

Automorphism groups of tetravalent Cayley graphs on minimal non-abelian groups

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Communicated by D. Simson

ABSTRACT. A Cayley graph $X = \text{Cay}(G, S)$ is called *normal* for G if the right regular representation $R(G)$ of G is normal in the full automorphism group $\text{Aut}(X)$ of X . In the present paper it is proved that all connected tetravalent Cayley graphs on a minimal non-abelian group G are normal when $(|G|, 2) = (|G|, 3) = 1$, and X is not isomorphic to either $\text{Cay}(G, S)$, where $|G| = 5^n$, and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or $\text{Cay}(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5 \cdot q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.

1. Introduction

Throughout this paper, graphs are finite and simple. For a graph X , we denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ the vertex set, the edge set and the automorphism group of X , respectively, and X is said to be *vertex-transitive* if $\text{Aut}(X)$ is transitive on vertices.

A finite group G is called *minimal non-abelian* if G is not abelian, but every proper subgroup of G is abelian. For a finite group G and a subset S of G such that $1_G \notin S$, and S is symmetric, that is $S^{-1} = \{s^{-1} | s \in S\}$ is equal to S . The *Cayley graph* $X = \text{Cay}(G, S)$ on G with respect to S is defined as the graph with vertex set G and edge set $\{[g, sg] | g \in G, s \in S\}$. Clearly the right regular representation $R(G)$ of G acts regularly on

2000 Mathematics Subject Classification: 05C25, 20B25.

Key words and phrases: Cayley graph, normal Cayley graph, minimal non-abelian group.

vertices, that is, $R(G)$ is transitive on vertices and only the identity element of $R(G)$ fixes a vertex. Furthermore, the group $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\text{Aut}(X)$. Actually, $\text{Aut}(G, S)$ is a subgroup of $\text{Aut}(X)_1$, the stabilizer of the vertex 1 in $\text{Aut}(X)$. For two groups M and N , $N \rtimes M$ denotes a semidirect product of N by M .

A Cayley graph $\text{Cay}(G, S)$ is said to be *normal* on G if the right regular representation $R(G)$ of G is normal in $\text{Aut}(X)$. Let $N_{\text{Aut}(X)}(R(G))$ be the normalizer of $R(G)$ in $\text{Aut}(X)$. By Godsil [8], or Xu [14], $N_{\text{Aut}(X)}(R(G)) = R(G) \rtimes \text{Aut}(G, S)$. Thus, $\text{Cay}(G, S)$ is normal on G if and only if $\text{Aut}(X) = R(G) \rtimes \text{Aut}(G, S)$. If $N_{\text{Aut}(X)}(R(G))$ is transitive on edges then $\text{Cay}(G, S)$ is called a *normal edge-transitive* Cayley graph by Praeger [13]. Clearly, if $\text{Cay}(G, S)$ is normal on G and edge-transitive then it is normal edge-transitive. Note that being a normal Cayley graph is not invariant under graph isomorphism, and so strictly depends upon which group the graph is a Cayley graph on. For example, the three-dimensional hypercube Q_3 is a Cayley graph on either the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or the group $\mathbb{Z}_4 \times \mathbb{Z}_2$. The Cayley graph on the first group is normal, but the Cayley graph on the second group is not.

A Cayley graph X of a group G is called a *graphical regular representation* if $\text{Aut}(X) = G$. It implies that $\text{Aut}(X)_1 = 1$ and so $\text{Aut}(G, S) = 1$. Therefore, a necessary condition for X to be a graphical regular representation of G is $\text{Aut}(G, S) = 1$. In some circumstance, this necessary condition is also sufficient for X to be a graphical regular representation. However, it is of course not the case in general, that is, there exist groups G and Cayley graphs $\text{Cay}(G, S)$ such that $\text{Aut}(G, S) = 1$ but $\text{Cay}(G, S)$ is not a graphical regular representation of G . Hence the following natural problem was proposed by Godsil [9].

