

Fully invariant subgroups of an infinitely iterated wreath product

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ABSTRACT. The article deals with the infinitely iterated wreath product of cyclic groups C_p of prime order p . We consider a generalized infinite wreath product as a direct limit of a sequence of finite n th wreath powers of C_p with certain embeddings and use its tableau representation. The main result are the statements that this group doesn't contain a nontrivial proper fully invariant subgroups and doesn't satisfy the normalizer condition.

Introduction

A *wreath product* of permutation groups is a group-theoretical construction, which is widely used for building groups with certain special properties. Given two permutation groups (G_1, X_1) and (G_2, X_2) , where G_1 acts on X_1 and G_2 acts on X_2 , we denote their wreath product to be the permutation group $G_1 \wr G_2 = \{[g_1(x), g_2] \mid g_2 \in G_2, g_1 : X_2 \rightarrow G_1\}$, which acts on the direct product $X_1 \times X_2$ (imprimitive action). The notion of the wreath product of two groups can be easily generalized to an arbitrary finite number of factors. If all factors are isomorphic to G then the corresponding wreath product is often called the *wreath power* of G (or the n -iterated wreath product of G).

Given a residue field \mathbb{Z}_p (p is prime) we consider its additive group C_p (without loss of generality we can assume that C_p acts on itself by the right translations). The finite n th wreath power of C_p (which is isomorphic to the Sylow p -subgroup of the finite symmetric group S_{p^n}) was studied

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by L. A. Kaloujnine [7]. In particular, in [7] the author investigated the structure of finite wreath powers of cyclic permutation groups (characteristic subgroups, upper and lower central series and derived series were described). Later, similar results for wreath powers of elementary abelian groups were obtained by V. I. Sushchansky in [15].

The notion of iterated wreath product admits various generalizations in the case of an infinite number of factors. In [4] P. Hall introduced a general construction

$$W = \text{wr}_{\lambda \in \Lambda} G_\lambda$$

of the "restricted" (in the sense of an action as a permutation group) wreath product of permutation groups indexed by a totally ordered set. In the same article the author used this wreath product construction to obtain the examples of characteristically simple groups.

Similar approaches can be found in papers of I. D. Ivanuta [6], W. C. Holland [5] (unrestricted wreath product of permutation groups indexed by a partially ordered set), M. Dixon and T. A. Fournelle ([1] and [2]).

For example, in [6] the author adapted the Hall's general construction with factors indexed by a totally ordered set to describe the main (transitive) infinite Sylow p -subgroup of the finitary symmetric group. To operate with considered wreath product I. D. Ivanuta also used a tableau representation of its elements.

The normalizer condition for the direct limits of finite standard wreath products (so called Kargapolov groups) was studied in [13] by Yu. I. Merzlyakov. Also in [14] a criterion of self normalizability for some classes of subgroups of the finitary unitriangular group was established.

In this article we consider the infinite wreath product construction (denoted by U_p^ω) as a direct limit of a sequence of finite n th wreath powers of C_p with certain embeddings. We also use the so called tableau representation of U_p^ω for the study of its properties.

In the first section the generalized infinite wreath product U_p^ω is defined. Then, in section 2 we present a review of known results on the characteristic structure of U_p^ω . The main results of the article are given in the last section:

- 1) if $p \neq 2$ and R is a fully invariant subgroup of U_p^ω then either $R = E$ (the identity subgroup), or $R = U_p^\omega$; in other words, U_p^ω is fully invariantly simple (theorem 2);
- 2) U_p^ω doesn't satisfy the normalizer condition (theorem 3).

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1. Generalized wreath product

In this section we consider a group of infinite tableaux of reduced polynomials (an approach similar to what was proposed by L.A. Kaloujnine in [7]) and then define it as a direct limit of finitely iterated wreath products (generalized wreath product).

1.1. The tableau representation

Let p be a prime ($p \neq 2$) and C_p be the additive group of the residue field \mathbb{Z}_p . In other words, C_p is the cyclic additive group of order p , which acts on itself by the right translations. Define U_p^ω as a group of infinite almost zero tableaux

$$[a_1(x_2, \dots, x_k), \dots, a_n(x_{n+1}, \dots, x_k), 0, \dots], \quad k, n \in \mathbb{N}, \quad k > n, \quad (1)$$

where $a_i(x_{i+1}, \dots, x_k)$ is a polynomial over \mathbb{Z}_p reduced (degree of each variable $\leq p-1$) modulo the ideal

$$\langle x_{i+1}^p - x_{i+1}, x_{i+2}^p - x_{i+2}, \dots, x_k^p - x_k \rangle.$$

The group U_p^ω acts on the direct product

$$X = \prod_{i=1}^{\infty} \mathbb{Z}_p = \{(t_1, \dots, t_m, 0, \dots) \mid t_i \in \mathbb{Z}_p, m \in \mathbb{N}\} \quad (2)$$

