j$  is not countable. Then it follows from inequalities (1) that  $X_j$  is not separable. Therefore  $wr_{i=1}^\infty X_i$  is not separable.

(D) The proof directly follows from Lemma 2.  $\square$

#### 4. The isometry group

For the next theorem we need a few definitions. Let  $(G_1, X_1)$ ,  $(G_2, X_2), \dots$  be an infinite sequence of transformation groups. Following [8] the transformation group  $(G, \prod_{i=1}^\infty X_i) = \wr_{i=1}^\infty (G_i, X_i)$  is called infinitely iterated wreath product of groups  $(G_1, X_1), (G_2, X_2), \dots$  if for all elements  $u \in G$  the following conditions hold:

1) if  $(x_1, \dots, x_n, \dots)^u = (y_1, \dots, y_n, \dots)$ , then for all  $i \geq 1$  the value of  $y_i$  depends only on  $x_1, \dots, x_i$ ;

2) for fixed  $x_1, \dots, x_{i-1}$  the mapping  $g_i(x_1, \dots, x_{i-1})$  defined by the equality

$$g_i(x_1, \dots, x_{i-1})(x_i) = y_i, \quad x_i \in X_i$$

is a transformation of the set  $X_i$  that belongs to  $G_i$ .

It follows from this definition that each element  $u \in G$  can be written as an infinite sequence, called tableaux:

$$u = [g_1, g_2(x_1), g_3(x_1, x_2), \dots],$$

where  $g_1 \in G_1$ ,  $g_i(x_1, \dots, x_{i-1}) \in G_i^{X_1 \times \dots \times X_{i-1}}$ ,  $i \geq 2$ . Each element  $u \in G$  acts on  $(m_1, m_2, m_3 \dots) \in \prod_{i=1}^\infty X_i$  by the rule

$$(m_1, m_2, m_3 \dots)^u = (m_1^{g_1}, m_2^{g_2(m_1)}, m_3^{g_3(m_1, m_2)}, \dots).$$

**Theorem 1.** *The isometry group of the infinitely iterated wreath product of metric spaces  $(X_n, d_n)$ ,  $n \geq 1$ , is isomorphic as a transformation group to the infinitely iterated wreath product of isometry groups of these spaces*

$$(\text{Isom}(wr_{i=1}^\infty X_i), \prod_{i=1}^\infty X_i) \simeq \wr_{i=1}^\infty (\text{Isom} X_i, X_i).$$

*Proof.* Consider arbitrary transformation

$$u = [g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1}), \dots] \in \wr_{i=1}^{\infty}(G_i, X_i).$$

We shall show that  $u$  is an isometry of the space  $\wr_{i=1}^{\infty} X_i$ . By the definition of the wreath product of permutation groups the element  $u$  acts on  $\prod_{i=1}^{\infty} X_i$ . Therefore, it is sufficient to show that  $u$  preserves the metric  $\rho_{\alpha}$ . Indeed, from (3) we have

$$\begin{aligned} \rho_{\alpha}((a_1, a_2, a_3, \dots)^u, (b_1, b_2, b_3, \dots)^u) &= \\ &= \rho_{\alpha}((a_1^{g_1}, a_2^{g_2(a_1)}, a_3^{g_3(a_1, a_2)}, \dots), (b_1^{g_1}, b_2^{g_2(b_1)}, b_3^{g_3(b_1, b_2)}, \dots)) = \\ &= \begin{cases} d_1(a_1^{g_1}, b_1^{g_1}), & \text{if } a_1^{g_1} \neq b_1^{g_1}; \\ s_2(d_2(a_2^{g_2(a_1)}, b_2^{g_2(b_1)})), & \text{if } a_1^{g_1} = b_1^{g_1} \text{ and } a_2^{g_2(a_1)} \neq b_2^{g_2(b_1)}; \\ s_3(d_3(a_3^{g_3(a_1, a_2)}, b_3^{g_3(b_1, b_2)})), & \text{if } a_1^{g_1} = b_1^{g_1}, a_2^{g_2(a_1)} = b_2^{g_2(b_1)}, \\ & a_3^{g_3(a_1, a_2)} \neq b_3^{g_3(b_1, b_2)}; \\ \dots \quad \dots \quad \dots \end{cases} \quad (4) \end{aligned}$$

