Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 14 **(2012)**. Number 2. pp. 297 – 306

On radical square zero rings

Claus Michael Ringel and Bao-Lin Xiong

Communicated by Yu. A. Drozd

Dedicated to the memory of Andrei Roiter

Abstract. Let Λ be a connected left artinian ring with radical square zero and with n simple modules. If Λ is not self-injective, then we show that any module M with $\mathrm{Ext}^i(M,\Lambda)=0$ for $1 \le i \le n+1$ is projective. We also determine the structure of the artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\mathrm{Ext}^i(M,\Lambda)=0$ for $1 \le i \le n$.

Xiao-Wu Chen [C] has recently shown: given a connected artin algebra Λ with radical square zero then either Λ is self-injective or else any CM module is projective. Here we extend this result by showing: If Λ is a connected artin algebra with radical square zero and n simple modules then either Λ is self-injective or else any module M with $\mathrm{Ext}^i(M,\Lambda)=0$ for $1 \le i \le n+1$ is projective. Actually, we will not need the assumption on Λ to be an artin algebra; it is sufficient to assume that Λ is a left artinian ring. And we show that for artin algebras the bound $n+1$ is optimal by determining the structure of those artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\mathrm{Ext}^i(M,\Lambda)=0$ for $1 \le i \le n$.

From now on, let Λ be a left artinian ring with radical square zero, this means that Λ has an ideal I with $I^2=0$ (the radical) such that

2010 MSC: 16D90, 16G10; 16G70.

Key words and phrases: Artin algebras; left artinian rings; representations, modules; Gorenstein modules, CM modules; self-injective algebras; radical square zero algebras.

Λ/I is semisimple artinian. We also assume that Λ is connected (the only central idempotents are 0 and 1). The modules to be considered are usually finitely generated left Λ-modules. Let n be the number of (isomorphism classes of) simple modules.

Given a module M, we denote by PM a projective cover, by QM an injective envelope of M. Also, we denote by ΩM a syzygy module for M, this is the kernel of a projective cover $PM \to M$. Since Λ is a ring with radical square zero, all the syzygy modules are semisimple. Inductively, we define $\Omega_0 M = M$, and $\Omega_{i+1}M = \Omega(\Omega_i M)$ for $i \geq 0$.

Lemma 1. *If M is a non-projective module with $\operatorname{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq d+1$ (and $d \geq 1$), then there exists a simple non-projective module S with $\operatorname{Ext}^i(S,\Lambda) = 0$ for $1 \leq i \leq d$.*

Proof. The proof is obvious: We have $\operatorname{Ext}^i(M,\Lambda) \simeq \operatorname{Ext}^{i-1}(\Omega M,\Lambda)$, for all $i \geq 2$. Since M is not projective, $\Omega M \neq 0$. Now ΩM is semisimple. If all simple direct summands of ΩM are projective, then also ΩM is projective, but then the condition $\operatorname{Ext}^1(M,\Lambda) = 0$ implies that $\operatorname{Ext}^1(M,\Omega M) = 0$ in contrast to the existence of the exact sequence $0 \to \Omega M \to PM \to M \to 0$. Thus, let S be a non-projective simple direct summand of ΩM. $\square$

Lemma 2. *If S is a non-projective simple module with $\operatorname{Ext}^1(S,\Lambda) = 0$, then PS is injective and ΩS is simple and not projective.*

Proof. First, we show that PS has length 2. Otherwise, ΩS is of length at least 2, thus there is a proper decomposition $\Omega S = U \oplus U'$ and then there is a canonical exact sequence

$$0 \to PS \to PS/U \oplus PS/U' \to S \to 0,$$

which of course does not split. But since $\operatorname{Ext}^1(S,\Lambda) = 0$, we have $\operatorname{Ext}^1(S,P) = 0$, for any projective module P. Thus, we obtain a contradiction.

This shows also that ΩS is simple. Of course, ΩS cannot be projective, again according to the assumption that $\operatorname{Ext}^1(S,P) = 0$, for any projective module P.

