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Semigroups with certain finiteness conditions and Chernikov groups*

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Dedicated to the 100th birthday of Sergei Nikolaevich Chernikov

Abstract. The main purpose of this short survey is to show how groups of special structure, which are accepted to be called Chernikov groups, appeared in the considerations of semigroups with certain finiteness conditions. A structure of groups with several such conditions has been described (they turned out to be special types of Chernikov groups). Lastly, a question concerning some special type of Chernikov groups is recalled; this question was raised by the author more than 35 years ago, and it is still open.

1. *Introduction.* One of the main lines in investigations of the present writer was devoted to study of semigroups with finiteness conditions. I recall that, given a class of algebraic systems, by a *finiteness condition* is meant any property which is possessed by all finite systems of this class. In the works of the mentioned line, such conditions were formulated in terms of subsemigroups or in terms of the lattice of subsemigroups of a semigroup. When describing a structure of systems with a non-trivial finiteness condition, one should clarify, so to say, a character and a degree of "deviations" from the property of being a finite system. Applying to the semigroups having been considered, the revealed

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deviations almost always took place in their maximal subgroups. A key general result obtained by the author was a result giving a complete reduction to the group case; it will be formulated below. Thereby a further clearing the structure of semigroups under consideration or exact interrelations between classes of such semigroups required either references to known results about groups or obtaining new group-theoretic results or, if there was success neither in the former nor in the latter, raising open questions concerning groups. All these three situations occurred in reality in a massive of effected investigations. A detailed presentation of the relevant rather rich material is given in Chapter IV of the monographs [1] and [2] (the latter is not simply an English translation of the former but its revised and enlarged version), and the interested reader is referred to this chapter. The main purpose of this paper is to narrate briefly how groups of special structure, which are accepted to be called Chernikov groups (see a definition in Section 3 below), appeared in the considerations mentioned. At the end of the paper, I want to recall a question concerning some special type of Chernikov groups; this question was raised by the author more than 35 years ago, and it is still open.

2. *Reduction Theorem.* It is assumed that the reader is acquainted with more or less standard algebraic notions used in the paper. I give only definitions of several notions which are not well-known. An element of a semigroup is called a *group element* if it is contained in a subgroup of the given semigroup. A semigroup is called an *epigroup* if for any of its elements some power of this element is a group element. A semigroup is called *finitely assembled* if it has finitely many non-group elements and finitely many idempotents. Thereby a semigroup S is finitely assembled if and only if it has finitely many maximal subgroups, say, $G_1, \ldots, G_n$ (as is known, maximal subgroups are mutually disjoint), and the set $S \setminus (G_1 \cup \ldots \cup G_n)$ is finite. If here all groups $G_1, \ldots, G_n$ possess some property θ (in other words, are θ-groups), then we say that S is finitely assembled *from θ-groups.* Any finitely assembled semigroup is obviously an epigroup.

By a *basis* of a semigroup S we mean an irreducible, i.e. minimal, generating set of S.

We need the following requirements which may be satisfied for a semigroup-theoretic property θ:

A) any subsemigroup of a θ-semigroup is a θ-semigroup;
B) any θ-semigroup has no unique infinite basis;
C) any semigroup covered by a finite set of θ-subsemigroups is itself a θ-semigroup.

A general result mentioned in Section 1 is formulated as follows.

Let θ be a finiteness condition satisfying the requirements A)–C) *listed above. An epigroup is a θ-semigroup if and only if it is finitely assembled from θ-groups.*

This statement (let us call it Reduction Theorem) gives, under highly general requirements on a property θ, a description of θ-epigroups effecting a complete reduction to groups. This reduction, in particular, can be taken into account when constructing counter-examples for proving the distinction between some finiteness conditions examined. Namely, if, for some properties θ_1 and θ_2 embraced by Reduction Theorem, there exists a θ_1-semigroup S which is not a θ_2-semigroup, then, according to this theorem, S necessarily contains a subgroup which is a θ_1-group but not a θ_2-group; therefore it is clear that a search of corresponding counter-examples has to be done only for groups.

