

Length of the inverse symmetric semigroup

Olexandr Ganyushkin and Ivan Livinsky

Communicated by V. Mazorchuk

ABSTRACT. The length of the lattice of subsemigroups of the inverse symmetric semigroup \mathcal{IS}_n is calculated.

1. Introduction

In the theory of inverse semigroups the inverse symmetric semigroup $\mathcal{IS}(M)$ plays a role similar to the role of the symmetric group in group theory. Especially interesting is the case $|M| = n < \infty$, since apart from specific semigroup problems, a lot of combinatorial problems arise there. A great number of papers, and even specialized monographs (see [1], [2] and literature cited there) are dedicated to the study of the semigroup \mathcal{IS}_n .

An essential question in the study of any semigroup is about the structure of the lattice of its subsemigroups. Although such lattices were studied actively (see e.g., [3]), not very much is known about the structure of lattices of subsemigroups of particular semigroups. The main reason is that lattices of subsemigroups have quite a complex structure. Even for monogenic semigroups this question is nontrivial. For groups, it is hard too: the paper [4] uses the classification of finite simple groups to calculate the length of the lattice of subgroups of the symmetric group S_n .

In the case of \mathcal{IS}_n a little is known as well. Though the lattice of two-sided ideals of \mathcal{IS}_n is quite simple (a chain of length n), the structure of the lattice of left (right) ideals is considerably more complex (see [2,

2000 Mathematics Subject Classification: 20M18, 20M10.

Key words and phrases: length of a semigroup, inverse semigroup, semigroup, Brandt semigroup.

Chapter 4.3]). Clearly, additional difficulties arise when considering the lattice of all subsemigroups.

Recall that the length $l(S)$ of a semigroup S is defined as the length of its lattice of subsemigroups, i.e., the maximal integer n , for which there exists a strictly increasing chain of subsemigroups

$$\emptyset \neq H_0 \subset H_1 \subset H_2 \subset \dots \subset H_n = S. \quad (1)$$

The objective of the present paper is the calculation of the length of the inverse symmetric semigroup \mathcal{IS}_n (Theorem 8). As an auxiliary result, a description of maximal subsemigroups of a finite Brandt semigroup is obtained (Theorem 6).

In this paper only finite semigroups and groups are considered.

2. The main Lemma

Lemma 1. *For every ideal I of a finite semigroup S the following equality holds*

$$l(S) = l(I) + l(S/I), \quad (2)$$

where S/I denotes the Rees quotient modulo the ideal I .

Proof. Let S be a finite semigroup and I an ideal of S . Assume that (1) gives a longest chain of subsemigroups in S . First we prove that for every extension $H_k \subset H_{k+1}$ the set $H_{k+1} \setminus H_k$ either is a subset of the ideal I or is disjoint with it. Indeed, if this is not the case, we would have the strict inclusions

$$H_k \subset H_k \cup (H_{k+1} \cap I) \subset H_{k+1}. \quad (3)$$

The subsemigroup $H_{k+1} \cap I$ is an ideal of H_{k+1} ; therefore, the set $H_k \cup (H_{k+1} \cap I)$, being a union of an ideal and a subsemigroup, would be a subsemigroup too. Then (3) would contradict the maximality of chain (1).

Extend chain (1) by the empty subsemigroup $H_{-1} = \emptyset$ and transform the chain

$$H_{-1} = \emptyset \subset H_0 \subset H_1 \subset H_2 \subset \dots \subset H_n = S \quad (4)$$

in the following way: if this chain contains a fragment $H_{k-1} \subset H_k \subset H_{k+1}$, where $H_k \setminus H_{k-1} \not\subseteq I$ and $H_{k+1} \setminus H_k \subseteq I$, replace it by the fragment

$$H_{k-1} \subset H_{k-1} \cup (H_{k+1} \cap I) \subset H_{k+1}$$

(strictness of the inclusions is obvious, and, similarly to the above, one shows that the set $H_{k-1} \cup (H_{k+1} \cap I)$ is a subsemigroup). By a finite number of such transformations we get the chain

$$H_{-1} = \emptyset \subset H'_0 \subset H'_1 \subset H'_2 \subset \cdots \subset H'_{n-1} \subset H_n = S, \quad (5)$$

in which all extensions by the elements of ideal I come first, followed by extensions by elements of $S \setminus I$. From this property and maximality of chain (5) it follows that for some m equality $H'_m = I$ must hold. Thus, from (5) we get the chain

$$\emptyset \subset H'_0 \subset H'_1 \subset H'_2 \subset \cdots \subset H'_m$$

of subsemigroups of I and the chain

$$0 = I/I = H'_m/I \subset H'_{m+1}/I \subset \cdots \subset H'_{n-1}/I \subset H_n/I = S/I$$

of subsemigroups of S/I . From the definition of Rees quotient it follows that in the last chain all extensions are strict too. This gives us the inequality $l(S) \leq l(I) + l(S/I)$.

