On radical square zero rings

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Abstract. Let \( \Lambda \) be a connected left artinian ring with radical square zero and with \( n \) simple modules. If \( \Lambda \) is not self-injective, then we show that any module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n + 1 \) is projective. We also determine the structure of the artin algebras with radical square zero and \( n \) simple modules which have a non-projective module \( M \) such that \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \).

Xiao-Wu Chen [C] has recently shown: given a connected artin algebra \( \Lambda \) with radical square zero then either \( \Lambda \) is self-injective or else any CM module is projective. Here we extend this result by showing: If \( \Lambda \) is a connected artin algebra with radical square zero and \( n \) simple modules then either \( \Lambda \) is self-injective or else any module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n + 1 \) is projective. Actually, we will not need the assumption on \( \Lambda \) to be an artin algebra; it is sufficient to assume that \( \Lambda \) is a left artinian ring. And we show that for artin algebras the bound \( n + 1 \) is optimal by determining the structure of those artin algebras with radical square zero and \( n \) simple modules which have a non-projective module \( M \) such that \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \).

From now on, let \( \Lambda \) be a left artinian ring with radical square zero, this means that \( \Lambda \) has an ideal \( I \) with \( I^2 = 0 \) (the radical) such that

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Λ/I is semisimple artinian. We also assume that Λ is connected (the only central idempotents are 0 and 1). The modules to be considered are usually finitely generated left Λ-modules. Let n be the number of (isomorphism classes of) simple modules.

Given a module M, we denote by PM a projective cover, by QM an injective envelope of M. Also, we denote by ΩM a syzygy module for M, this is the kernel of a projective cover PM → M. Since Λ is a ring with radical square zero, all the syzygy modules are semisimple. Inductively, we define Ω₀M = M, and Ωᵢ+₁M = Ω(ΩᵢM) for i ≥ 0.

**Lemma 1.** If M is a non-projective module with Extᵢ(M, Λ) = 0 for 1 ≤ i ≤ d + 1 (and d ≥ 1), then there exists a simple non-projective module S with Extᵢ(S, Λ) = 0 for 1 ≤ i ≤ d.

*Proof.* The proof is obvious: We have Extᵢ(M, Λ) ≃ Extᵢ−1(ΩM, Λ), for all i ≥ 2. Since M is not projective, ΩM ≠ 0. Now ΩM is semisimple. If all simple direct summands of ΩM are projective, then also ΩM is projective, but then the condition Ext₁(M, Λ) = 0 implies that Ext₁(M, ΩM) = 0 in contrast to the existence of the exact sequence 0 → ΩM → PM → M → 0. Thus, let S be a non-projective simple direct summand of ΩM.

**Lemma 2.** If S is a non-projective simple module with Ext₁(S, Λ) = 0, then PS is injective and ΩS is simple and not projective.

*Proof.* First, we show that PS has length 2. Otherwise, ΩS is of length at least 2, thus there is a proper decomposition ΩS = U ⊕ U′ and then there is a canonical exact sequence

0 → PS → PS/U ⊕ PS/U′ → S → 0,

which of course does not split. But since Ext₁(S, Λ) = 0, we have Ext₁(S, P) = 0, for any projective module P. Thus, we obtain a contradiction.

This shows also that ΩS is simple. Of course, ΩS cannot be projective, again according to the assumption that Ext₁(S, P) = 0, for any projective module P.

Now let us consider the injective envelope Q of ΩS. It contains PS as a submodule (since PS has ΩS as socle). Assume that Q is of length at least 3. Take a submodule I of Q of length 2 which is different from PS.
and let $V = PS + I$, this is a submodule of $Q$ of length 3. Thus, there are the following inclusion maps $u_1, u_2, v_1, v_2$:

$$
\begin{array}{c}
\Omega S \xrightarrow{u_1} PS \\
v_1 \downarrow \quad \downarrow u_2 \\
I \xrightarrow{v_2} V
\end{array}
$$

The projective cover $p : PI \to I$ has as restriction a surjective map $p' : \text{rad } PI \to \Omega S$. But $\text{rad } PI$ is semisimple, thus $p'$ is a split epimorphism, thus we obtain a map $w : \Omega S \to PI$ such that $pw = v_1$. We consider the exact sequence induced from the sequence $0 \to \Omega S \to PS \to S \to 0$ by the map $w$:

$$
\begin{array}{c}
0 \longrightarrow \Omega S \xrightarrow{u_1} PS \xrightarrow{e_1} S \longrightarrow 0 \\
0 \longrightarrow PI \xrightarrow{u'_1} N \xrightarrow{e'_1} S \longrightarrow 0
\end{array}
$$

