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RESEARCH ARTICLE

The detour hull number of a graph[1]

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Abstract. For vertices u and v in a connected graph $G = (V, E)$, the set $I_D[u, v]$ consists of all those vertices lying on a $u - v$ longest path in G. Given a set S of vertices of G, the union of all sets $I_D[u, v]$ for $u, v \in S$, is denoted by $I_D[S]$. A set S is a detour convex set if $I_D[S] = S$. The detour convex hull $[S]_D$ of S in G is the smallest detour convex set containing S. The detour hull number $d_h(G)$ is the minimum cardinality among the subsets S of V with $[S]_D = V$. A set S of vertices is called a detour set if $I_D[S] = V$. The minimum cardinality of a detour set is the detour number $dn(G)$ of G. A vertex x in G is a detour extreme vertex if it is an initial or terminal vertex of any detour containing x. Certain general properties of these concepts are studied. It is shown that for each pair of positive integers r and s, there is a connected graph G with r detour extreme vertices, each of degree s. Also, it is proved that every two integers a and b with $2 \leq a \leq b$ are realizable as the detour hull number and the detour number respectively, of some graph. For each triple D, k and n of positive integers with $2 \leq k \leq n - D + 1$ and $D \geq 2$, there is a connected graph of order n, detour diameter D and detour hull number k. Bounds for the detour hull number of a graph are obtained. It is proved that $dn(G) = dh(G)$ for a connected graph G with detour diameter at most 4. Also, it is proved that for positive integers a, b and $k \geq 2$ with $a < b \leq 2a$, there exists a connected graph G with detour radius a, detour diameter b and detour hull number k. Graphs G for which $d_h(G) = n - 1$ or $d_h(G) = n - 2$ are characterized.

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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to [1, 9]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G. An u - v path of length $d(u, v)$ is called an u - v *geodesic*. The set $I[u, v]$ consists of all vertices lying on some u - v geodesic of G; while for $S \subseteq V$, $I[S] = \bigcup_{u,v\in S} I[u, v]$. The set S is *convex* if $I[S] = S$. The *convex hull* $[S]$ is the smallest convex containing S. A set S of vertices of G is a *hull set* of G if $[S] = V$. The *hull number* $h(G)$ of G is the minimum cardinality of a hull set and any hull set of cardinality $h(G)$ is called a *minimum hull set* of G. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and a geodetic set of minimum cardinality is a *minimum geodetic set* of G. The cardinality of a minimum geodetic set of G is the *geodetic number* $g(G)$. These concepts were studied in [1, 3, 4, 5, 6, 11]

For vertices u and v in a nontrivial connected graph G, the *detour distance* $D(u, v)$ is the length of a longest $x - v$ path in G. An $u - v$ path of length $D(u, v)$ is an $u - v$ detour. It is known that the detour distance is a metric on the vertex set V of G. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G. The detour distance of a graph was studied in [2].

The set $I_D[u, v]$ consists of all vertices lying on some $u - v$ detour of G; while for $S \subseteq V$, $I_D[S] = \bigcup_{u,v\in S} I_D[u, v]$. A set $S \subseteq V$ is called a *detour set* if $I_D[S] = V$. The *detour number* $dn(G)$ of G is the minimum cardinality of a detour set and any detour set of cardinality $dn(G)$ is called a *minimum detour set* of G. The detour number of a graph was introduced in [7] and further studied in [14]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [8, 12]. In [13], the distance martix and the detour matrix of a connected graph are used to discuss the applications of the graph parameters Wiener index, the detour index, the hyper-Wiener index and the hyper-detour index to a class of graphs viz. bridge and chain graphs, which in turn, capture different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry.

For more applications of these paramters, one may refer to [13] and the references therein.

The following theorems are used in the sequel.

Theorem 1.1. [7] Every end vertex of a connected graph G belongs to each detour set of G. Also if the set S of all end vertices of G is a detour set, then S is the unique minimum detour set of G.

Theorem 1.2. [7] If G is a connected graph of order n and detour diameter D, then $dn(G) \leq n - D + 1$.

Theorem 1.3. [7] Let G be a connected graph of order $n \geq 4$. Then $dn(G) = n - 1$ if and only if $G = K_{1,n-1}$.

Theorem 1.4. [7] Let G be a connected graph of order $n \geq 5$. Then $dn(G) = n - 2$ if and only if G is a double star or $K_{1,n-1} + e$.

2. Detour hull number of a graph

A set S of vertices is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ of S in G is the smallest detour convex set containing S. The detour convex hull of S can also be formed from the sequence $\{I_D^k[S]\}$ $(k \geq 0)$, where $I_D^0[S] = S, I_D^1[S] = I_D[S]$ and $I_D^k[S] = I_D[I_D^{k-1}[S]]$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_D^p[S] = I_D^{p+1}[S]$. Then $I_D^p[S]$ is the *detour convex hull* $[S]_D$. A set S of vertices of G is a *detour hull set* if $[S]_D = V$ and a detour hull set of minimum cardinality is the *detour hull number* $d_h(G)$.