The opposite inequality follows from the following fact: if

$$\emptyset \neq H_0 \subset H_1 \subset \cdots \subset H_p = I \quad \text{and} \quad \emptyset \neq K_0 \subset K_1 \subset \cdots \subset K_q = S/I$$

are chains of subsemigroups for I and S/I correspondingly, and $\pi : S \rightarrow S/I$ is the canonical epimorphism, then

$$\emptyset \neq H_0 \subset H_1 \subset \cdots \subset H_p \subseteq \pi^{-1}(K_0) \subset \pi^{-1}(K_1) \subset \cdots \subset \pi^{-1}(K_q) = S$$

is a chain of subsemigroups for S . □

3. The length of the Brandt semigroup

Recall that the Brandt semigroup $B(n, G)$ over a group G is the semigroup of all matrices of dimension n , all entries of which are zero, except for at most one entry which is supposed to be an element of G . The matrix, in which an element $g \in G$ is in k -th row and l -th column, is denoted by $(g)_{kl}$. The zero matrix is the zero in $B(n, G)$ and the multiplication of nonzero elements from $B(n, G)$ is the usual matrix multiplication: $(g)_{kl} \cdot (h)_{pq}$ is equal to $(gh)_{kq}$ if $l = p$, and to 0 otherwise.

We will use the following notation: e is the unit of the group G ; for an arbitrary subset $H \subseteq G$ denote by $(H)_{kl}$ the set $\{(h)_{kl} \mid h \in H\}$; for an arbitrary subset $S \subseteq B(n, G)$ denote $S_{kl} = S \cap (G)_{kl}$.

For an arbitrary subgroup $H \leq G$ and elements $g_1 = e, g_2, \dots, g_n$ of G define $S(H; g_2, \dots, g_n)$ by

$$S(H; g_2, \dots, g_n) = \left(\bigcup_{k,l} (g_k^{-1} H g_l)_{kl} \right) \cup \{0\}. \quad (6)$$

Lemma 2. *For an arbitrary subgroup $H \leq G$ and elements $g_1 = e, g_2, \dots, g_n$ of G the set $S(H; g_2, \dots, g_n)$ is a subsemigroup of the semigroup $B(n, G)$.*

Proof. That $S(H; g_2, \dots, g_n)$ is closed under multiplication is verified by a direct calculation. \square

Lemma 3. *Let S be a subsemigroup of $B(n, G)$ such that for every k and l $S \cap (G)_{kl} \neq \emptyset$. Then S is equal to $S(H; g_2, \dots, g_n)$ for some g_2, \dots, g_n .*

Proof. Let S be a subsemigroup of $B(n, G)$ such that $S_{kl} \neq \emptyset$ for arbitrary k and l . In every S_{kl} choose an element $s_{kl} = (g_{kl})_{kl}$. Obviously, S_{11} is a subgroup in $B(n, G)$; thus, S_{11} is equal to $(H)_{11}$, where H is some subgroup of G . In particular, we can take $g_{11} = e$. From the inequalities

$$s_{lk} \cdot S_{kk} \cdot s_{kl} \subseteq S_{ll}, \quad S_{kk} \cdot s_{kl} \subseteq S_{kl}, \quad S_{kl} \cdot s_{lk} \subseteq S_{kk} \quad (7)$$

it follows that all of the sets S_{kl} are equinumerous. Hence, inclusions (7) are equalities.