As  $g_1 \in \text{Isom} X_1$ ,  $a_1^{g_1} = b_1^{g_1}$  iff  $a_1 = b_1$ . Hence,  $a_1^{g_1} \neq b_1^{g_1}$  iff  $a_1 \neq b_1$ . As  $g_2(a_1) \in \text{Isom} X_2$ , from equalities  $a_1 = b_1$  and  $a_2^{g_2(a_1)} = b_2^{g_2(b_1)}$  it follows that  $a_2 = b_2$  and so on. Then similarly, using (4), we get

$$\begin{aligned} \rho_{\alpha}((a_1, a_2, a_3, \dots)^u, (b_1, b_2, b_3, \dots)^u) &= \\ &= \begin{cases} d_1(a_1, b_1), & \text{if } a_1 \neq b_1; \\ s_2(d_2(a_2, b_2)), & \text{if } a_1 = b_1 \text{ and } a_2 \neq b_2; \\ s_3(d_3(a_3, b_3)), & \text{if } a_1 = b_1, a_2 = b_2, a_3 \neq b_3; \\ \dots \quad \dots \quad \dots \end{cases} = \\ &= \rho_{\alpha}((a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots)). \end{aligned}$$

Therefore,  $u$  is an isometry of  $\wr_{i=1}^{\infty} X_i$ .

Let now  $\varphi$  be an isometry of  $\wr_{i=1}^{\infty} X_i$ . Consider points  $(a_1, a_2, a_3, \dots)$ ,  $(b_1, b_2, b_3, \dots)$  of  $\wr_{i=1}^{\infty} X_i$  such that

$$\varphi((a_1, a_2, a_3, \dots)) = (y_1, y_2, y_3, \dots), \quad \varphi((b_1, b_2, b_3, \dots)) = (z_1, z_2, z_3, \dots),$$

for some  $(y_1, y_2, y_3, \dots), (z_1, z_2, z_3, \dots) \in \wr_{i=1}^{\infty} X_i$ . We have

$$\rho_{\alpha}((a_1, a_2, \dots, a_n, \dots), (b_1, b_2, \dots, b_n, \dots)) = s_j(d_j(a_j, b_j)), \quad (5)$$

where  $j$  is those number for which  $a_1 = b_1, \dots, a_{j-1} = b_{j-1}, a_j \neq b_j$ . Similarly

$$\begin{aligned} \rho_\alpha(\varphi(a_1, a_2, \dots, a_n, \dots), \varphi(b_1, b_2, \dots, b_n, \dots)) &= \\ &= \rho_\alpha((y_1, y_2, \dots, y_n, \dots), (z_1, z_2, \dots, z_n, \dots)) = s_l(d_l(y_l, z_l)), \end{aligned} \quad (6)$$

where  $l$  is the number for which  $y_1 = z_1, \dots, y_{l-1} = z_{l-1}, y_l \neq z_l$ . Using (3), (5), (6), we get  $l = j$ .

Hence, for all  $i \geq 1$  the values of  $y_i$  depend only of the values of  $a_1, \dots, a_i$ . Therefore, there exists a tableaux  $[g_1, g_2(x_1), g_3(x_1, x_2), \dots]$  such that  $g_1 \in IsomX_1, g_i(x_1, \dots, x_{i-1}) \in (IsomX_i)^{X_1 \times \dots \times X_{i-1}}, i > 1$ . Moreover, the  $n$ -coordinate tableaux  $[g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})]$  acts on  $X$  as  $\varphi$  does. This completes the proof.  $\square$

The next corollaries follow immediately from Theorem 1.

**Corollary 1.** *Let  $(G_1, X_1), (G_2, X_2), \dots$  be an infinite sequence of transformation groups. If each of the groups  $(G_i, X_i), i \geq 1$  is the isometry group of some uniformly discrete metric space with finite diameter then the wreath product  $\wr_{i=1}^\infty (G_i, X_i)$  is isomorphic as a transformation group to the isometry group of a metric space of finite diameter.*

**Corollary 2.** *Let  $G_1, G_2, \dots$  be an infinite sequence of finite groups. Then the wreath product  $\wr_{i=1}^\infty G_i$  of these groups is isomorphic to the isometry group of a totally disconnected compact metric space of finite diameter.*