Now let us consider the injective envelope Q of ΩS. It contains PS as a submodule (since PS has ΩS as socle). Assume that Q is of length at least 3. Take a submodule I of Q of length 2 which is different from PS

and let $V = PS + I$, this is a submodule of Q of length 3. Thus, there are the following inclusion maps u_1, u_2, v_1, v_2:

$$\begin{tikzcd}
\Omega S \arrow[r, "u_1"] \arrow[d, "v_1"'] & PS \arrow[d, "u_2"] \\
I \arrow[r, "v_2"] & V
\end{tikzcd}$$

The projective cover $p: \ PI \to I$ has as restriction a surjective map $p' : \operatorname{rad} PI \to \Omega S$. But $\operatorname{rad} PI$ is semisimple, thus p' is a split epimorphism, thus we obtain a map $w: \ \Omega S \to PI$ such that $pw = v_1$. We consider the exact sequence induced from the sequence $0 \to \Omega S \to PS \to S \to 0$ by the map w:

$$\begin{tikzcd}
0 \arrow[r] & \Omega S \arrow[r, "u_1"] \arrow[d, "w"'] & PS \arrow[r, "e_1"] \arrow[d, "w'"] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & PI \arrow[r, "u_1'"] & N \arrow[r, "e_1'"] & S \arrow[r] & 0
\end{tikzcd}$$

Here, N is the pushout of the two maps u_1 and w. Since we know that $u_2u_1 = v_2v_1 = v_2pw$, there is a map $f: \ N \to V$ such that $fu_1' = v_2p$ and $fw' = u_2$. Since the map $\begin{bmatrix} v_2p & u_2 \end{bmatrix} : \ PI \oplus PS \to V$ is surjective, also f is surjective.

But recall that we assume that $\operatorname{Ext}^1(S, \Lambda) = 0$, thus $\operatorname{Ext}^1(S, PI) = 0$. This means that the lower exact sequence splits and therefore the socle of $N = PI \oplus S$ is a maximal submodule of N (since I is a local module, also PI is a local module). Now f maps the socle of N into the socle of V, thus it maps a maximal submodule of N into a simple submodule of V. This implies that the image of f has length at most 2, thus f cannot be surjective. This contradiction shows that Q has to be of length 2, thus $Q = PS$ and therefore PS is injective. $\square$

Lemma 3. *If S is a non-projective simple module with* $\operatorname{Ext}^i(S, \Lambda) = 0$ *for $1 \le i \le d$, then the modules $S_i = \Omega_i S$ with $0 \le i \le d$ are simple and not projective, and the modules $P(S_i)$ are injective for $0 \le i < d$.*

Proof. The proof is again is obvious, we use induction. If $d \ge 2$, we know by induction that the modules S_i with $0 \le i \le d-1$ are simple and not projective, and that the modules $P(S_i)$ are injective for $0 \le i < d-1$.

But $\mathrm{Ext}^1(\Omega_{d-1}S,\Lambda) \simeq \mathrm{Ext}^d(S,\Lambda) = 0$, thus Lemma 2 asserts that also S_d is simple and not projective and that $P(S_{d-1})$ is injective. $\square$

Lemma 4. *Let $S_0, S_1, \ldots, S_b$ be simple modules with $S_i = \Omega_i(S_0)$ for $1 \le i \le b$. Assume that there is an integer $0 \le a < b$ such that the modules S_i with $a \le i < b$ are pairwise non-isomorphic, whereas S_b is isomorphic to S_a. In addition, we asssume that the modules $P(S_i)$ for $a \le i < b$ are injective. Then $S_a, \ldots, S_{b-1}$ is the list of all the simple modules and Λ is self-injective.*