Here, $N$ is the pushout of the two maps $u_1$ and $w$. Since we know that $u_2u_1 = v_2v_1 = v_2pw$, there is a map $f : N \to V$ such that $fu'_1 = v_2p$ and $fw' = u_2$. Since the map $[v_2p \quad u_2] : PI \oplus PS \to V$ is surjective, also $f$ is surjective.

But recall that we assume that $\text{Ext}^1(S, \Lambda) = 0$, thus $\text{Ext}^1(S, PI) = 0$. This means that the lower exact sequence splits and therefore the socle of $N = PI \oplus S$ is a maximal submodule of $N$ (since $I$ is a local module, also $PI$ is a local module). Now $f$ maps the socle of $N$ into the socle of $V$, thus it maps a maximal submodule of $N$ into a simple submodule of $V$. This implies that the image of $f$ has length at most 2, thus $f$ cannot be surjective. This contradiction shows that $Q$ has to be of length 2, thus $Q = PS$ and therefore $PS$ is injective.

**Lemma 3.** If $S$ is a non-projective simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_iS$ with $0 \leq i \leq d$ are simple and not projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$.

**Proof.** The proof is again obvious, we use induction. If $d \geq 2$, we know by induction that the modules $S_i$ with $0 \leq i \leq d - 1$ are simple and not projective, and that the modules $P(S_i)$ are injective for $0 \leq i < d - 1$. 

But $\text{Ext}^1(\Omega_{d-1}S, \Lambda) \simeq \text{Ext}^d(S, \Lambda) = 0$, thus Lemma 2 asserts that also $S_d$ is simple and not projective and that $P(S_{d-1})$ is injective.

**Lemma 4.** Let $S_0, S_1, \ldots, S_b$ be simple modules with $S_i = \Omega_i(S_0)$ for $1 \leq i \leq b$. Assume that there is an integer $0 \leq a < b$ such that the modules $S_i$ with $a \leq i < b$ are pairwise non-isomorphic, whereas $S_b$ is isomorphic to $S_a$. In addition, we assume that the modules $P(S_i)$ for $a \leq i < b$ are injective. Then $S_a, \ldots, S_{b-1}$ is the list of all the simple modules and $\Lambda$ is self-injective.

**Proof.** Let $S$ be the subcategory of all modules with composition factors of the form $S_i$, where $a \leq i < b$. We claim that this subcategory is closed under projective covers and injective envelopes. Indeed, the projective cover of $S_i$ for $a \leq i < b$ has the composition factors $S_i$ and $S_{i+1}$ (and $S_b = S_a$), thus is in $S$. Similarly, the injective envelope for $S_i$ with $a < i < b$ is $Q(S_i) = P(S_{i-1})$, thus it has the composition factors $S_{i-1}$ and $S_i$, and $Q(S_a) = Q(S_b) = P(S_{b-1})$ has the composition factors $S_{b-1}$ and $S_a$. Since we assume that $\Lambda$ is connected, we know that the only non-trivial subcategory closed under composition factors, extensions, projective covers and injective envelopes is the module category itself. This shows that $S_a, \ldots, S_{b-1}$ are all the simple modules. Since the projective cover of any simple module is injective, $\Lambda$ is self-injective.

**Theorem 1.** Let $\Lambda$ be a connected left artinian ring with radical square zero. Assume that $\Lambda$ is not self-injective. If $S$ is a non-projective simple module such that $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_iS$ with $0 \leq i \leq d$ are pairwise non-isomorphic simple and non-projective modules and the modules $P(S_i)$ are injective for $0 \leq i < d$.

**Proof.** According to Lemma 3, the modules $S_i$ (with $0 \leq i \leq d$) are simple and non-projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$. If at least two of the modules $S_0, \ldots, S_d$ are isomorphic, then Lemma 4 asserts that $\Lambda$ is self-injective, but this we have excluded.