Example 2.1. For the graph G given in Figure 1, and $S = \{v_1, v_6\}$, $I_D[S] = V - \{v_7\}$ and $I_D^2[S] = V$. Thus S is a minimum detour hull set of G and so $d_h(G) = 2$. Since S is not a detour set and $S \cup \{v_7\}$ is a detour set of G, it follows from Theorem 1.1 that $dn(G) = 3$. Hence the detour number and detour hull number of a graph are different. Note that the sets $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_3, v_4, v_5, v_7\}$ are detour convex sets in G.

Definition 2.2. A vertex v in a connected graph G is a *detour extreme vertex* if it is an initial or terminal vertex of any detour in G containing the vertex v.

Observation 2.3. A vertex v is detour extreme vertex if and only if $V(G) - \{v\}$ is a detour convex set in G.

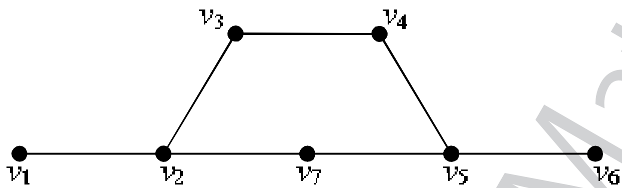

FIGURE 1. G

Remark 2.4. Every end vertex of a graph G is a detour extreme vertex. However, there are detour extreme vertices which are not end vertices. For the graph G in Figure 2, w is a detour extreme vertex of G of degree 2.

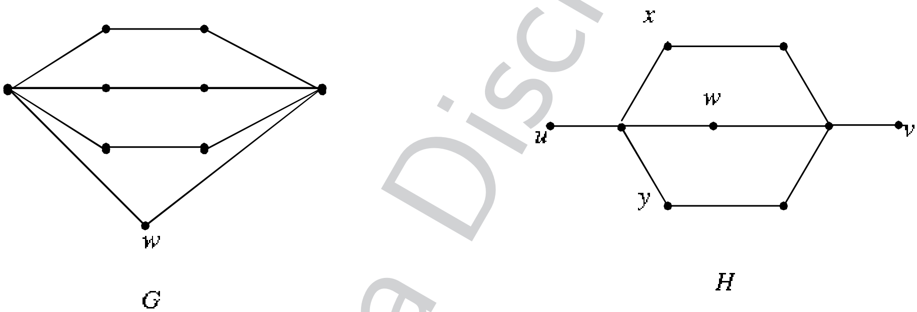

FIGURE 2. G

It is defined in [7] that a vertex v in a connected graph G is a *detour vertex* if it belongs to every minimum detour set of G. It is clear that every detour extreme vertex is a detour vertex. However, a detour vertex need not be a detour extreme vertex. For the graph H in Figure 2, the set $S = \{u, v, w\}$ is the unique detour set of H and so w is a detour vertex of G. Since w lies on an $x - y$ detour in H, w is not a detour extreme vertex.

Proposition 2.5. Each detour extreme vertex of a nontrivial connected graph G belongs to every detour hull set of G. In particular, each detour extreme vertex belongs to every detour set of G.

Proof. Let x be a detour extreme vertex of G. Then x is either an initial or terminal vertex of any detour containing the vertex x in G. Hence it follows that x belongs to every detour hull set of G. Also, since every detour set is a detour hull set, we see that each detour extreme vertex belongs to every detour set of G. □

It is clear that the set of all end vertices of a nontrivial tree is a detour set as well as a detour hull set and so we have the following corollary.

Corollary 2.6. If T is a tree with k end vertices, then $d_h(T) = dn(T) = k$.

Proposition 2.7. For a connected graph G of order n, $2 \leq d_h(G) \leq dn(G) \leq n$.

Proof. This follows from the fact that every detour set is a detour hull set and any detour hull set contains at least 2 vertices. □

Proposition 2.8. If a connected graph $G \neq K_2$ has a full degree vertex v, then v is not a detour extreme vertex of G.

Proof. Suppose that v is a detour extreme vertex of G. Let u, u' be two vertices such that $D(u, u') = diam_D(G)$. Let $P : u = u_0, u_1, \ldots, u_k = u'$ be a detour diameteral path in G. Then $N(u) \subseteq V(P)$ and $N(u') \subseteq V(P)$. If $u = v$ or $u' = v$, say $u = v$, then P is a Hamiltonian path. Hence the path P together with the edge vu' is a Hamiltonian cycle in G and so $v \in I_D[u_1, u_2]$, which is a contradiction to the fact that v is a detour extreme vertex of G. So, assume that $u \neq v$ and $u' \neq v$. This implies that $v \in N(u) \subseteq V(P)$, which is again a contradiction. Hence the result follows. □

Theorem 2.9. For each pair of positive integers r and s, there is a connected graph G with r detour extreme vertices each of degree s.