Proof. Let $\mathcal{S}$ be the subcategory of all modules with composition factors of the form S_i, where $a \le i < b$. We claim that this subcategory is closed under projective covers and injective envelopes. Indeed, the projective cover of S_i for $a \le i < b$ has the composition factors S_i and S_{i+1} (and $S_b = S_a$), thus is in $\mathcal{S}$. Similarly, the injective envelope for S_i with $a < i < b$ is $Q(S_i) = P(S_{i-1})$, thus it has the composition factors S_{i-1} and S_i, and $Q(S_a) = Q(S_b) = P(S_{b-1})$ has the composition factors S_{b-1} and S_a. Since we assume that Λ is connected, we know that the only non-trivial subcategory closed under composition factors, extensions, projective covers and injective envelopes is the module category itself. This shows that $S_a, \ldots, S_{b-1}$ are all the simple modules. Since the projective cover of any simple module is injective, Λ is self-injective. $\square$

Theorem 1. *Let Λ be a connected left artinian ring with radical square zero. Assume that Λ is not self-injective. If S is a non-projective simple module such that $\mathrm{Ext}^i(S,\Lambda) = 0$ for $1 \le i \le d$, then the modules $S_i = \Omega_i S$ with $0 \le i \le d$ are pairwise non-isomorphic simple and non-projective modules and the modules $P(S_i)$ are injective for $0 \le i < d$.*

Proof. According to Lemma 3, the modules S_i (with $0 \le i \le d$) are simple and non-projective, and the modules $P(S_i)$ are injective for $0 \le i < d$. If at least two of the modules $S_0, \ldots, S_d$ are isomorphic, then Lemma 4 asserts that Λ is self-injective, but this we have excluded. $\square$

Theorem 2. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules. The following conditions are equivalent:*

(i) *Λ is self-injective, but not a simple ring.*

(ii) *There exists a non-projective module M with* $\operatorname{Ext}^i(M,\Lambda) = 0$ *for* $1 \le i \le n+1$.

(iii) *There exists a non-projective simple module S with* $\operatorname{Ext}^i(S,\Lambda) = 0$ *for* $1 \le i \le n$.

Proof. First, assume that Λ is self-injective, but not simple. Since Λ is not semisimple, there is a non-projective module M. Since Λ is self-injective, $\operatorname{Ext}^i(M,\Lambda) = 0$ for all $i \ge 1$. This shows the implication (i) $\Longrightarrow$ (ii). The implication (ii) $\Longrightarrow$ (iii) follows from Lemma 1. Finally, for the implication (iii) $\Longrightarrow$ (i) we use Theorem 1. Namely, if Λ is not self-injective, then Theorem 1 asserts that the simple modules $S_i = \Omega_i S$ with $0 \le i \le n$ are pairwise non-isomorphic. However, these are $n+1$ simple modules, and we assume that the number of isomorphism classes of simple modules is n. This completes the proof of Theorem 2. □

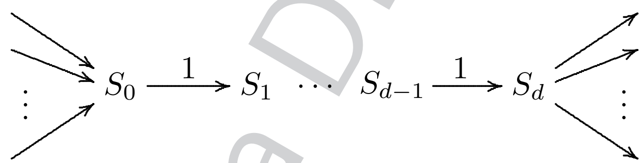
Note that the implication (ii) $\Longrightarrow$ (i) in Theorem 2 asserts in particular that either Λ is self-injective or else that any CM module is projective, as shown by Chen [C]. Let us recall that a module M is said to be a CM module provided $\operatorname{Ext}^i(M,\Lambda) = 0$ and $\operatorname{Ext}^i(\operatorname{Tr} M,\Lambda) = 0$, for all $i \ge 1$ (here Tr denotes the transpose of the module); these modules are also called Gorenstein-projective modules, or totally reflexive modules, or modules of G-dimension equal to 0. Note that in general there do exist modules M with $\operatorname{Ext}^i(M,\Lambda) = 0$ for all $i \ge 1$ which are not CM modules, see [JS].