*Proof.* Each finite group is isomorphic to the isometry group of some finite metric space. This fact follows, for example, from Frucht's theorem [9]. Let  $(X_1, d_1), (X_2, d_2), \dots,$  be an infinite sequence of finite metric spaces such that  $G_i \simeq IsomX_i$ . Then it follows from Theorem 1 that

$$\wr_{i=1}^\infty IsomX_i \simeq Isom(wr_{i=1}^\infty X_i).$$

Therefore  $\wr_{i=1}^\infty G_i \simeq Isom(wr_{i=1}^\infty X_i)$ . Moreover, it follows from Proposition 4 that the space  $wr_{i=1}^\infty X_i$  is totally disconnected compact and has finite diameter.  $\square$

**Corollary 3.** *If for each  $i \geq 1$  the space  $X_i$  is homogeneous, then the space  $wr_{i=1}^\infty X_i$  is homogeneous too.*

## 5. Examples

### 5.1. Wreath products of Hamming spaces

Denote by  $H_m$  the  $m$ -dimensional cube equipped with the Hamming distance, i.e. the set of all binary  $m$ -tuples  $(a_1, \dots, a_m)$ ,  $a_i \in \{0, 1\}$ ,  $1 \leq i \leq m$ , with the Hamming metric  $d_{H_m}$ :

$$d_{H_m}(\bar{x}, \bar{y}) = \sum_{i=1}^m |x_i - y_i|,$$

where  $\bar{x}, \bar{y} \in \{0, 1\}^m$ . Let  $\Theta$  be the set of all infinite increasing sequences of natural numbers. For any  $\bar{m} \in \Theta$ ,  $\bar{m} = (m_1, m_2, \dots)$ , we can fix an infinite sequence of scales

$$\alpha(\bar{m}) = (s_2(x), s_3(x), s_4(x), \dots),$$

such that

$$s_k(t) = \frac{t}{\prod_{i=2}^k (m_i + 1)}, \quad k \geq 2.$$

As  $\text{diam} H_{m_i} = m_i$ , inequalities (2) hold for the sequence of Hamming spaces

$$H_{m_1}, H_{m_2}, \dots$$

Hence, we can consider the space  $\overrightarrow{wr}_{i=1}^{\infty}(\alpha)H_{m_i}$ . Denote this space by  $UH(\bar{m})$ .

Since for every  $i$ ,  $i \geq 1$ , the space  $H_{m_i}$  is finite, the space  $UH(\bar{m})$  is countable. From (3) it follows that  $\text{diam}(UH(\bar{m})) = m_1$ .

**Proposition 5.** *Let  $\bar{m}, \bar{k} \in \Theta$ ,  $\bar{m} \neq \bar{k}$ . Then spaces  $UH(\bar{m})$  and  $UH(\bar{k})$  are not isomorphic.*

*Proof.* Let  $\bar{m} = (m_1, m_2, \dots)$ ,  $\bar{k} = (k_1, k_2, \dots)$ . Assume that  $m_1 = k_1, \dots, m_{i-1} = k_{i-1}$ , but  $m_i \neq k_i$ . Then the  $(m_1 + \dots + m_{i-1} + 1)$ th largest values of metrics in spaces  $UH(\bar{m})$  and  $UH(\bar{k})$  are achieved on different numbers of points. Hence, the spaces  $UH(\bar{m})$  and  $UH(\bar{k})$  are not isomorphic.  $\square$

**Theorem 2.** *Let  $\bar{m} \in \Theta$ . Then  $UH(\bar{m})$  is a homogeneous countable metric space, which contains an isomorphic copy of arbitrary finite metric space. Any countable residually finite group  $G$  is isomorphic to some subgroup of the isometry group of the space  $UH(\bar{m})$ .*

*Proof.* Let  $\bar{m} = (m_1, m_2, \dots)$ . Each space  $H_{m_i}$  is homogeneous. Hence, from Corollary 3 we obtain that the space  $UH(\bar{m})$  is homogeneous.