A general scheme given in Reduction Theorem embraces many concrete finiteness conditions. And for each of them the fulfillment of requirement A) is practically always trivial; requirement B) is often fulfilled even in a stronger variant: any θ-semigroup has no infinite basis at all (and usually it is easy to see); only verification of the fulfillment of requirement C), as a rule, is not straightforward and sometimes turns out to be rather non-trivial.

Note that semigroups with the conditions being studied are in most cases automatically periodic, so, in concrete formulations for such cases, there is no need to mention the requirement for the semigroup under consideration to be an epigroup (since any periodic semigroup is an epigroup). Furthermore, since any subsemigroup of a periodic group is a subgroup, in formulations for these cases we may mention groups with the same condition applying to *subgroups*. A typical example of such a situation is presented by the minimal condition (for subsemigroups and, respectively, for subgroups). In accordance with one of the common terms, we shall call corresponding semigroups and groups *artinian*. Artinian semigroups are obviously periodic, so a concrete version of Reduction Theorem for them looks as follows.

A semigroup is artinian if and only if it is finitely assembled from artinian groups.

3. *On artinian groups.* S. N. Chernikov in his group-theoretic works devoted considerable attention to artinian groups. He revealed a special type of such groups having a very distinct structure; the term "Chernikov group" for such a group has become common for a long time (Chernikov in his papers used the term "extremal group"). A group is called a *Chernikov group* if it is an extension of a direct product of finitely many quasicyclic groups by a finite group. (For the sake of accuracy, it is worth noting that with zero quantity of quasicyclic factors in a Chernikov group we obtain merely a finite group.) Chernikov has found, in particular, that any locally soluble artinian group is necessarily a Chernikov group; the same takes place under a weaker condition, namely the minimal condition for abelian subgroups. The interested reader can find the proofs of the corresponding results in [3], Chapter 4, Section 2, in [4], Section 24, or in [5], Section 3.4. Related results concerning groups with one or another of the minimal conditions were obtained later by some pupils of S. N. Chernikov. The most principal of them is a result by V. P. Shunkov (1970) proving that a locally finite artinian group is a Chernikov group (the same result was obtained also by O. H. Kegel and B. A. F. Wehrfritz in the same year). Shunkov has established a stronger result that a locally finite group with the minimal condition for abelian subgroups is a Chernikov group; it gave the answer to one question posed by Chernikov in 1959.

The last result increased the interest to an old question whether in general artinian groups are exhausted by Chernikov groups; this question was posed by Chernikov as long ago as in 1940. Counter-examples were constructed only at the junction of the 1970s and 1980s by A. Yu. Ol'shanskii owing to the effective technique of geometric methods in combinatorial group theory (see his monograph [6]). Among them infinite groups whose proper subgroups have prime orders (and in the known examples of such groups these orders either distinct or the same). In [1] and [2] we called any such infinite group an Ol'shanskii group. The existence of Ol'shanskii groups allowed to obtain answers to certain other questions which were open for a long time.

4. ***On groups of finite breadth.*** Chernikov groups appeared also when the author of the present paper considered a number of other finiteness conditions which were not considered for groups before. The first of such conditions is finiteness of breadth. The notion of breadth came from lattice theory, and the initial definition is given in lattice-theoretic terms as applied to the lattice of subsemigroups of a semigroup and the lattice of subgroups of a group (as it was already noted, one may read more widely in Chapter IV of [1] or [2]). However there is an equivalent condition that can be formulated in terms of generating sets. Namely, a semigroup [group] is of *breadth* b if and only if for any $b+1$ of its elements at least one belongs to the subsemigroup [subgroup] generated by the others, and b is the least number with this property. The property of being a semigroup of finite breadth is embraced by the scheme given in Reduction Theorem, and it easy to ascertain that semigroups and groups of finite breadth are periodic. Therefore, according to this theorem, *a semigroup is of finite breadth if and only is it is finitely assembled from groups of finite breadth.*

Note that this reduction had been proved by the author separately for the property of having finite breadth and for several other finiteness conditions in the paper “Certain finiteness conditions in the theory of semigroups” (Izv. AN SSSR. Ser. Matem., 1965) published before a general reduction scheme was found, which took place considerably later. Before that paper the author published the paper “Semigroups of finite breadth” (In: “Theory of semigroups and its applications”, Saratov Univ., 1965), where semigroups and groups satisfying this condition and some related conditions were first studied.