**Theorem 2.** Let $\Lambda$ be a connected left artinian ring with radical square zero and with $n$ simple modules. The following conditions are equivalent:

(i) $\Lambda$ is self-injective, but not a simple ring.
(ii) There exists a non-projective module $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$.

(iii) There exists a non-projective simple module $S$ with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n$.

Proof. First, assume that $\Lambda$ is self-injective, but not simple. Since $\Lambda$ is not semisimple, there is a non-projective module $M$. Since $\Lambda$ is self-injective, $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$. This shows the implication $(i) \implies (ii)$. The implication $(ii) \implies (iii)$ follows from Lemma 1. Finally, for the implication $(iii) \implies (i)$ we use Theorem 1. Namely, if $\Lambda$ is not self-injective, then Theorem 1 asserts that the simple modules $S_i = \Omega_i S$ with $0 \leq i \leq n$ are pairwise non-isomorphic. However, these are $n + 1$ simple modules, and we assume that the number of isomorphism classes of simple modules is $n$. This completes the proof of Theorem 2. \qed

Note that the implication $(ii) \implies (i)$ in Theorem 2 asserts in particular that either $\Lambda$ is self-injective or else that any CM module is projective, as shown by Chen [C]. Let us recall that a module $M$ is said to be a CM module provided $\text{Ext}^i(M, \Lambda) = 0$ and $\text{Ext}^i(\text{Tr} M, \Lambda) = 0$, for all $i \geq 1$ (here $\text{Tr}$ denotes the transpose of the module); these modules are also called Gorenstein-projective modules, or totally reflexive modules, or modules of G-dimension equal to 0. Note that in general there do exist modules $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$ which are not CM modules, see [JS].

We also draw the attention to the generalized Nakayama conjecture formulated by Auslander-Reiten [AR]. It asserts that for any artin algebra $\Lambda$ and any simple $\Lambda$-module $S$ there should exist an integer $i \geq 0$ such that $\text{Ext}^i(S, \Lambda) \neq 0$. It is known that this conjecture holds true for algebras with radical square zero. The implication $(iii) \implies (i)$ of Theorem 2 provides an effective bound: If $n$ is the number of simple $\Lambda$-modules, and $S$ is simple, then $\text{Ext}^i(S, \Lambda) \neq 0$ for some $0 \leq i \leq n$. Namely, in case $S$ is projective or $\Lambda$ is self-injective, then $\text{Ext}^0(S, \Lambda) \neq 0$. Now assume that $S$ is simple and not projective and that $\Lambda$ is not self-injective. Then there must exist some integer $1 \leq i \leq n$ with $\text{Ext}^i(S, \Lambda) \neq 0$, since otherwise the condition $(iii)$ would be satisfied and therefore condition $(i)$. Theorem 1 may be interpreted as a statement concerning the Ext-quiver of $\Lambda$. Recall that the Ext-quiver $\Gamma(R)$ of a left artinian ring $R$ has as vertices the (isomorphism classes of the) simple $R$-modules, and if $S, T$
are simple $R$-modules, there is an arrow $T \to S$ provided $\text{Ext}^1(T, S) \neq 0$, thus provided that there exists an indecomposable $R$-module $M$ of length 2 with socle $S$ and top $T$. We may add to the arrow $\alpha : T \to S$ the number $l(\alpha) = ab$, where $a$ is the length of soc $PT$ and $b$ is the length of $QS/soc$ (note that $b$ may be infinite). The properties of $\Gamma(R)$ which are relevant for this note are the following: the vertex $S$ is a sink if and only if $S$ is projective; the vertex $S$ is a source if and only if $S$ is injective; finally, if $R$ is a radical square zero ring and $S, T$ are simple $R$-modules then $PT = QS$ if and only if there is an arrow $\alpha : T \to S$ with $l(\alpha) = 1$ and this is the only arrow starting at $T$ and the only arrow ending in $S$.

Theorem 1 assert the following: Let $\Lambda$ be a connected left artinian ring with radical square zero. Assume that $\Lambda$ is not self-injective. Let $S$ be a non-projective simple module such that $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, and let $S_i = \Omega_i S$ with $0 \leq i \leq d$. Then the local structure of $\Gamma(\Lambda)$ is as follows:

such that there is at least one arrow starting in $S_d$ (but maybe no arrow ending in $S_0$). To be precise: the picture is supposed to show all the arrows starting or ending in the vertices $S_0, \ldots, S_d$ (and to assert that the vertices $S_0, \ldots, S_d$ are pairwise different).