Problem A. Determine classes of groups G and Cayley graphs $\text{Cay}(G, S)$ for which $\text{Cay}(G, S)$ is a graphical regular representation of G if and only if $\text{Aut}(G, S) = 1$.

The solution of Problem A is known only for some classes of groups. Godsil solved the problem in [8, Theorem 3.8] for p -groups with p prime which have no homomorphism onto $\mathbb{Z}_p \times \mathbb{Z}_p$, and Li [10], solved the problem A for arbitrary cubic Cayley graphs of 2-power order, also Li and Sim solved the problem A for metacyclic p -groups with prime p (see [11]). In the present paper, it is proved that for most finite minimal non-abelian groups G , a tetravalent Cayley graph $\text{Cay}(G, S)$ is a graphical regular representation if and only if $\text{Aut}(G, S) = 1$.

Fang et al. [5] proved that the vast majority of connected cubic

Cayley graphs on non-abelian simple groups are normal. It was shown in [1, 2, 6], that most connected Cayley graphs of small valency on abelian groups are normal, and that all connected tetravalent Cayley graphs on p -groups of nilpotency class 2 with p an odd prime are normal. Also the normality of tetravalent Cayley graphs on regular p -groups was done by Feng and Xu [7]. In the present paper, we shall prove that all connected tetravalent Cayley graphs on a minimal non-abelian group are normal when $(|G|, 2) = (|G|, 3) = 1$, and X is not isomorphic to either $\text{Cay}(G, S)$, where $|G| = 5^n$, and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5^n$ where $m \in \{2, 3\}$ and $n \geq 3$, or $\text{Cay}(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5 \cdot q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.

2. Preliminaries

In this section we give some results which will be used later. The following fact is basic for Cayley graphs.

Proposition 2.1. [3, Lemma 16.3] *A graph X is a Cayley graph of a group G if and only if $\text{Aut}(X)$ contains a regular subgroup isomorphic to G .*

We have the following result for minimal non-abelian p -groups.

Proposition 2.2. [12] *Let G be a finite minimal non-abelian group. Then one of the following holds:*

- (1) G is a minimal non-abelian p -group;
- (2) G is a semi-direct product of an elementary abelian q -group Q of order q^α by a cyclic group $P = \langle b \rangle$ of order p^β , where p, q are distinct primes and the action of b on Q is an automorphism of Q of order p .

Let $X = \text{Cay}(G, S)$ be a Cayley graph on a finite group G and H a normal subgroup of G . Let X_H be the quotient graph corresponding to the cosets of H in G , with two cosets adjacent in X_H , whenever there is an edge between those cosets in X .

Proposition 2.3 ([13]). *The quotient graph X_H is a Cayley graph and*

$$X_H = \text{Cay}(G/H, SH/H),$$

where $SH/H = \{sH | s \in S\}$.

The *socle* of a group G is the subgroup generated by the set of all minimal normal subgroups of G , and it is denoted by $\text{soc}(G)$. By [4, Theorem 4.3], we have the following proposition:

Proposition 2.4. *Let G be a nontrivial finite group. Then $\text{soc}(G) = T_1 \times \dots \times T_k$, where T_i ($1 \leq i \leq k$) are simple normal subgroups and $T_i \cong T_j$ ($1 \leq i, j \leq k$).*

3. Main result

The main purpose of this paper is to prove the following theorem.