What can one say about groups of finite breadth? Some general properties of such groups had been revealed in the paper “Semigroups of finite breadth” mentioned. Among other properties, it was established there that any Chernikov group has finite breadth, and for abelian groups (as well as for groups of a certain wider class) finiteness of breadth is equivalent to the property of being an artinian group. Examples of groups showing difference of these two properties were unknown, which impelled the author to pose two questions about the fulfillment of possible implications connecting these finiteness conditions. These questions were formulated in the paper “Semigroups of finite breadth” and were included in a list of seven open questions about possible implications between several finiteness conditions given in the paper “Certain finiteness conditions in the theory of semigroups” mentioned above. They were included also in the first edition of “Kourovka Notebook (Unsolved problems of group theory)”, 1965; these are questions 1.81 a) and b). Counter-examples giving negative answers to them were constructed in the 1980s by Ol’shanskii’s pupils G. S. Deryabina and S. V. Ivanov. Note that Ol’shanskii groups did not give answers to these questions; indeed, any Ol’shanskii group is artinian and has breadth 2. Soon Ivanov and V. N. Obraztsov (also Ol’shanskii’s pupil) constructed examples of groups giving negative answers to a number of other questions about possible implications between the finiteness conditions (and their combinations) considered by the author; see details and references in Chapter IV of [1] or [2], an indication to the solution of the first two question of the author is also given in [6].

Taking into account known properties of groups of finite breadth and the result by Shunkov mention above, one can prove that *a locally finite group has finite breadth if and only if it is a Chernikov group.* So for locally finite groups finiteness of breadth and the property of being an artinian group are equivalent.

The author have also established that *a locally finite group has at most countable the subgroup lattice if and only if it is a Chernikov group.*

In view of Reduction Theorem we obtain the following consequence from the group-theoretic results given above. *The following conditions for a locally finite semigroup S are equivalent:* (1) *S is artinian,* (2) *S has finite breadth,* (3) *S has at most countable the subsemigroup lattice,* (4) *S is finitely assembled from Chernikov groups.*

5. *Three more lattice finiteness conditions.* The first two of the conditions considered below distinguish special types of Chernikov groups in the class of all groups; the third one distinguishes a special type of such groups in the class of locally finite groups. We start with necessary definitions concerning partially ordered sets. We call a partially ordered set (poset) *narrow* if in it any antichain is finite. If the powers of such a poset P are bounded by a natural number, then P is said to be *of finite width* and the greatest power of antichains is called the *width* of P. The *dimension* of a poset $\langle P, \leqslant \rangle$ is the least cardinal number δ such that the relation $\leqslant$ is the set intersection of δ linear orders on P (or, equivalent, P is embedded in a direct product of δ chains).

When considering semigroups [groups] with just the listed restrictions on the subsemigroup [subgroup] lattice, let us arrange to transfer the predicate in the corresponding term from the lattice mentioned to a semigroup [group] itself. So, we shall speak about narrow semigroups and groups as well as about semigroups and groups of finite width. Note that the term "narrow" as applied to lattices, semigroups and groups was introduced in [2], while in the monograph [1] and two preceding works, where narrow semigroups and groups were considered (a supplement to the paper "Semigroups of finite breadth", 1971, and the survey "Semigroups and their subsemigroup lattices" written jointly with A. J. Ovsyannikov, 1983), instead of the adjective "narrow" the prefix NF was used (which was definitely less successful), i.e. the terms "NF-lattice", "NF-semigroup", "NF-group" appeared there.

As to the property of having finite dimension, it is advisable to speak not "semigroups of finite dimension" but "semigroups of finite lattice dimension" (and the same for groups); the reason is that the first term for groups and semigroups was used in literature in another sense, see comments in Chapter IV of [1] or [2].

5.1. *Narrow semigroups and groups.* The property of being a narrow semigroup is embraced by the general reduction scheme described in

Section 2, and it is easy to ascertain that narrow semigroups are periodic and artinian. An exhaustive description of infinite narrow groups is given by the following theorem.