We also draw the attention to the generalized Nakayama conjecture formulated by Auslander-Reiten [AR]. It asserts that for any artin algebra Λ and any simple Λ-module S there should exist an integer $i \ge 0$ such that $\operatorname{Ext}^i(S,\Lambda) \neq 0$. It is known that this conjecture holds true for algebras with radical square zero. The implication (iii) $\Longrightarrow$ (i) of Theorem 2 provides an effective bound: *If n is the number of simple Λ-modules, and S is simple, then* $\operatorname{Ext}^i(S,\Lambda) \neq 0$ *for some* $0 \le i \le n$. Namely, in case S is projective or Λ is self-injective, then $\operatorname{Ext}^0(S,\Lambda) \neq 0$. Now assume that S is simple and not projective and that Λ is not self-injective. Then there must exist some integer $1 \le i \le n$ with $\operatorname{Ext}^i(S,\Lambda) \neq 0$, since otherwise the condition (iii) would be satisfied and therefore condition (i).

Theorem 1 may be interpreted as a statement concerning the Ext-quiver of Λ. Recall that the Ext-*quiver* $\Gamma(R)$ of a left artinian ring R has as vertices the (isomorphism classes of the) simple R-modules, and if S, T

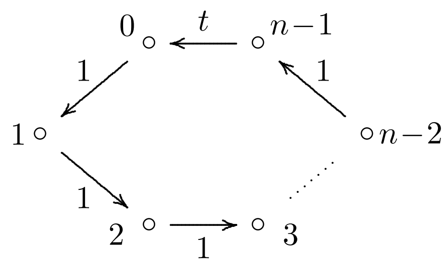
are simple R-modules, there is an arrow $T \to S$ provided $\mathrm{Ext}^1(T,S) \neq 0$, thus provided that there exists an indecomposable R-module M of length 2 with socle S and top T. We may add to the arrow $\alpha : T \to S$ the number $l(\alpha) = ab$, where a is the length of $\operatorname{soc} PT$ and b is the length of QS/soc (note that b may be infinite). The properties of $\Gamma(R)$ which are relevant for this note are the following: the vertex S is a sink if and only if S is projective; the vertex S is a source if and only if S is injective; finally, if R is a radical square zero ring and S, T are simple R-modules then $PT = QS$ if and only if there is an arrow $\alpha : T \to S$ with $l(\alpha) = 1$ and this is the only arrow starting at T and the only arrow ending in S.

Theorem 1 assert the following: Let Λ be a connected left artinian ring with radical square zero. Assume that Λ is not self-injective. Let S be a non-projective simple module such that $\mathrm{Ext}^i(S,\Lambda) = 0$ for $1 \le i \le d$, and let $S_i = \Omega_i S$ with $0 \le i \le d$. Then the local structure of $\Gamma(\Lambda)$ is as follows:



such that there is at least one arrow starting in S_d (but maybe no arrow ending in S_0). To be precise: the picture is supposed to show all the arrows starting or ending in the vertices $S_0, \ldots, S_d$ (and to assert that the vertices $S_0, \ldots, S_d$ are pairwise different).

Let us introduce the quivers $\Delta(n,t)$, where n, t are positive integers. The quiver $\Delta(n,t)$ has n vertices and also n arrows, namely the vertices labeled $0, 1, \ldots, n-1$, and arrows $i \to i{+}1$ for $0 \le i \le n-1$ (modulo n) (thus, we deal with an oriented cycle); in addition, let $l(\alpha) = t$ for the arrow $\alpha : n-1 \to 0$ and let $l(\beta) = 1$ for the remaining arrows β:



Note that the Ext-quiver of a connected self-injective left artinian ring with radical square zero and n vertices is just $\Delta(n,1)$. Our further interest lies in the cases $t>1$.

Theorem 3. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules.*

(a) *If there exists a non-projective simple modules S with $\mathrm{Ext}^i(S,\Lambda)=0$ for $1\le i\le n-1$, or if there exists a non-projective module M with $\mathrm{Ext}^i(M,\Lambda)=0$ for $1\le i\le n$, then $\Gamma(\Lambda)$ is of the form $\Delta(n,t)$ with $t>1$.*

(b) *Conversely, if $\Gamma(\Lambda)=\Delta(n,t)$ and $t>1$, then there exists a unique simple module S with $\mathrm{Ext}^i(S,\Lambda)=0$ for $1\le i\le n-1$, namely the module $S=S(0)$ (and it satisfies $\mathrm{Ext}^n(S,\Lambda)\neq 0$).*