Theorem 3.1. *Let p be a prime and G a minimal non-abelian group which $(|G|, 2) = (|G|, 3) = 1$. Let $X = \text{Cay}(G, S)$ be a connected tetravalent Cayley graph on G . Then we have $\text{Aut}(\text{Cay}(G, S)) = R(G) \rtimes \text{Aut}(G, S)$, except when X is not isomorphic to either $\text{Cay}(G, S)$, where $|G| = 5^n$, and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or $\text{Cay}(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5 \cdot q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.*

Let $Q_8 = \langle a, b \mid a^4 = 1, bab^{-1} = a^{-1}, a^2 = b^2 \rangle$, be a quaternion group, and $S = \{a, a^3, b, b^3\}$ a subset of Q_3 . Now $\sigma = (ab, a^3b) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and hence $\text{Cay}(Q_8, \{a, a^3, b, b^3\})$ is not normal. Also if $D_6 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, and $S = \{a, a^2, b, ab\}$, then $\delta = (a, b)(a^2, ab) \in A_1$, but $\delta \notin \text{Aut}(G, S)$ and hence $\text{Cay}(D_6, \{a, a^2, b, ab\})$ is not normal. It follows that Theorem 3.1 is not true generally when $(|G|, 2) \neq 1$, and $(|G|, 3) \neq 1$. As a consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *Let G be a minimal non-abelian group, and $(|G|, 2) = (|G|, 3) = 1$. Also let X is not isomorphic to either $\text{Cay}(G, S)$, where $|G| = 5^n$, and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or $\text{Cay}(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5 \cdot q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$. Then a connected Cayley graph $X = \text{Cay}(G, S)$ of valency 4 is a graphical regular representation of G if and only if $\text{Aut}(G, S) = 1$.*

Proof of Theorem 3.1. Recall that $\text{Aut}(\text{Cay}(G, S)) = R(G) \rtimes \text{Aut}(G, S)$ if and only if $\text{Cay}(G, S)$ is normal on G . Let G be a counterexample of least order, that is, G has the smallest order with the following properties: G is a minimal non-abelian group and there exists a subset $S = \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$ of G such that $X = \text{Cay}(G, S)$ is not a normal 4-valent connected Cayley graph. Let $A = \text{Aut}(X)$ and let A_1 be the stabilizer of 1 in A . Since X has valency 4, the stabilizer A_1 of the vertex 1 in A is a $\{2, 3\}$ -group. By Proposition 2.2, first assume that G is a p -group. By our assumption G is not 2- or 3-group. By Proposition 2.4, $\text{soc}(A) = T_1 \times \dots \times T_k$, where

$k \geq 1$ and $T_i \cong T_j$ ($1 \leq i, j \leq k$). T is a $\{2, 3, p\}$ -group because A is $\{2, 3, p\}$ -group.

