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RESEARCH ARTICLE

Partitions of groups into sparse subsets

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Abstract. A subset A of a group G is called sparse if, for every infinite subset X of G, there exists a finite subset $F \subset X$, such that $\bigcap_{x\in F} xA$ is finite. We denote by $\eta(G)$ the minimal cardinal such that G can be partitioned in $\eta(G)$ sparse subsets. If $|G| > (\kappa^+)^{\aleph_0}$ then $\eta(G) > \kappa$, if $|G| \leqslant \kappa^+$ then $\eta(G) \leqslant \kappa$. We show also that $cov(A) \geqslant cf|G|$ for each sparse subset A of an infinite group G, where $cov(A) = \min\{|X| : G = XA\}$.

A subset A of a group G with the identity e is called

- *large* if there exists a finite subset F such that $G = FA$;
- *small* if $L \setminus A$ is large for each large subset L of G;
- *thin* if $gA \cap A$ is finite for every $g \in G \setminus \{e\}$;
- *sparse* if, for every infinite subset X of G, there exists a finite subset $F \subset X$ such that $\bigcap_{x\in F} xA$ is finite.

We note that large, small, and thin subsets can be considered as asymptotic counterparts of dense, nowhere dense and discrete subsets of a topological space [7, Chapter 9]. The sparse subsets were introduced in [2] to characterize the strongly prime ultrafilters in the Stone-Čech compactification βG of G. If G is infinite then each thin subset is sparse, and each sparse subset is small [3].

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By [4], every infinite group G can be partitioned in $\aleph_0$ large subsets, and if G is amenable then G can not be partitioned in $> \aleph_0$ large subsets. By [5], every infinite group G can be partitioned in $\aleph_0$ small subsets.

For a group G, we denote by $\mu(G)$ and $\eta(G)$ respectively the minimal cardinals such that G can be partitioned in $\mu(G)$ thin subsets and in $\eta(G)$ sparse subsets. By [6], $\mu(G) = |G|$ if $|G|$ is a limit cardinal, and $\mu(G) = \kappa$ if G is infinite and $|G| = \kappa^+$, where κ^+ is the cardinal-successor of κ. In Theorem 1, we evaluate the cardinal $\eta(G)$.

A covering number of a subset A of G is the cardinal $cov(A) = \min\{|X| : G = XA\}$. In Theorem 2, we show that $cov(A) \geqslant cf|G|$ for every sparse subset A of an infinite group G, where $cf|G|$ is the cofinality of $|G|$.

Lemma 1. *A subset S of a group G is not sparse if and only if there exists an infinite subset X of G such that, for each infinite subset F of X, the set $\{x \in G : F^{-1}x \subseteq A\}$ is infinite.*

Proof. It suffices to note that $x \in \bigcap_{g\in F} gS$ if and only if $F^{-1}x \subseteq A$. □

We say that a subset S of a group G is *rectangle free* if $XY \nsubseteq S$ for any infinite subsets X, Y of G.

Lemma 2. *Every sparse subset S of a group G is rectangle free.*

Proof. Apply Lemma 1. □

Lemma 3. *Let X and Y be infinite sets of cardinality $|X| = \kappa^+$ and $|Y| > (\kappa^+)^\lambda$ for some non-zero cardinal $\lambda \leqslant \kappa^+$. For any κ-coloring $\chi : X \times Y \to \kappa$, there are subsets $A \subseteq X$ and $Z \subseteq Y$ such that $|A| = \lambda$, $|Z| > (\kappa^+)^\lambda$ and the set $A \times Z$ is monochrome.*

Proof. [1, Lemma 1]. □

Theorem 1. *Let G be an infinite group, κ be an infinite cardinal. If $|G| > (\kappa^+)^{\aleph_0}$ then $\eta(G) > \kappa$. If $|G| \leqslant \kappa^+$ then $\eta(G) \leqslant \kappa$.*