(c) *If $\Gamma(\Lambda)=\Delta(n,t)$ and $t>1$, and if we assume in addition that Λ is an artin algebra, then there exists a unique indecomposable module M with $\mathrm{Ext}^i(M,\Lambda)=0$ for $1\le i\le n$, namely $M=\mathrm{Tr}\,D(S(0))$ (and it satisfies $\mathrm{Ext}^{n+1}(M,\Lambda)\neq 0$).*

Here, for Λ an artin algebra, D denotes the k-duality, where k is the center of Λ (thus $D=\mathrm{Hom}_k(-,E)$, where E is a minimal injective cogenerator in the category of k-modules); thus $D\,\mathrm{Tr}$ is the Auslander-Reiten translation and $\mathrm{Tr}\,D$ the reverse.

Proof of Theorem 3. Part (a) is a direct consequence of Theorem 1, using the interpretation in terms of the Ext-quiver as outlined above. Note that we must have $t>1$, since otherwise Λ would be self-injective.

(b) We assume that $\Gamma(\Lambda)=\Delta(n,t)$ with $t>1$. For $0\le i<n$, let $S(i)$ be the simple module corresponding to the vertex i, let $P(i)$ be its projective cover, $I(i)$ its injective envelope. We see from the quiver that all the projective modules $P(i)$ with $0\le i\le n-2$ are injective, thus $\mathrm{Ext}^j(-,\Lambda)=\mathrm{Ext}^j(-,P(n-1))$ for all $j\ge 1$. In addition, the quiver shows that $\Omega S(i)=S(i+1)$ for $0\le i\le n-2$. Finally, we have $\Omega S(n-1)=S(0)^a$ for some positive integer a dividing t and the injective envelope of $P(n-1)$ yields an exact sequence

$$0\to P(n-1)\to I(P(n-1))\to S(n-1)^{t-1}\to 0 \qquad (*)$$

(namely, $I(P(n-1))=I(\mathrm{soc}\,P(n-1))=I(S(0)^a)=I(S(0))^a$ and $I(S(0))/\,\mathrm{soc}$ is the direct sum of b copies of $S(n-1)$, where $ab=t$; thus the cokernel of the inclusion map $P(n-1) \to I(P(n-1))$ consists of $t-1$ copies of $S(n-1)$).

Since $t > 1$, the exact sequence $(*)$ shows that $\operatorname{Ext}^1(S(n-1), P(n-1)) \neq 0$. It also implies that $\operatorname{Ext}^1(S(i), P(n-1)) = 0$ for $0 \leq i \leq n-2$, and therefore that

$$\begin{aligned}\operatorname{Ext}^i(S(0), P(n-1)) &= \operatorname{Ext}^1(\Omega_{i-1}S(0), P(n-1)) \\ &= \operatorname{Ext}^1(S(i-1), P(n-1)) \\ &= 0\end{aligned}$$

for $1 \leq i \leq n-1$.

Since $\Omega_{n-i-1}S(i) = S(n-1)$ for $0 \leq i \leq n-1$, we see that

$$\begin{aligned}\operatorname{Ext}^{n-i}(S(i), P(n-1)) &= \operatorname{Ext}^1(\Omega_{n-i-1}S(i), P(n-1)) \\ &= \operatorname{Ext}^1(S(n-1), P(n-1)) \\ &\neq 0\end{aligned}$$

for $0 \leq i \leq n-1$. Thus, on the one hand, we have $\operatorname{Ext}^n(S(0), \Lambda) \neq 0$, this concludes the proof that $S(0)$ has the required properties. On the other hand, we also see that $S = S(0)$ is the only simple module with $\operatorname{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n-1$. This completes the proof of (b).