Let  $X$  be an  $n$ -point metric space. It follows from [10] that there exists an isomorphic embedding of  $X$  into the Hamming space  $H_m$ , where  $m = \frac{1}{2}((\binom{n}{2})^2 - (\binom{n}{2}) - 2)(n^2 - 2n + 7)$ . Note, that for finite metric spaces the definitions of their isomorphism used in this paper and in [10] coincide. Since  $\bar{m}$  is an infinite increasing sequences of natural numbers there exists a number  $m_j$  such that the space  $X$  is isomorphic to some subspace  $H_{m_j}$ , and hence it is isomorphic to some subspace of  $UH(\bar{m})$ .

The isometry group  $IsomH_m$  of the space  $H_m$  is isomorphic to the wreath product  $S_m \wr S_2$  of symmetric groups  $S_m$  and  $S_2$  (see, e.g. [11]). Hence, from Theorem 1 we have  $IsomUH(\bar{m}) \simeq \varprojlim_{i=1}^{\infty} (S_{m_i} \wr S_2)$ . Therefore,  $\varprojlim_{i=1}^{\infty} (S_{m_i}) < IsomUH(\bar{m})$ . Since  $\bar{m}$  is an infinite increasing sequences of natural numbers arbitrary countable residually finite group  $G$  is isomorphic to some subgroup of  $\varprojlim_{i=1}^{\infty} (S_{m_i})$  (see [12]).  $\square$

## 5.2. Self-similarity

For definitions and basic properties of self-similar sets see, for instance, [13].

Let  $(W, d_W)$  be a finite metric space,  $W = \{w_1, \dots, w_n\}$ , such that

$$d = \min_{x, y \in W, x \neq y} d_W(x, y) > 1.$$

Assume that  $D$  is positive real number such that  $D = diam(W)$ .

Consider the sequence of spaces  $W, W, \dots$ . Let  $X = \prod_{i=1}^{\infty} W$ . Fix the sequence of scales

$$\alpha = \left( \frac{d}{D+1}t, \frac{d}{(D+1)^2}t, \frac{d}{(D+1)^3}t, \dots \right).$$

Then inequalities (2) hold and we can consider the infinitely iterated wreath product  $wr_{i=1}^{\infty}(\alpha)W$ .

Since  $W$  is finite, the space  $wr_{i=1}^{\infty}(\alpha)W$  is compact. If space  $W$  is discrete then  $wr_{i=1}^{\infty}(\alpha)W$  and the space  $\partial T$  of rooted paths in the  $n$ -regular rooted tree  $T$  are isometric. Hence, in this case we immediately obtain that the space  $wr_{i=1}^{\infty}(\alpha)W$  is self-similar (see, for instance, [5]). But if  $W$  is not a discrete metric space, then we can similarly show that  $wr_{i=1}^{\infty}(\alpha)W$  is self-similar as well. Indeed, for all  $i$ ,  $1 \leq i \leq n$ , we define a map  $f_i : X \rightarrow X$  by the rule:

$$f_i(u_1, u_2, u_3, \dots) = (w_i, u_1, u_2, u_3, \dots).$$

Then  $f_i$  is a contraction with respect to the metric  $\rho_\alpha$ . Since  $X = \bigcup_{i=1}^n f_i(X)$  the space  $wr_{i=1}^\infty(\alpha)W$  is self-similar with respect to  $f_1, \dots, f_n$  (see [13, Theorem 1.1.4]).

From Theorem 1 it follows that the isometry group  $Isom wr_{i=1}^\infty(\alpha)W$  of the space  $wr_{i=1}^\infty(\alpha)W$  is isomorphic to the infinite wreath product  $\wr_{i=1}^\infty Isom W$ . Since

$$Isom(wr_{i=1}^\infty(\alpha)W) = Isom W \wr Isom(wr_{i=1}^\infty(\alpha)W)$$

the group  $Isom(wr_{i=1}^\infty(\alpha)W)$  acts on  $X$  self-similarly (see [14, Definition 1.5.4] for details).

Note that all spaces  $wr_{i=1}^\infty(\alpha)W$  are homeomorphic to the Cantor space. If  $(W, d_W)$  is the space with trivial isometry group, then from Theorem 1 it follows that  $wr_{i=1}^\infty(\alpha)W$  is a self-similar set with trivial isometry group.

**Proposition 6.** *Let  $G$  be a finite group. Then there exists a self-similar metric space  $X$  such that  $\wr_{i=1}^\infty G \simeq Isom(X)$ .*

The proof of this proposition follows from Corollary 2.

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