Proof. If $s = 1$, then $G = K_{1,r}$ has the desired properties. Assume that $s = 2$. For each $i = 1, 2, 3$, let P_4^i be a $u_i - v_i$ vertex disjoint paths of order 4. Let H_1 be the graph obtianed from P_4^i 's by identifying the vertices u_1, u_2, u_3 as u and identifying the vertices v_1, v_2, v_3 as v. Let H_2 be the totally disconnected graph $\overline{K_r}$ on r vertices such that H_1 and H_2 are vertex disjoint. Let G be the graph obtained from H_1 and H_2 by joining each vertex of H_2 to both u and v. The graph G is shown in Figure 3. We claim that $V(G) - \{w\}$ is a detour convex set for each $w \in V(H_2)$. Let $w \in V(H_2)$. If $r = 1$ or $r = 2$, then $w \notin I_D[x, y]$ for $x, y \in V(H_2)$ with $w \neq x, y$. For $r \geq 3$, it is clear that $D(x, y) = 5$ for all $x, y \in V(H_2)$ and any $x - y$ path which contains w with $w \neq x, y$ has length 4. Hence $w \notin I_D[x, y]$ for all $x, y \in V(H_2) - \{w\}$. Also, we have $D(u, v) = 3$ and $P : u, w, v$ is the unique $u - v$ path which contains w. Thus $w \notin I_D[u, v]$. Let $x, y \in V(H_1) - \{u, v\}$. If x and y are adjacent, then $D(x, y) = 5$ and any $x - y$ path that contains w has length 4. Also, if x and y are

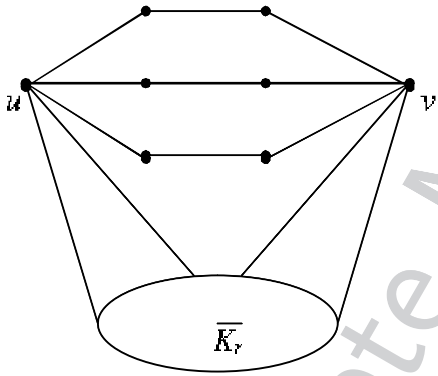

FIGURE 3. G

nonadjacent, then $D(x,y) = 6$ or $D(x,y) = 7$; and any $x - y$ path that contains w has length $D(x,y) - 1$. Hence it follows that $V(G) - \{w\}$ is a detour convex set and so w is a detour extreme vertex of G. Thus G has r detour extreme vertices, each of degree 2.

Assume that $s \geq 3$. Let M_s be the complete multigraph with $V(M_s) = \{w_1, w_2, \ldots, w_s\}$ such that there are exactly two edges between every pair of distinct vertices of M_s. Subdividing each edges of M_s twice, we obtain a graph $S_2(M_s)$. For each pair i, j of integers with $1 \leq i < j \leq s$, let $w_i, u_{ijl}, v_{ijl}, w_j$ $(l = 1, 2)$ be the two $w_i - w_j$ path of length 3 in $S_2(M_s)$. Let H be the totally disconnected graph on r vertices $\overline{K_r}$ such that $S_2(M_s)$ and H are vertex disjoint. Let G be the graph obtained from $S_2(M_s)$ and H by joining each vertex w of H to each of the vertices $w_i (1 \leq i \leq s)$. The graph G is shown in Figure 4 for $s = 3$. We show that every vertex of H is detour extreme. We prove this for the case when $s = 3$ only, since the argument for $s \geq 4$ is similar. Let $w \in V(H)$. If $r = 1$ or $r = 2$, then it is clear that $w \notin I_D[x,y]$, where $x, y \in V(H)$ with $w \neq x, y$. For $r \geq 3$, it is clear that any $x - y$ path containing the vertex w has length at most 7 for $x, y \in V(H) - \{w\}$. Since $D(x,y) = 8$ for all $x, y \in V(H)$, it follows that $w \notin I_D[x,y]$ for all $x, y \in V(H) - \{w\}$. Also, $D(u_{ijl}, v_{ijl}) = 8$ and any $u_{ijl} - v_{ijl}$ path containing the vertex w is of length at most 7. Similarly, $D(u_{ij1}, v_{ij2}) = 10$ and any $u_{ij1} - v_{ij2}$ path containing the vertex w is of length at most 9. Similarly, for the other vertices, it can be easily checked that any $x - y$ path containing the vertex w with $w \neq x, y$ is of length at most $D(x,y) - 1$. Hence it follows that w is a detour extreme vertex of G. Thus, G has r detour extreme vertices each of degree s. □

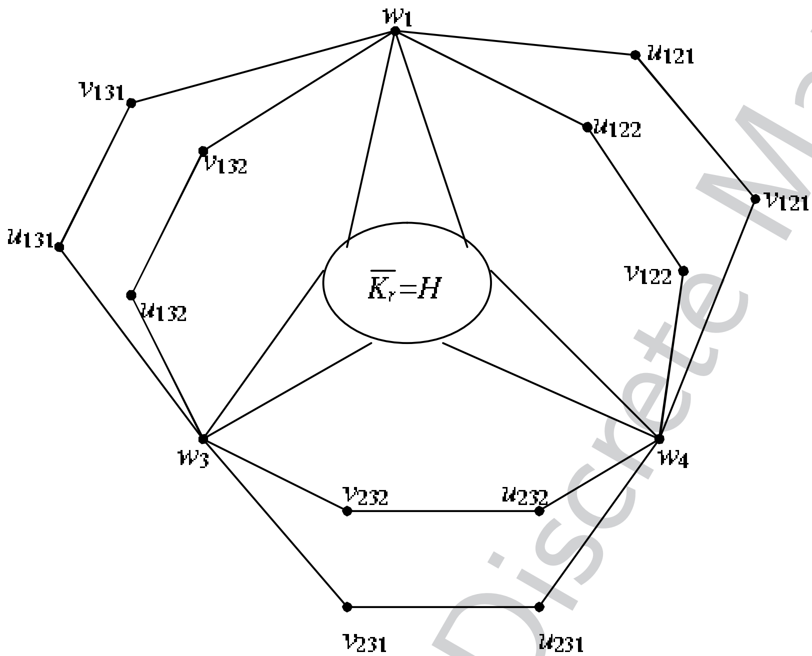

Figure 4. G

Theorem 2.10. For each pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G with $d_h(G) = a$ and $dn(G) = b$.