(c) Assume now in addition that Λ is an artin algebra. As usual, we denote the Auslander-Reiten translation $D\operatorname{Tr}$ by τ. Let M be a non-projective indecomposable module with $\operatorname{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$. The shape of $\Gamma(\Lambda)$ shows that $\Omega M = S^c$ for some simple module S (and we have $c \geq 1$), also it shows that no simple module is projective. Now $\operatorname{Ext}^i(S, \Lambda) = 0$ for $1 \leq i < n$, thus according to (b) we must have $S = S(0)$. It follows that PM has to be a direct sum of copies of $P(n-1)$, say of d copies. Thus a minimal projective presentation of M is of the form

$$P(0)^c \to P(n-1)^d \to M \to 0,$$

and therefore a minimal injective copresentation of τM is of the form

$$0 \to \tau M \to I(0)^c \to I(n-1)^d.$$

In particular, $\operatorname{soc} \tau M = S(0)^c$ and $(\tau M)/\operatorname{soc}$ is a direct sum of copies of $S(n-1)$.

Assume that $\tau M \neq S(0)$, thus it has at least one composition factor of the form $S(n-1)$ and therefore there exists a non-zero map $f : P(n-1) \to \tau M$. Since τM is indecomposable and not injective, any map from an injective module to τM maps into the socle of τM. But the image of f is not contained in the socle of τM, therefore f cannot be factored through an injective module. It follows that

$$\operatorname{Ext}^1(M, P(n-1)) \simeq D\overline{\operatorname{Hom}}(P(n-1), \tau M) \neq 0,$$

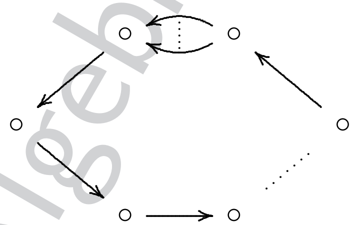
which contradicts the assumption that $\operatorname{Ext}^1(M, \Lambda) = 0$. This shows that $\tau M = S(0)$ and therefore $M = \operatorname{Tr} DS(0)$.

Of course, conversely we see that $M = \operatorname{Tr} DS(0)$ satisfies $\operatorname{Ext}^i(M, P(n-1)) = 0$ for $1 \leq i \leq n$, and $\operatorname{Ext}^{n+1}(M, P(n-1)) \neq 0$. $\square$

Remarks.

(1) The module $M = \operatorname{Tr} DS(0)$ considered in (c) has length $t^2 + t - 1$, thus the number t (and therefore $\Delta(n, t)$) is determined by M.

(2) If Λ is an artin algebra with Ext-quiver $\Delta(n, t)$, the number t has to be the square of an integer, say $t = m^2$. A typical example of such an artin algebra is the path algebra of the following quiver



with altogether $n + m - 1$ arrows, modulo the ideal generated by all paths of length 2. Of course, if Λ is a finite-dimensional k-algebra with radical square zero and Ext-quiver $\Delta(n, m^2)$, and k is an algebraically closed field, then Λ is Morita-equivalent to such an algebra.

Also the following artin algebras with radical square zero and Ext-quiver $\Delta(1, m^2)$ may be of interest: the factor rings of the polynomial ring $\mathbb{Z}[T_1, \ldots, T_{m-1}]$ modulo the square of the ideal generated by some prime number p and the variables $T_1, \ldots, T_{m-1}$.

References

[AR] M. Auslander, I. Reiten: On a generalized version of the Nakayama conjecture. Proc. Amer. Math. Soc. 52 (1975), 69–74.

[ARS] M. Auslander, I. Reiten, S. Smalø: Representation Theory of Artin Algebras. 1995. Cambridge University Press.

[C] X.-W. Chen: Algebras with radical square zero are either self-injective or CM-free. Proc. Amer. Math. Soc. Vol. 140, 93–98.

[JS] D. A. Jorgensen, L. M. Şega: Independence of the total reflexivity conditions for modules. Algebras and Representation Theory 9 (2006), 217–226.

Contact information

C. M. Ringel
Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, P. R. China and King Abdulaziz University, P O Box 80200, Jeddah, Saudi Arabia
E-Mail: ringel@math.uni-bielefeld.de

B.-L. Xiong
Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, P. R. China
E-Mail: xiongbaolin@gmail.com

Received by the editors: 24.05.2012
and in final form 17.01.2013.