(X is the set of all almost zero sequences over \mathbb{Z}_p). If $u \in U_p^\omega$ and $t = (t_i)_{i=1}^{\infty} \in X$ then

$$t^u = (t_1 + a_1(t_2, \dots, t_k), \dots, t_n + a_n(t_{n+1}, \dots, t_k), t_{k+1}, t_{k+2}, \dots). \quad (3)$$

For simplicity, we introduce some auxiliary notation. Let

$$a_i(\bar{x}_{i+1,k}) = a_i(x_{i+1}, \dots, x_k)$$

and $[u]_i = a_i(\bar{x}_{i+1,k})$ – the i th coordinate of the tableau u . Also, let $[a_i(\bar{x}_{i+1,k})]_{i=1}^{\infty}$ be a short notation of the tableau (1) and $f(\bar{x}^u)$ denote the reduced polynomial, which is equivalent to the polynomial

$$f(\dots, x_j + a_j(x_{j+1}, \dots, x_k), \dots).$$

Thus, according to (3), if $u = [a_i(\bar{x}_{i+1,k})]_{i=1}^{\infty}$ and $v = [b_i(\bar{x}_{i+1,k})]_{i=1}^{\infty}$ then

$$uv = [a_i(\bar{x}_{i+1,k}) + b_i(\bar{x}_{i+1,k}^u)]. \quad (4)$$

If $[u]_n \neq 0$ and $[u]_i = 0$ for all $i > n$ then n is called the *depth of the tableau* u .

1.2. U_p^ω as a direct limit of wreath powers

Recall that a sequence $\{G_n\}_{n=1}^\infty$ of groups with a corresponding sequence

$$\{\varphi_n : G_n \rightarrow G_{n+1}\}_{n=1}^\infty$$

of embeddings is called *direct system* and denoted by $\langle G_n, \varphi_n \rangle_{n=1}^\infty$.

Let P_n be a Sylow p -subgroup of the symmetric group S_{p^n} (p is prime and $n \in \mathbb{N}$). In [7] the group P_n was described by L.A. Kaloujnine as a group of tableaux

$$[a_1, a_2(x_1), \dots, a_n(x_1, x_2, \dots, x_{n-1})], \quad (5)$$

where $a_1 \in \mathbb{Z}_p$, $a_i(x_1, \dots, x_{i-1})$ is a polynomial (over the residue field \mathbb{Z}_p) reduced modulo the ideal

$$\langle x_1^p - x_1, x_2^p - x_2, \dots, x_{i-1}^p - x_{i-1} \rangle.$$

If we denote by $a_i(x_1, \dots, x_{i-1})$ and $b_i(x_1, \dots, x_{i-1})$ the i th coordinates of tableaux u and v respectively then the i th coordinate of $u \cdot v$ can be found as follows

$$a_i(x_1, \dots, x_{i-1}) + b_i(x_1 + a_1, \dots, x_{i-1} + a_{i-1}(x_1, \dots, x_{i-2})).$$

Also P_n can be considered as the n th *wreath power* of a cyclic group C_p , i.e. $P_n = C_p \wr \dots \wr C_p$ (n factors).

Given P_n and P_{n+1} ($n \in \mathbb{N}$) define the mapping $\delta_n : P_n \rightarrow P_{n+1}$. If u is a tableau of type (5) then

$$\delta_n(u) = [0, a_1, a_2(x_2), \dots, a_n(x_2, \dots, x_n)] \in P_{n+1}.$$

By the direct calculations (or see [9], Lemma 4) it is easy to show that δ_n is a *strictly diagonal* (in the sense of the article [8]) embedding. Moreover, δ_n is, actually, the embedding of P_n onto the *diagonal* of the wreath product $P_{n+1} = P_n \wr C_p$, where C_p is the active group.

Lemma 1. [9] *The group U_p^ω is isomorphic to the direct limit of the direct system $\langle P_n, \delta_n \rangle_{n=1}^\infty$.*

2. Characteristic subgroups of U_p^ω

This section is devoted to some necessary statements regarding description of characteristic subgroups of U_p^ω . All results are taken from [10] and [11], where they are proved for the case of a generalized infinitely iterated wreath product of elementary abelian groups.

Definition 1. The weighted degree of the monomial $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ is the positive rational number

$$h = \sum_{i=1}^n k_i p^{-i} + 1.$$

The weighted degree of a polynomial is the maximum among the weighted degrees of its monomials. Let also $h[0] = 0$.

Thus, if $u = [a_i(\bar{x}_{i+1,k})]_{i=1}^{\infty} \in U_p^{\omega}$ then $h[a_i(\bar{x}_{i+1,k})] \in \{0\} \cup [1; 1 + p^{-i})$.