First let T be an abelian group. Then $\text{soc}(A)$ is an elementary abelian q -group, where $q = 2, 3$, or p . If $H = \text{soc}(A)$ is transitive on $V(X)$, then $H = \text{soc}(A)$ acts regularly on $V(X)$. Thus $|\text{soc}(A)| = |G|$, and hence $\text{soc}(A) = G$, a contradiction. So $H = \text{soc}(A)$ is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, \dots, B_t\}$ is the set of orbits of H on $V(X)$. By the normality of H in A we have that t and $|B_i|$ ($1 \leq i \leq t$) are powers of p . It follows that H is a p -group and $H \leq G$. Consider the quotient graph X_H defined by $V(X_H) = \Sigma$, and $(B_i, B_j) \in E(X_H)$ if and only if there exists $v_i \in B_i, v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. Since the quotient group G/H acts regularly on Σ , therefore by Proposition 2.1 X_H must be connected Cayley graph of G/H . So by Proposition 2.3 $X_H \cong \text{Cay}(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Since $|V(X_H)|$ is odd, the valency $v(X_H)$ of X_H could not be 3 and hence $v(X_H) = 2$, or $v(X_H) = 4$. If $v(X_H) = 4$ and G/H be a non-abelian, then by the minimality of $|G|$, X_H is normal as a Cayley graph of G/H . Thus $G/H \trianglelefteq \text{Aut}(X_H)$. Let K be the kernel of A on Σ . Also let K_1 be the stabilizer of the vertex 1 in K . Then K_1 fixes the neighborhood $N(1)$ of 1 in X pointwise. Since X is connected, K_1 fixes each vertex in X . Thus $K_1 = 1$, and hence $K = H$. Therefore $A/H \leq \text{Aut}(X_H)$, and G/H is normal in A/H , which implies that $G \trianglelefteq \text{Aut}(X)$, a contradiction. Now if $v(X_H) = 4$ and G/H is an abelian, then by [2], X_H is normal except when $X_H = \text{Cay}(G/H, SH/H)$, where $G/H \cong \mathbb{Z}_5$. If X_H is normal, then $G/H \trianglelefteq \text{Aut}(X_H)$ and so $G \trianglelefteq \text{Aut}(X)$, a contradiction. Now suppose that $X_H = K_5$, and $\text{Aut}(X_H) = S_5$. Therefore $|A/H| \mid 2^3 \cdot 3 \cdot 5$ and hence $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5^n$, where $m \in \{2, 3\}$ and $n \geq 3$. If $3 \nmid |A|$, then clearly $R(G) \trianglelefteq A$, a contradiction. Thus $3 \mid |A|$. Now $X = \text{Cay}(G, S)$ is not normal. Suppose to the contrary that X is normal. Hence $A_1 = \text{Aut}(G, S)$, and so $3 \mid |\text{Aut}(G, S)|$. Therefore there exists an element α of order 3 in $\text{Aut}(G, S)$. Then $|S| = 4$ implies that α fixes an element in S , say s . Consequently, α fixes s^{-1} . Since $s, s^{-1} \in S$ and $s \neq s^{-1}$, we have that α fixes S pointwise and hence $\langle S \rangle = G$ implies $\alpha = 1$, a contradiction. Now assume that $v(X_H) = 2$. Then X_H is a cycle of length $p^m = |G/H|$, for some positive integer m . Also $\text{Aut}(X_H) \cong D_{2p^m}$. We have $SH/H = \{Hs_1, Hs_2\}$. Clearly $Hs_1 \neq Hs_1^{-1}$ and $Hs_2 \neq Hs_2^{-1}$. Therefore $Hs_1 = Hs_2^{-1}$, and $s_1s_2 \in H$. If $H \cap S \neq \emptyset$, then $G = H$, a contradiction. So $H \cap S = \emptyset$, and hence $G = \langle H, s_1 \rangle$. Now since G/H is a cyclic group and $\{Hs_1\}$ is a minimal generating set, by [14, Proposition 3.1], we have $\text{Aut}(X) = G/H$, a contradiction.

Now let T is not an abelian group. Assume that $H = \text{soc}(A) \cap G$. By

considering the order of A , one has $H \neq 1$. Clearly H is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, \dots, B_s\}$ is the set of orbits of H on $V(X)$. By the normality of H in G we have that s and $|B_i|$ ($1 \leq i \leq s$) are powers of p . Considering the quotient graph X_H defined by $V(X_H) = \Sigma$, and $(B_i, B_j) \in E(X_H)$ if and only if there exists $v_i \in B_i$, $v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. Since the quotient group G/H acts regularly on Σ , therefore X_H must be connected Cayley graph of G/H . So $X_H \cong \text{Cay}(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Now with the same arguments as before we get same result.