Let us introduce the quivers $\Delta(n, t)$, where $n, t$ are positive integers. The quiver $\Delta(n, t)$ has $n$ vertices and also $n$ arrows, namely the vertices labeled $0, 1, \ldots, n-1$, and arrows $i \to i+1$ for $0 \leq i \leq n-1$ (modulo $n$) (thus, we deal with an oriented cycle); in addition, let $l(\alpha) = t$ for the arrow $\alpha : n-1 \to 0$ and let $l(\beta) = 1$ for the remaining arrows $\beta$:  

\[ 
\begin{array}{ccc}
\vdots & & \vdots \\
S_0 & \xrightarrow{1} & S_1 & \ldots & S_{d-1} & \xrightarrow{1} & S_d & \xrightarrow{1} & \vdots \\
\end{array} 
\]
Note that the Ext-quiver of a connected self-injective left artinian ring with radical square zero and \( n \) vertices is just \( \Delta(n, 1) \). Our further interest lies in the cases \( t > 1 \).

**Theorem 3.** Let \( \Lambda \) be a connected left artinian ring with radical square zero and with \( n \) simple modules.

(a) If there exists a non-projective simple modules \( S \) with \( \text{Ext}^i(S, \Lambda) = 0 \) for \( 1 \leq i \leq n - 1 \), or if there exists a non-projective module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \), then \( \Gamma(\Lambda) \) is of the form \( \Delta(n, t) \) with \( t > 1 \).

(b) Conversely, if \( \Gamma(\Lambda) = \Delta(n, t) \) and \( t > 1 \), then there exists a unique simple module \( S \) with \( \text{Ext}^i(S, \Lambda) = 0 \) for \( 1 \leq i \leq n - 1 \), namely \( S = S(0) \) (and it satisfies \( \text{Ext}^n(S, \Lambda) \neq 0 \)).

(c) If \( \Gamma(\Lambda) = \Delta(n, t) \) and \( t > 1 \), and if we assume in addition that \( \Lambda \) is an artin algebra, then there exists a unique indecomposable module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \), namely \( M = \text{Tr} D(S(0)) \) (and it satisfies \( \text{Ext}^{n+1}(M, \Lambda) \neq 0 \)).

Here, for \( \Lambda \) an artin algebra, \( D \) denotes the \( k \)-duality, where \( k \) is the center of \( \Lambda \) (thus \( D = \text{Hom}_k(-, E) \), where \( E \) is a minimal injective cogenerator in the category of \( k \)-modules); thus \( D \text{Tr} \) is the Auslander-Reiten translation and \( \text{Tr} D \) the reverse.

**Proof of Theorem 3.** Part (a) is a direct consequence of Theorem 1, using the interpretation in terms of the Ext-quiver as outlined above. Note that we must have \( t > 1 \), since otherwise \( \Lambda \) would be self-injective.

(b) We assume that \( \Gamma(\Lambda) = \Delta(n, t) \) with \( t > 1 \). For \( 0 \leq i < n \), let \( S(i) \) be the simple module corresponding to the vertex \( i \), let \( P(i) \) be its projective cover, \( I(i) \) its injective envelope. We see from the quiver that all the projective modules \( P(i) \) with \( 0 \leq i \leq n - 2 \) are injective, thus \( \text{Ext}^j(-, \Lambda) = \text{Ext}^j(-, P(n-1)) \) for all \( j \geq 1 \). In addition, the quiver shows that \( \Omega S(i) = S(i+1) \) for \( 0 \leq i \leq n - 2 \). Finally, we have \( \Omega S(n-1) = S(0)^a \) for some positive integer \( a \) dividing \( t \) and the injective envelope of \( P(n-1) \) yields an exact sequence

\[
0 \to P(n-1) \to I(P(n-1)) \to S(n-1)^t \to 0 \quad (*)
\]

(namely, \( I(P(n-1)) = I(\text{soc} P(n-1)) = I(S(0)^a) = I(S(0))^a \) and \( I(S(0))/\text{soc} \) is the direct sum of \( b \) copies of \( S(n-1) \), where \( ab = t \); thus...
the cokernel of the inclusion map $P(n-1) \to I(P(n-1))$ consists of $t-1$ copies of $S(n-1)$.