Proof. If $a = b$, then $K_{1,a}$ has the desired properties. So, assume that $a < b$. Let $G_i (1 \leq i \leq b-a)$ be the graph given in Figure 5. If $b - a = 1$, then $H = G_1$. If $b - a \geq 2$, then let H be the graph obtained from the $G_i{}'$ s by identifying the vertices $v_i, u_{i+1} (1 \leq i \leq b-a-1)$. Let G be the graph obtained from H by adding $a-1$ new vertices $s_1, s_2, \ldots, s_{a-1}$ and joining each $s_i (1 \leq i \leq a-1)$ to v_{b-a}. We show that the graph G has the desired properties. Let $S = \{u_1, s_1, s_2, \ldots, s_{a-1}\}$ be the set of end vertices of G. Then it is clear that $I_D[S] = V - \{w_1, w_2, \ldots, w_{b-a}\}$ and $I_D^2[S] = V$. Hence by Proposition 2.5, S is a minimum detour hull set of G so that $d_h(G) = a$. Now, each $w_i (1 \leq i \leq b-a)$ lies only on the $x_i - z_i, x_i - t_i, y_i - t_i$ and $y_i - z_i$ detours and so w_i is a detour vertex of G. Since $S \cup \{w_1, w_2, \ldots, w_{b-a}\}$ is a detour set of G, it follows from Proposition 2.5 that $dn(G) = b$. □

Lemma 2.11. Let S be a minimum detour hull set of a connected graph G and let $u, v \in S$. If w is a vertex distinct from u and v such that it lies on a $u - v$ detour in G, then $w \notin S$.

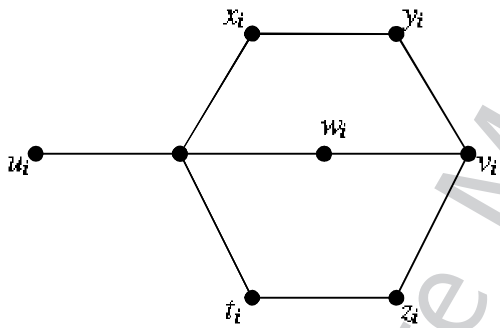

FIGURE 5. G_i

Proof. If $w \in S$, then $S \subseteq I_D[S - \{w\}]$ and hence $S - \{w\}$ is a detour hull set of G, which is a contradiction to S a minimum detour hull set of G. Thus the result follows. □

Theorem 2.12. Let G be a connected graph with a cut vertex v and S a detour hull set of G. Then
(i) Every component of $G - v$ contains an element of S.
(ii) If S is a minimum detour hull set of G, then no cut vertex of G belongs to S.

Proof. (i). Let C be a component of $G - v$. Since v is a cut vertex, it is clear that $V(G) - V(C)$ is a detour convex set of G. Hence it follows that $V(C) \cap S \neq \phi$.

(ii). Let S be a minimum detour hull set of G and let $C_1, C_2, \ldots, C_k$ $(k \geq 2)$ be the components of $G - v$. By (i), we see that $V(C_i) \cap S \neq \phi$ for $i = 1, 2, \ldots, k$. Since v is a cut vertex of G, it follows that $v \in I_D[u_1, u_2]$, where $u_1 \in V(C_1) \cap S$ and $u_2 \in V(C_2) \cap S$. Now, it follows from Lemma 2.11 that $v \notin S$. This completes the proof. □

Corollary 2.13. If G is a connected graph having $k \geq 2$ end-blocks, then $d_h(G) \geq k$.

Theorem 2.14. Let G be a connected graph with p end vertices and k end-blocks $B_1, B_2, \ldots, B_k$ such that $|V(B_i)| \geq 3$ for $1 \leq i \leq k$. If $d_h(B_i) = k_i$, then $d_h(G) \geq p + (\sum_{i=1}^{k} k_i) - k$.