Proof. Suppose that $|G| > (\kappa^+)^{\aleph_0}$, take an arbitrary partition of G into κ subsets and denote by χ' corresponding κ-coloring. Then we choose a subset X of G with $|X| = \kappa^+$, put $Y = G$ and define a coloring $\chi : X \times Y \to \kappa$ by the rule $\chi((x, y)) = \chi'((x, y))$. Applying Lemma 3 with $\lambda = \aleph_0$, we get $A \subseteq X$ and $Z \subseteq Y$ such that $|A| = \aleph_0$, $|Z| > (\kappa^+)^{\aleph_0}$ and $A \times Z$ is monochrome. By Lemma 2, $A \times Z$ is not sparse, so at least one cell of the partition is not sparse and $\eta(G) > \kappa$.

If $|G| \leqslant \kappa^+$, by [6, Lemma 2], G can be partitioned in $\leqslant \kappa$ thin subsets. Since every thin subset is sparse, $\eta(G) \leqslant \kappa$. □

Corollary 1. *If $|G| > 2^\kappa$ then $\eta(G) > \kappa$.*

Proof. It suffices to note that, for any infinite cardinal κ, $(\kappa^+)^{\aleph_0} \leqslant (2^\kappa)^{\aleph_0} = 2^\kappa$. □

Corollary 2. *If $\eta(G) = \aleph_0$ then $\aleph_0 \leqslant |G| \leqslant 2^{\aleph_0}$.*

Question 1. *Does $|G| = 2^{\aleph_0}$ imply $\eta(G) = \aleph_0$?*

Under CH, Theorem 1 gives an affirmative answer to this question. To answer this question negatively under $\neg$CH, it suffices to show that, for any $\aleph_0$-coloring of $\aleph_2 \times \aleph_2$, there is a monochrome subset $A \times B$, $A \subset \aleph_2, B \subset \aleph_2$, $|A| = |B| = \aleph_0$.

Theorem 2. *For every sparse subset A of an infinite group G, $cov(A) \geqslant cf|G|$.*

Proof. We suppose the contrary and choose $X \subset G$ such that $G = XA$ and $|X| < cf|G|$. Clearly, $|A| = |G|$. Since $|X| < cf|A|$ and $A = \bigcup_{x\in X}(A \cap xA)$, there is $x_0 \in X$ such that $|A \cap x_0A| = |G|$. We put $A_0 = A \cap x_0A$ so $x_0A_0 \subseteq A$. Suppose that we have chosen distinct elements $x_0, x_1, \dots, x_n$ of X and the subsets $A_0 \supseteq A_1 \supseteq \dots \supseteq A_n$ of A such that $|A_0| = |A_1| = \dots = |A_n| = |G|$ and $x_0A_0 \subseteq A_1, x_1A_1 \subseteq A_2, \dots, x_nA_n \subseteq A_{n+1}$. We take an arbitrary element $g \in G$ such that $g^{-1}X \cap \{x_0, \dots, x_n\} = \varnothing$. Since $|gA_n| = |G|$, $gA_n \subseteq \bigcup_{x\in X} xA$ and $|X| < cf|G|$, there is $x \in X$ such that $|gA_n \cap xA| = |G|$. We put $x_{n+1} = g^{-1}X$, $A_{n+1} = A_n \cap g^{-1}XA$. Then $x_{n+1} \notin \{x_0, x_1, \dots, x_n\}$, $x_{n+1}A_{n+1} \subseteq A_n$.

After ω steps we get a countable set $X' = \{x_n : n \in \omega\}$ and an increasing chain $\{A_n : n \in \omega\}$ of subsets of cardinality $|G|$ such that $A_{n+1} \subseteq \{g \in G : \{x_0, x_1, \dots, x_n\}g \subseteq A\}$. By Lemma 1, A is not sparse. □

Question 2. *Is $cov(A) = |G|$ for every sparse subset A of an arbitrary infinite group G? By Theorem 2, this is so if $|G|$ is regular.*

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