An infinite group is a narrow group if and only if it is the direct product of a finite group and finitely many quasicyclic groups for distinct primes not dividing the order of the finite factor.

Note that in the proof of this theorem some results of Chernikov's paper "Infinite groups with finite layers" (Matem. Sb., 1948) are essentially used.

As it is seen, narrow groups represent a special type of Chernikov groups and, in particular, have finite breadth. In view of Reduction Theorem we obtain that any narrow semigroup is a semigroup of finite breadth. It is worth noting that this implication cannot be deduced only from lattice-theoretic considerations; indeed, there exist examples of narrow lattices which are not of finite breadth.

5.2. ***Semigroups and groups of finite width.*** From definitions it directly follows that any semigroup of finite width is a narrow semigroup. But the property of being a semigroup of finite width is not embraced by our general reduction scheme; namely, requirement C) is not fulfilled for this property (there exist counter-examples). However requirements A) and B) are obviously fulfilled. Then, in view of the necessity part of Reduction Theorem (I did not distinguish it separately above), any semigroup of finite width is finitely assembled from groups of finite width. This necessary condition is not sufficient for the given property (counter-examples exist), so the search of a necessary and sufficient condition is retained as an unsolved problem.

An exhaustive description of infinite groups of finite width is given by the following theorem based naturally on the theorem from Section 5.1.

An infinite group is a group of finite width if and only if it is the direct product of a finite group and a quasicyclic group for some prime not dividing the order of the finite factor.

5.3. ***Semigroups and groups of finite lattice dimension.*** It can be proved that any lattice of finite dimension has finite breadth (which is not greater than dimension), therefore any semigroup [group] of finite lattice dimension is a semigroup [group] of finite breadth and, in particular, periodic. The property of being a semigroup of finite lattice dimension is embraced by our general reduction scheme, so, in view of Reduction Theorem, *a semigroup is of finite lattice dimension if and only if it is finitely assembled from groups of finite lattice dimension.*

The problem of describing arbitrary groups of finite lattice dimension seems unrealistic; I may note in this connection that any Ol'shanskii group is of lattice dimension 2. However for locally finite groups, the problem indicated, in all likelihood, can be solved in an exhaustive way. This claim is explained by the following theorem together with a hypothesis attached to it.

Any locally finite group of finite lattice dimension is an extension of a direct product of finitely many quasicyclic groups for distinct primes by a finite group.

Hypothesis: the converse statement is valid as well, i.e. *any group of the indicated structure is of finite lattice dimension.*

The formulated result was obtained by the author in the mid 1970s and reported (together with the hypothesis) in a special course "Periodic semigroups" that was being delivered at Ural State University those years. It was announced (again with the hypothesis) in "Abstracts of the XI All-Union Symposium on Group Theory", 1989, and its proof was first published in [1]. A question about the validity of this hypothesis was included in the 11th edition of "Kourovka Notebook", 1990 (question 11.116) as well as in the monographs [1] and [2] (question 13.10.4). It is still open.

References

[1] Shevrin L. N., Ovsyannikov A. J. *Semigroups and Their Subsemigroup Lattices*. Sverdlovsk, Ural University Press, Part 1, 1990, Part 2, 1991 (in Russian).

[2] Shevrin L. N., Ovsyannikov A. J. *Semigroups and Their Subsemigroup Lattices*. Dordrecht, Kluwer Academic Publishers, 1996.

[3] Chernikov S. N. *Groups with Given Properties of System of Subgroups*, Moscow, Nauka, 1980 (in Russian).

[4] Kargapolov M. I., Merzlyakov Yu. I. *Fundamentals of the Theory of Groups*. 3rd edition, Moscow, Nauka, 1982 (in Russian). English transl. of the 2nd edition: New-York, Springer-Verlag, 1979.

[5] Robinson D. J. S. *Finiteness Conditions and Generalized Soluble Groups*. Parts 1, 2. Berlin, Springer-Verlag, 1972.

[6] Ol'shanskii A. Yu. *Geometry of Defining Relations in Groups*. Moscow, Nauka, 1989 (in Russian). English transl.: Dordrecht, Kluwer Academic Publishers, 1991.

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