Proof. If $k = 0$, then by Proposition 2.5, the result follows. If $p = 0$ and $k = 1$, then the graph G itself is a block and the result follows. For the remaining cases, assume to the contrary that there exists a connected graph G with p end vertices and k end-blocks $B_1, B_2, \ldots, B_k$ such that $|V(B_i)| \geq 3$ and $d_h(B_i) = k_i$ $(1 \leq i \leq k)$ for which $d_h(G) \leq p + (\sum_{i=1}^{k} k_i) - k - 1$. Then G contains a detour hull set S of cardinality $p + (\sum_{i=1}^{k} k_i) - k - 1$. Consequently, at least one of the end-blocks B_i contains no more than $k_i - 2$ vertices of S. Without loss of generality, let B_1 contain at most $k_1 - 2$ vertices of S and let $S_1 = S \cap V(B_1) \cup \{v\}$, where $v \in V(B_1)$ is the cut vertex of G. Then $|S_1| \leq k_1 - 1$ and S_1 is not a detour hull set of B_1. Hence there exists $u \in V(B_1)$ such that $u \notin I_D^n[S_1]_{B_1}$ for any $n \geq 0$. Now, we claim that $I_D^n[S] \cap V(B_1) \subseteq I_D^n[S_1]$ for any $n \geq 0$. We prove this by induction on n. If $n = 0$, then the claim is obvious. Let $x \in I_D[S] \cap V(B_1)$. If $x \in S$, then $x \in S_1 \subseteq I_D[S_1]$. So, assume that $x \notin S$. We have $x \in I_D[y, z]$ for some $y, z \in S$. If $x = v$, then $x \in S_1$ and so $x \in I_D[S_1]$. Let $x \neq v$. Since B_1 is an end-block, it follows that at least one of y and z, say y belongs to B_1 and so $y \in S_1$. If $z \in V(B_1)$, then $z \in S_1$ and so $x \in I_D[S_1]$. So, assume that $z \notin V(B_1)$. Let P be a $y - z$ detour which contains the vertex x. Then the $y - v$ subpath Q of P is a $y - v$ detour in G. Hence $x \in I_D[y, v] \subseteq I_D[S_1]$. Thus $I_D[S] \cap V(B_1) \subseteq I_D[S_1]$. Now, assume that the result is true for $n = k$. Then $I_D^k[S] \cap V(B_1) \subseteq I_D^k[S_1]$. Let $x \in I_D^{k+1}[S] \cap V(B_1)$. If $x \in I_D^k[S]$, then by induction hypothesis, $x \in I_D^k[S_1]$, which is a subset of $I_D^{k+1}[S_1]$, and so we are through. So, assume that $x \notin I_D^k[S]$. Then $x \in I_D[y, z]$ for some $y, z \in I_D^k[S]$. Since $x \in V(B_1)$, as above, we see that at least one of y and z, say y belongs to $V(B_1)$. Hence by induction hypothesis, $y \in I_D^k[S_1]$. If $z \in V(B_1)$, then again by induction hypothesis, $z \in I_D^k[S_1]$ and so $x \in I_D^{k+1}[S_1]$. If $z \notin V(B_1)$, then $x \in I_D[y, v]$ with $y \in I_D^k[S_1]$ and $v \in S_1$. Since $S_1 \subseteq I_D^k[S_1]$, we see that $x \in I_D^{k+1}[S_1]$. Thus by induction $I_D^n[S] \cap V(B_1) \subseteq I_D^n[S_1]$ for all $n \geq 0$. Now, since S is a detour hull set of G, there is an integer $r \geq 0$ such that $I_D^r[S] = V(G)$. This implies that $V(B_1) \subseteq I_D^r[S_1]$. Also, since B_1 is an end-block of G, it is clear that $I_D^n[S_1] = I_D^n[S_1]_{B_1}$ for all $n \geq 0$ and so $I_D^r[S]_{B_1} = V(B_1)$. This is a contradiction to the fact that $u \notin I_D^n[S_1]_{B_1}$ for any $n \geq 0$. Hence the result follows. □

Remark 2.15. The lower bound in Theorem 2.14 is strict. For the graph G in Figure 6, each of the k end-blocks B_i is such that $d_h(B_i) = 2$. Note that x is a detour extreme vertex of G. Since the set $S = \{u_1, u_2, \ldots, u_p, x, v_1, v_2, \ldots, v_k\}$ is a detour hull set, it follows from Proposition 2.5 and Theorem 2.12 that $d_h(G) = p + k + 1$ and so the bound in Theorem 2.14 is strict. Also, for the graph $H = G - x$, we have $d_h(G) = p + k$ and so the lower bound in Theorem 2.14 is sharp.

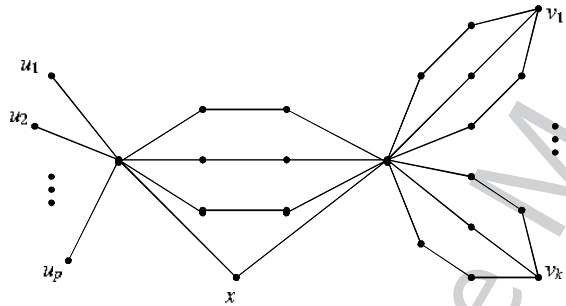

FIGURE 6. G

Theorem 2.16. Let G be a unicyclic graph with the cycle C and $k \geq 1$ end vertices. Then

$$d_h(G) = \begin{cases} k+1 & \text{if exactly one vertex of } C \text{ has degree } \geq 3 \\ k & \text{otherwise.} \end{cases}$$