Finally assume that $G = Q \rtimes P$, where P and Q are Sylow p -subgroup and Sylow q -subgroup of G , respectively such that P is a cyclic group and $Q \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \dots \times \mathbb{Z}_q$. Now considering $\text{soc}(A) = T_1 \times \dots \times T_k$, where $k \geq 1$ and $T_i \cong T_j$ ($1 \leq i, j \leq k$). If $\text{soc}(A)$ is $\{2, 3\}$ -group, Then $\text{Soc}(A)$ is abelian 2-group or abelian 3-group. Clearly $\text{Soc}(A)$ is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, \dots, B_s\}$ is the set of orbits of $\text{soc}(A)$ on $V(X)$. By the normality of $\text{soc}(A)$ in A , p , q or pq divides s and $|B_i|$ ($1 \leq i \leq s$). This is a contradiction. Thus $H = \text{soc}(A) \cap G \neq 1$ and H is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, \dots, B_s\}$ is the set of orbits of H on $V(X)$. By the normality of H in G , p , q , or pq divides s and $|B_i|$ ($1 \leq i \leq s$). Since the quotient group G/H acts regularly on Σ , therefore X_H must be connected Cayley graph of G/H . So $X_H \cong \text{Cay}(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Since $|V(X_H)|$ is odd, the valency $v(X_H)$ of X_H could not be 3 and hence $v(X_H) = 2$, or $v(X_H) = 4$. If $v(X_H) = 4$ and G/H be a non-abelian, then by the minimality of $|G|$, X_H is normal as a Cayley graph of G/H . Thus $G/H \trianglelefteq \text{Aut}(X_H)$. Let K be the kernel of A on Σ . Also let K_1 be the stabilizer of the vertex 1 in K . Then K_1 fixes the neighborhood $N(1)$ of 1 in X pointwise. Since X is connected, K_1 fixes each vertex in X . Thus $K_1 = 1$, and hence $K = H$. Therefore $A/H \leq \text{Aut}(X_H)$, and G/H is normal in A/H , which implies that $G \trianglelefteq \text{Aut}(X)$, a contradiction. Now if $v(X_H) = 4$ and G/H is an abelian, then by [2], X_H is normal except when $X_H = \text{Cay}(G/H, SH/H)$, where $G/H \cong \mathbb{Z}_5$. If X_H is normal, then $G/H \trianglelefteq \text{Aut}(X_H)$ and so $G \trianglelefteq \text{Aut}(X)$, a contradiction. Now suppose that $X_H = K_5$, and $\text{Aut}(X_H) = S_5$. Therefore $|\text{Aut}(X)| = 2^m \cdot 3 \cdot 5 \cdot q^n$, where $m \in \{2, 3\}$ and $n \geq 1$. If $3 \nmid |A|$, then $|A| = 2^m \cdot 5 \cdot q^n$. Clearly $PQ/Q \trianglelefteq A/Q$ and so $PQ \trianglelefteq A$. Thus $R(G) \trianglelefteq A$, a contradiction. Thus $3 \mid |A|$. Now $X = \text{Cay}(G, S)$ is not normal. Suppose to the contrary that X is normal. Hence $A_1 = \text{Aut}(G, S)$, and so $3 \mid |\text{Aut}(G, S)|$. Therefore there exists an element α of order 3 in $\text{Aut}(G, S)$. Then $|S| = 4$ implies that α fixes an element in S , say s . Consequently, α fixes s^{-1} . Since $s, s^{-1} \in S$ and

$s \neq s^{-1}$, we have that α fixes S pointwise and hence $\langle S \rangle = G$ implies $\alpha = 1$, a contradiction. Now assume that $v(X_H) = 2$. Then X_H is a cycle of length $p^m q^n = |G/H|$, for some positive integer m, n . Also $\text{Aut}(X_H) \cong D_{2p^m q^n}$. We have $SH/H = \{Hs_1, Hs_2\}$. Clearly $Hs_1 \neq Hs_1^{-1}$ and $Hs_2 \neq Hs_2^{-1}$. Therefore $Hs_1 = Hs_2^{-1}$, and $s_1 s_2 \in H$. If $H \cap S \neq \emptyset$, then $G = H$, a contradiction. So $H \cap S = \emptyset$, and hence $G = \langle H, s_1 \rangle$. Now since G/H is a cyclic group and $\{Hs_1\}$ is a minimal generating set, by [14, Proposition 3.1], we have $\text{Aut}(X) = G/H$, a contradiction.

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Received by the editors: 13.10.2011
and in final form 27.11.2011.