Given $u = [a_i(\bar{x}_{i+1,k})]_{i=1}^{\infty} \in U_p^{\omega}$ we denote the weighted degree of $a_i(\bar{x}_{i+1,k})$ by $|u|_i$. The sequence $|u| = (|u|_i)_{i=1}^{\infty}$ is called the *multidegree* of the tableau u . The set of all multidegrees can be partially ordered as follows: $|u| \preceq |v|$ if and only if $|u|_i \leq |v|_i$ (with respect to the natural order on \mathbb{Q}) for all $i \in \mathbb{N}$.

Definition 2. A subgroup R of the group U_p^{ω} is called a *parallelotopic subgroup* if $u \in R$ and $|v| \preceq |u|$ yield $v \in R$.

For every parallelotopic subgroup R we put in correspondence the sequence $|R| = (\chi_i^{\varepsilon})_{i=1}^{\infty}$ such that

- 1) $\chi_i = \sup_{u \in R} |u|_i$;
- 2) if R contains such a tableau u that $|u|_i = \chi_i$, then $\varepsilon = "+"$;
- 3) otherwise, $\varepsilon = "-"$.

This sequence is called the *indicatrix* of R . If $\chi_i \neq 0$ for finitely many indices i only then $d(R) = \max\{i \mid \chi_i \neq 0\}$ is called the *depth of the parallelotopic subgroup* R . Otherwise, we put $d(R) = +\infty$.

Definition 3. A group G is called *characteristically* (fully invariantly or verbally) *simple* if only its characteristic (fully invariant or verbal) subgroups are E (the identity subgroup) or G .

In [12] it was shown that U_p^{ω} is verbally complete (a group G is called *verbally complete* if for an arbitrary $g \in G$ and for an arbitrary non-trivial word $w(x_1, x_2, \dots, x_n)$ there are $g_1, g_2, \dots, g_n \in G$ such that $w(g_1, g_2, \dots, g_n) = g$). Consequently, U_p^{ω} is verbally simple.

Characteristic subgroups of U_p^{ω} were investigated in [10] and [11]. Moreover, in these papers even more general case (the infinitely iterated wreath powers (we denote it by $U_{p,n}^{\infty}$) of the elementary abelian groups of rank n) was considered. If we put $n = 1$ then $U_{p,1}^{\infty} \cong U_p^{\omega}$. Thereby, from [10] and [11] it is known that U_p^{ω} has non-trivial proper characteristic subgroups, i.e. U_p^{ω} is not characteristically simple.

Theorem 1. [11] *If $p \neq 2$ and R is a characteristic (fully invariant or verbal) subgroup of the group U_p^ω , then*

- 1) R is a parallelotopic subgroup of U_p^ω ;
- 2) $d(R) < +\infty$ (R has finite depth).

3. Fully invariant subgroups of U_p^ω

Recall that a subgroup H of a group G is called *fully invariant* if it is invariant under all endomorphisms (homomorphisms of G into G) of G .

Lemma 2. *Let $u = [a_i(\bar{x}_{i+1,k})]_{i=1}^\infty \in U_p^\omega$. Then the mapping*

$$\varphi : U_p^\omega \rightarrow U_p^\omega,$$

which acts on the elements of U_p^ω by the rule:

$$[\varphi(u)]_1 = 0, \quad [\varphi(u)]_{i+1} = a_i(x_{i+2}, \dots, x_{k+1}) \text{ for all } i \in \mathbb{N}$$

is an endomorphism of U_p^ω .

Remark 1. For any $u \in U_p^\omega$ the mapping φ actually is a coordinate-wise translation to the right (all variables also must be shifted: $x_j \rightarrow x_{j+1}$).

Proof of lemma 2. Let

$$u = [a_i(\bar{x}_{i+1,k})]_{i=1}^\infty \in U_p^\omega \quad \text{and} \quad v = [b_i(\bar{x}_{i+1,k})]_{i=1}^\infty \in U_p^\omega$$

(without loss of generality we assume that $[u]_i = [v]_i = 0$ for all $i > n$, where $n < k$). Obviously, $[\varphi(uv)]_1 = [\varphi(u)\varphi(v)]_1 = 0$. Therefore, we consider $[\varphi(uv)]_i$ and $[\varphi(u)\varphi(v)]_i$, where $i \geq 2$. According to the formula (4) we have

$$\begin{aligned} [uv]_i &= a_i(\bar{x}_{i+1,k}) + b_i(\bar{x}_{i+1,k}^u) = \\ &= a_i(\dots, x_j, \dots) + b_i(\dots, x_j + a_j(x_{j+1}, \dots, x_k), \dots), \end{aligned}$$

where $j \in \{i+1, \dots, k\}$. Thus

$$[\varphi(uv)]_{i+1} = a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + a_j(x_{j+2}, \dots, x_{k+1}), \dots),$$

where $j \in \{i+1, \dots, k\}$.