Proof. Let $S = \{u_1, u_2, \ldots, u_k\}$ be the set of end vertices of G. For each vertex u_i, there exists a unique vertex v_i in C such that $d(u_i, v_i)$ is minimum. If exactly one vertex of C has degree ≥ 3, then $v_1 = v_2 = \cdots = v_k = v$, say. Then it can be easily seen that $[S]_D$ contains at most the vertex v from C and so S is not a detour hull set. Let v' be a vertex in C such that v' is adjacent to v. Then $I_D[S \cup \{v'\}] = V$ and so it follows from Proposition 2.5 that $S \cup \{v'\}$ is a minimum detour hull set of G and so $d_h(G) = |S| + 1 = k + 1$. Now, assume that C has at least two vertices of degree ≥ 3. Since G is unicyclic, it is clear that $I_D[v_i, v_j] \subseteq I_D[u_i, u_j]$ for $v_i \neq v_j$. Let P_i be the $u_i - v_i$ path in G and let Q_{ij} be a $v_i - v_j$ detour in G. Then $V(Q_{ij}) \subseteq V(C)$, and for $v_i \neq v_j$, P_i together with Q_{ij} followed by P_j is a $u_i - u_j$ detour in G. Now, let x be a vertex of G. If $x \notin V(C)$, then $x \in V(P_i)$ for some i with $1 \leq i \leq k$. Since C has at least two vertices of degree ≥ 3, it follows that $x \in I_D[u_i, u_j]$ for some j with $1 \leq i \neq j \leq k$. Now, let $x \in V(C)$. Let v and v' be vertices in C such that $deg(v) \geq 3$ and $deg(v') \geq 3$. Then $v = v_r$ and $v' = v_s$ for some r, s with $1 \leq r \neq s \leq k$. If $x \in Q_{rs}$, then $x \in I_D[u_r, u_s]$. Otherwise, $x \in I_D[v', y]$, where $v', y \in V(Q_{rs})$ such that v' and y are adjacent. Thus we see that $x \in I_D^2[u_r, u_s]$. Hence it follows from Proposition 2.5 that S is a minimum detour hull set of G and so $d_h(G) = |S| = k$. □

3. The detour hull number and the detour number

The following theorem is an immediate consequence of Theorem 1.2.

Theorem 3.1. If G is a connected graph of order n and detour diameter D, then $d_h(G) \leq n - D + 1$.

We give below a characterization theorem for trees.

Theorem 3.2. For every non-trivial tree T of order n and detour diameter D, $d_h(T) = n - D + 1$ if only if T is a caterpillar.

Proof. Let T be any non-trivial tree. Let u, v be two vertices in T such that $D(u, v) = D$ and $P : u = v_0, v_1, \ldots, v_{D-1}, v_D = v$ be a detour diameteral path. Let k be the number of end-vertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D-1}$. Then $D - 1 + l + k = n$. By Corollary 2.6, $d_h(T) = k = n - D - l + 1$. Hence $d_h(T) = n - D + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the detour diameteral path P, if and only if T is a caterpillar. □

Theorem 3.3. For each triple D, k and n of positive integers with $2 \leq k \leq n - D + 1$ and $D \geq 3$, there is a connected graph G of order n, detour diameter D and detour hull number k.

Proof. Let G be the graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (1) adding $k - 1$ new vertices $v_1, v_2, \ldots, v_{k-1}$ and joining each vertex $v_i (1 \leq i \leq k - 1)$ to u_1 and (2) adding $n - D - k + 1$ new vertices $w_1, w_2, \ldots, w_{n-D-k+1}$ and joining each vertex $w_i (1 \leq i \leq n - D - k + 1)$ to both u_1 and u_3. The graph G has order n and detour diameter D and is shown in Figure 7. Now, we show that $d_h(G) = k$.

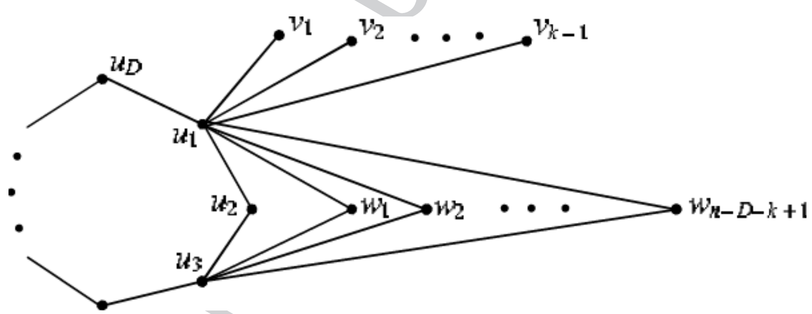

Figure 7. G

Let $S = \{v_1, v_2, \ldots, v_{k-1}\}$ be the set of end vertices of G. It is clear that $I_D[S] = S \cup \{u_1\}$ and $I_D^2[S] = I_D[S]$. Thus $[S]_D = S \cup \{u_1\} \neq V$ and so S is not a detour hull set of G. Since $I_D[S \cup \{u_D\}] = V$, it follows from Proposition 2.5 that $S \cup \{u_D\}$ is a minimum detour hull set of G so that $d_h(G) = |S| + 1 = k$. □

It is proved in [2] that the detour radius and detour diameter of a connected graph G satisfy $rad_D(G) \leq diam_D(G) \leq 2\,rad_D(G)$. It is also proved that every pair a, b of positive integers can be realized as the detour radius and detour diameter respectively of some connected graph, provided $a \leq b \leq 2a$. We extend this theorem so that the detour hull number can be prescribed as well, when $a < b \leq 2a$.

Theorem 3.4. For positive integers a, b and $k \geq 2$ with $a < b \leq 2a$, there exists a connected graph G with $rad_D(G) = a, diam_D(G) = b$ and $d_h(G) = b$.