Since $t > 1$, the exact sequence $(\ast)$ shows that $\text{Ext}^1(S(n-1), P(n-1)) \neq 0$. It also implies that $\text{Ext}^1(S(i), P(n-1)) = 0$ for $0 \leq i \leq n-2$, and therefore that

$$\text{Ext}^i(S(0), P(n-1)) = \text{Ext}^1(\Omega_{i-1}S(0), P(n-1))$$

$$= \text{Ext}^1(S(i-1), P(n-1))$$

$$= 0$$

for $1 \leq i \leq n-1$.

Since $\Omega_{n-i-1}S(i) = S(n-1)$ for $0 \leq i \leq n-1$, we see that

$$\text{Ext}^{n-i}(S(i), P(n-1)) = \text{Ext}^1(\Omega_{n-i-1}S(i), P(n-1))$$

$$= \text{Ext}^1(S(n-1), P(n-1))$$

$$\neq 0$$

for $0 \leq i \leq n-1$. Thus, on the one hand, we have $\text{Ext}^n(S(0), \Lambda) \neq 0$, this concludes the proof that $S(0)$ has the required properties. On the other hand, we also see that $S = S(0)$ is the only simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n-1$. This completes the proof of (b).

(c) Assume now in addition that $\Lambda$ is an artin algebra. As usual, we denote the Auslander-Reiten translation $D \text{Tr}$ by $\tau$. Let $M$ be a non-projective indecomposable module with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$. The shape of $\Gamma(\Lambda)$ shows that $\Omega M = S^c$ for some simple module $S$ (and we have $c \geq 1$), also it shows that no simple module is projective. Now $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i < n$, thus according to (b) we must have $S = S(0)$. It follows that $PM$ has to be a direct sum of copies of $P(n-1)$, say of $d$ copies. Thus a minimal projective presentation of $M$ is of the form

$$P(0)^c \to P(n-1)^d \to M \to 0,$$

and therefore a minimal injective copresentation of $\tau M$ is of the form

$$0 \to \tau M \to I(0)^c \to I(n-1)^d.$$

In particular, $\text{soc} \tau M = S(0)^c$ and $(\tau M) / \text{soc}$ is a direct sum of copies of $S(n-1)$.
Assume that $\tau M \neq S(0)$, thus it has at least one composition factor of the form $S(n-1)$ and therefore there exists a non-zero map $f : P(n-1) \to \tau M$. Since $\tau M$ is indecomposable and not injective, any map from an injective module to $\tau M$ maps into the socle of $\tau M$. But the image of $f$ is not contained in the socle of $\tau M$, therefore $f$ cannot be factored through an injective module. It follows that

$$\text{Ext}^1(M, P(n-1)) \simeq D\text{Hom}(P(n-1), \tau M) \neq 0,$$

which contradicts the assumption that $\text{Ext}^1(M, \Lambda) = 0$. This shows that $\tau M = S(0)$ and therefore $M = \text{Tr} DS(0)$.

Of course, conversely we see that $M = \text{Tr} DS(0)$ satisfies $\text{Ext}^i(M, P(n-1)) = 0$ for $1 \leq i \leq n$, and $\text{Ext}^{n+1}(M, P(n-1)) \neq 0$. □

Remarks.

(1) The module $M = \text{Tr} DS(0)$ considered in (c) has length $t^2 + t - 1$, thus the number $t$ (and therefore $\Delta(n, t)$) is determined by $M$.

(2) If $\Lambda$ is an artin algebra with Ext-quiver $\Delta(n, t)$, the number $t$ has to be the square of an integer, say $t = m^2$. A typical example of such an artin algebra is the path algebra of the following quiver

![Quiver Diagram]

with altogether $n + m - 1$ arrows, modulo the ideal generated by all paths of length 2. Of course, if $\Lambda$ is a finite-dimensional $k$-algebra with radical square zero and Ext-quiver $\Delta(n, m^2)$, and $k$ is an algebraically closed field, then $\Lambda$ is Morita-equivalent to such an algebra.

Also the following artin algebras with radical square zero and Ext-quiver $\Delta(1, m^2)$ may be of interest: the factor rings of the polynomial ring $\mathbb{Z}[T_1, \ldots, T_{m-1}]$ modulo the square of the ideal generated by some prime number $p$ and the variables $T_1, \ldots, T_{m-1}$.
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