On the other hand, since

$$[\varphi(u)]_{i+1} = a_i(x_{i+2}, \dots, x_{k+1}) \quad \text{and} \quad [\varphi(v)]_{i+1} = b_i(x_{i+2}, \dots, x_{k+1})$$

then

$$\begin{aligned} [\varphi(u)\varphi(v)]_{i+1} &= a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + [\varphi(u)]_{j+1}, \dots) = \\ &= a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + a_j(x_{j+2}, \dots, x_{k+1}), \dots), \end{aligned}$$

where $j \in \{i+1, \dots, k\}$.

Hence, $\varphi(uv) = \varphi(u)\varphi(v)$, i.e. φ is an endomorphism of U_p^ω . \square

Now we can prove the main result.

Theorem 2. *If $p \neq 2$ then U_p^ω is fully invariantly simple.*

Proof. Let us assume that R is a fully invariant subgroup of the group U_p^ω , $R \neq E$ and $R \neq U_p^\omega$. Then R is a parallelotopic subgroup of U_p^ω and $d(R) < +\infty$ (theorem 1). If $d(R) = r$ then R contains the tableau

$$u = [0, \dots, 0, 1, 0, \dots].$$

$\underbrace{\hspace{1.5cm}}_{r-1}$

Given the endomorphism φ , defined in lemma 2, we have $\varphi(u) \notin R$ (since the depth of $\varphi(u)$ is equal to $r + 1$). But, on the other hand, $\varphi(u) \in R$ (since R is a fully invariant subgroup of U_p^ω). This contradiction shows the falsity of the assumptions. \square

4. A note on the normalizer condition

A group G is said to satisfy the *normalizer condition* if every proper subgroup H is properly contained in its own normalizer, i.e. $H < N_G(H)$ for all $H < G$.

Lemma 3. [13] *If a group G satisfies the normalizer condition then every subgroup H of G also satisfies normalizer condition.*

Let

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Given a prime p we consider the set FM_p^ω of all almost identity infinite matrices over \mathbb{Z}_p (ω is the least infinite ordinal). In other words,

$$FM_p^\omega = \left\{ (a_{ij}) \mid \begin{array}{l} a_{ij} \in \mathbb{Z}_p; a_{ij} = \delta_{ij} \text{ for all but} \\ \text{finitely many } (i, j) \in \mathbb{N} \times \mathbb{N} \end{array} \right\}.$$

Also, FM_p^ω is called the set of all *finitary* matrices. Finitary matrices can be multiplied by the usual rule:

$$(ab)_{ij} = \sum_k a_{ik} b_{kj},$$

since the sum on the right side contains only a finite number of nonzero terms. The set of all finitary invertible matrices with operation of matrix multiplication forms a group, which is called the *finitary linear group*.

The *finitary (upper) unitriangular group* is the group UT_p^ω of all finitary matrices over \mathbb{Z}_p such that $a_{ij} = \delta_{ij}$ for all $i \geq j$ (see, for example, [14]). Similarly, we can consider the *lower* unitriangular group.

On the other hand, the group of all finitary invertible matrices can be considered as a permutation group on the set X of all almost zero sequences over \mathbb{Z}_p (see formula (2)) with natural action

$$t^A = t \cdot A = \left(\sum_{i=1}^k a_{i1}t_i, \sum_{i=1}^k a_{i2}t_i, \dots, \sum_{i=1}^k a_{in}t_i, \dots \right), \quad (6)$$

where $t = (t_i)_{i=1}^\infty \in X$ and $A = (a_{ij}) \in FM_p^\omega$, $i, j \in \mathbb{N}$.

In [14] (see theorem 1) it was shown that *the finitary unitriangular group UT_p^ω does not satisfy the normalizer condition.* And we get the following natural corollary.

Theorem 3. *The group U_p^ω does not satisfy the normalizer condition.*

Proof. According to lemma 3 it is sufficient to show that U_p^ω contains a group, which is isomorphic to the finitary (upper) unitriangular group.

Let $u \in U_p^\omega$ be a tableau with linear coordinates, i.e. $[u]_i$ is a homogeneous linear polynomial. By the direct computation it can be shown that the action of u on X (which is defined by the equation (3)) agrees with the action (6). Obviously, the subset of all tableaux with linear coordinates is a subgroup of U_p^ω and this subgroup is isomorphic to the finitary (lower) unitriangular matrix group.

Since the groups of upper and lower finitary unitriangular matrices are isomorphic (with isomorphism $x \rightarrow (x^T)^{-1}$, where $x \in UT_p^\omega$ and x^T is the transpose of x) the group U_p^ω does not satisfy the normalizer condition. \square

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