Proof. If $a = 1$ and $b = 2$, then $G = K_{1,k}$ has the desired properties. So, let $a \geq 2$. Let K_a and K_{b-a} be the complete graphs of order a and $b - a$ respectively such that both are vertex disjoint. Let H be the graph obtained by identifying a vertex v of K_a and K_{b-a}. Let H_1 be the graph obtained from H by adding $k-1$ new vertices $u_1, u_2, \ldots, u_{k-1}$ and joining each $u_i (1 \leq i \leq k-1)$ to a vertex $x \neq v$ of K_a. Now, if $b - a = 1$, then G be the graph obtained from H_1 by adding a new vertex u_k and joining it to v; if $b - a \geq 2$, then G be the graph obtained from H_1 by adding a new vertex u_k and joining it to a vertex $y \neq v$ of K_{b-a}. Then it is clear that the set $S = \{u_1, u_2, \ldots, u_k\}$ of end vertices of G is a detour hull set of G and so by Proposition 2.5, $d_h(G) = k$. Note that

$$D(v, z) = \begin{cases} a-1 & \text{if } z \in V(K_a) \text{ and } z \neq v, \\ a & \text{if } z = u_i (1 \leq i \leq k-1), \\ b-a-1 & \text{if } z \in V(K_{b-a}) \text{ and } z \neq u_k, \\ b-a & \text{if } x = u_k. \end{cases}$$

Since $b \leq 2a$, we have $b - a \leq a$. Hence it follows that $e_D(v) = a$. Similarly, it can be easily seen that $e_D(u_i) = b$ for $i = 1, 2, \ldots, k$ and $e_D(x) = b - 1$ for all $x \neq v, u_i (1 \leq i \leq k)$. Hence it follows that $rad_D(G) = a$ and $daim_D(G) = b$. □

A graph G is said to be *hypohamiltonian* if G does not itself have a Hamiltonian cycle but every graph formed by removing a single vertex from G is Hamiltonian.

Proposition 3.5. If G is a Hamiltonian or hypohamiltonian graph, then $dn(G) = d_h(G) = 2$.

Proof. If G is Hamiltonian, then G has a Hamiltonian cycle C. Then any two adjacent vertices in C is a detour set as well as a detour hull set of G and so $dn(G) = d_h(G) = 2$. If G is a hypohamiltonian graph, then for any vertex v, $G - v$ has a Hamiltonian cycle $C : u_1, u_2, \ldots, u_{n-1}, u_1$, where u_1

is adjacent to v. Now, $P : v, u_1, u_2, \ldots, u_{n-1}$ is a $v - u_{n-1}$ Hamiltonian path in G. Hence $S = \{v, u_{n-1}\}$ is a detour set as well as a detour hull set of G and so $dn(G) = d_h(G) = 2$. □

Now, we introduce two classes of graphs Γ and Ω given in Figures 8 and 9, respectively, which are used in the proof of Theorem 3.6.

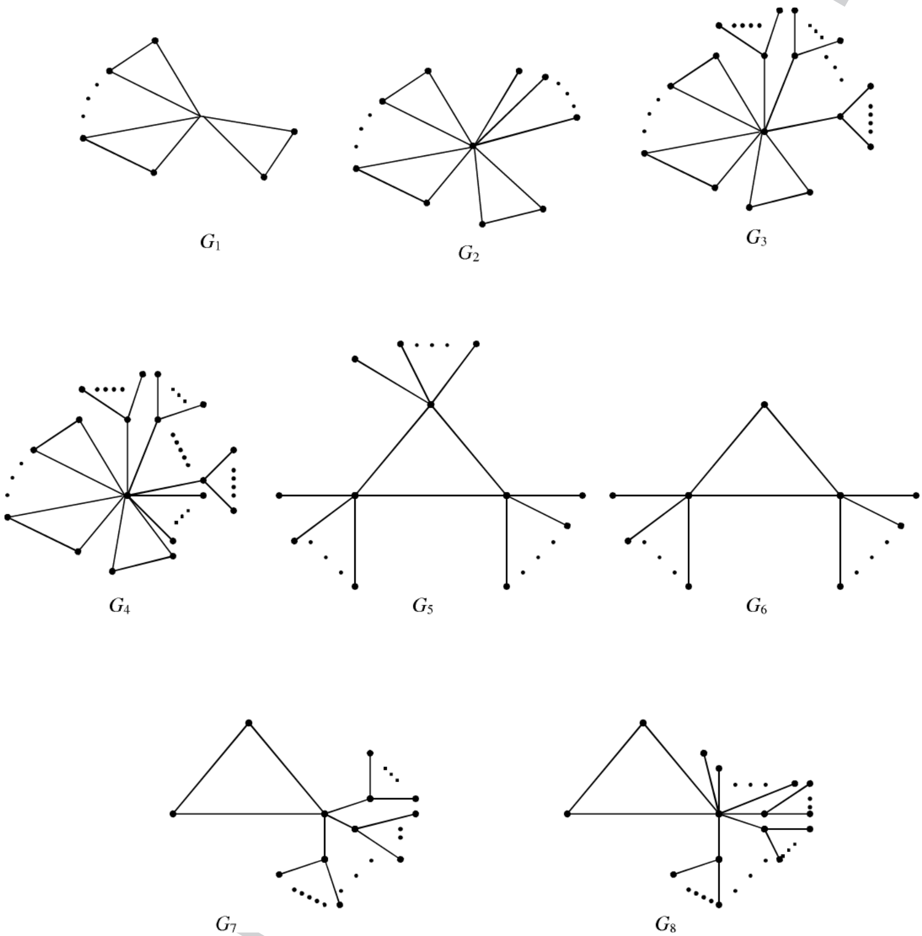

Figure 8. Γ

Theorem 3.6. If G is a connected graph with $diam_D(G) \leq 4$, then $dn(G) = d_h(G)$.

Proof. If $diam_D(G) = 1$, then $G = K_2$ and so the result follows. Also, if $diam_D(G) = 2$, then $G = K_{1,n} (n \geq 2)$ or $G = K_3$ and so $dn(G) = d_h(G)$.

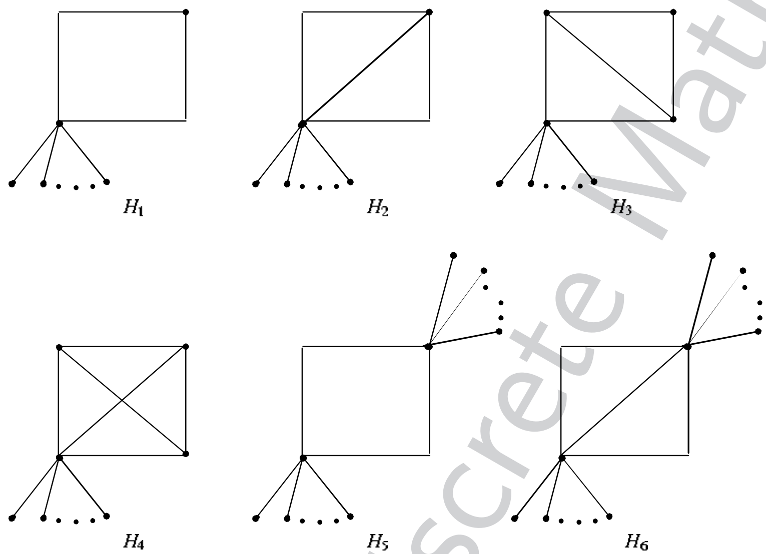

FIGURE 9. Ω

Now, let $diam_D(G) = 3$. If G is a tree, then by Corollary 2.6, $dn(G) = d_h(G)$. So, assume that $cir(G) \geq 3$, where $cir(G)$ denotes the length of a longest cycle in G. Since $diam_D(G) = 3$, it is clear that $cir(G) = 3$ or $cir(G) = 4$. If $cir(G) = 4$, then $G = C_4$ or $G = C_4 + e$ or $G = K_4$ and so the result follows from Proposition 3.5. Let $cir(G) = 3$. Then the graph G reduces to $G = K_{1,n-1} + e$ and so it is easily seen that $dn(G) = d_h(G)$. Now, let $diam_D(G) = 4$. If G is a tree, then the result follows. Assume that G is not a tree. Since $diam_D(G) = 4$, we have $cir(G) \leq 5$. If $cir(G) = 3$, then G belongs to the family Γ. Hence it follows from Proposition 2.5 and Theorem 2.12 that $dn(G) = d_h(G)$ for each G in Γ. Let $cir(G) = 4$. Then it is clear that G belongs to the family Ω. It follows from Proposition 2.5 and Theorem 2.12 that $dn(G) = d_h(G)$ for each G in Ω. Now, let $cir(G) = 5$. Since $diam_D(G) = 4$, it follows that order of G is 5 and hence G is Hamiltonian. Then it follows from Proposition 3.5 that $dn(G) = d_h(G)$. This completes the proof. □

Theorem 3.7. Let G be a connected graph of order $n \geq 4$. Then the following are equivalent:
(i) $d_h(G) = n - 1$
(ii) $dn(G) = n - 1$
(iii) $G = K_{1,n-1}$

Proof. By Theorem 1.3, it is enough to prove that (i) and (ii) are equivalent. Suppose that $d_h(G) = n-1$. Then by Theorem 3.1, $diam_D(G) \leq 2$ and so it follows from Theorem 3.6 that $dn(G) = n-1$. Conversely, Suppose that $dn(G) = n-1$. Then by Theorem 1.2, $diam_D(G) \leq 2$ and so it follows from Theorem 3.6 that $d_h(G) = n-1$. □

Theorem 3.8. Let G be a connected graph of order $n \geq 5$. Then the following are equivalent:
(i) $d_h(G) = n-2$
(ii) $dn(G) = n-2$
(iii) G is a double star or $G = K_{1,n-1} + e$

Proof. By Theorem 1.4, it is enough to prove that (i) and (ii) are equivalent. Suppose that $d_h(G) = n-2$. Then by Theorem 3.1, $diam_D(G) \leq 3$ and so it follows from Theorem 3.6 that $dn(G) = n-2$. Conversely, Suppose that $dn(G) = n-2$. Then by Theorem 1.2, $diam_D(G) \leq 3$ and so it follows from Theorem 3.6 that $d_h(G) = n-2$. □

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