Since $s_{1k} s_{k1} \in S_{11}$, then $g_{1k} g_{k1} \in H$. From the equality $S_{k1} = s_{k1} \cdot S_{11}$ it follows that, replacing g_{k1} by $g_{k1} \cdot (g_{1k} g_{k1})^{-1}$, we can assume $g_{k1} = g_{1k}^{-1}$. Then, from the equality $S_{kl} = s_{k1} \cdot S_{11} \cdot s_{1l}$ it follows that $S_{kl} = (g_{1k}^{-1} H g_{1l})_{kl}$ and the subsemigroup S is equal to $S(H; g_2, \dots, g_n)$, where $g_i = g_{1i}$, $2 \leq i \leq n$. \square

Corollary 4. *The semigroup $S(H; g_2, \dots, g_n)$ is isomorphic to the semigroup $B(n, H)$.*

Proof. The map $(g_k^{-1} h g_l)_{kl} \mapsto (h)_{kl}$ is an isomorphism between $S(H; g_2, \dots, g_n)$ and $B(n, H)$. \square

For arbitrary subsets $K, L \subseteq \{1, 2, \dots, n\}$ denote

$$S(K, L) = B(n, G) \setminus \bigcup_{i \in K, j \in L} (G)_{ij}. \quad (8)$$

Lemma 5. *If sets K and L form a covering of the set $\{1, 2, \dots, n\}$, then $S(K, L)$ is a subsemigroup of the semigroup $B(n, G)$.*

Proof. If $S(K, L)$ is not a subsemigroup, then there exist non-zero elements $(g)_{pq}, (h)_{rt} \in S(K, L)$ such that $(g)_{pq} \cdot (h)_{rt} \notin S(K, L)$. But then $p \in K, q \notin L, t \in L, r \notin K$ and $q = r$, which is impossible. \square

Theorem 6. *Let $n > 1$ and G be a finite group. A subsemigroup S of a Brandt semigroup $B(n, G)$ is maximal if and only if either it is equal to the subsemigroup $S(H; g_2, \dots, g_n)$, where H is a maximal subgroup of the group G , or to the subsemigroup $S(K, L)$, where sets K and L form a partition of the set $\{1, 2, \dots, n\}$.*

Proof. Let S be a subsemigroup of $B(n, G)$. There are two possible cases.

I. For arbitrary k and l we have $S_{kl} \neq \emptyset$. Then, by Lemma 3, S is equal to some $S(H; g_2, \dots, g_n)$. From the statement that all sets S_{kl} are equinumerous it follows that $S(H; g_2, \dots, g_n)$ is a maximal subsemigroup of $B(n, G)$ if and only if H is a maximal subgroup of G .

II. There exist k and l such that $S_{kl} = \emptyset$. Since from $S_{ij} \neq \emptyset$ and $S_{ji} \neq \emptyset$ it follows that $S_{ii} \supseteq S_{ij} \cdot S_{ji} \neq \emptyset$, we can assume that $k \neq l$. Denote

$$K = \{p \mid S_{kp} \neq \emptyset\} \cup \{k\}, \quad L = \{q \mid S_{ql} \neq \emptyset\} \cup \{l\}.$$

Sets K and L do not intersect. Indeed, if $i \in K \cap L$, then for arbitrary $(g)_{ki} \in S_{ki}$ and $(h)_{il} \in S_{il}$ we have $(gh)_{kl} \in S_{kl}$, which contradicts to the condition $S_{kl} = \emptyset$.

Note, that for arbitrary $i \in K, j \in L$ we have $S_{ij} = \emptyset$. Indeed, for S_{kj} and S_{il} this follows from the definition of the sets K and L . If $i \in K \setminus \{k\}, j \in L \setminus \{l\}$, this follows from the inclusion $S_{ki} \cdot S_{ij} \cdot S_{jl} \subseteq S_{kl}$.

Show that if S is a maximal subsemigroup, then $K \cup L = \{1, \dots, n\}$. Indeed, let $m \in \{1, \dots, n\} \setminus (K \cup L)$. Assume that there exists $p \in K$ such that $S_{pm} \neq \emptyset$. Then S is a proper subset of the subsemigroup $S_1 = \langle S, (G)_{km} \rangle$. On the other hand, from the inclusion $S_{pm} \cdot S_{mq} \subseteq S_{pq}$ it follows that $S_{mq} = \emptyset$ for all $q \in L$. Thus, S_1 does not contain elements of $(G)_{kl}$, i.e. it is a proper subsemigroup of $B(n, G)$. If for all $p \in K$ we have $S_{pm} = \emptyset$, then by the same argument it can be proved that S is a proper subsemigroup of $\langle S, (G)_{ml} \rangle$.

Therefore, if S is a maximal subsemigroup, then K and L form a partition of the set $\{1, 2, \dots, n\}$. Since S is a subset of $S(K, L)$, it must be equal to the last one, i.e. $S = S(K, L)$.

It is left to show that, if K and L form a partition, then the subsemigroup $S(K, L)$ is maximal. Indeed, $S(K, L)$ cannot be a subset of any other subsemigroup of this kind. Thus, if $S^* \not\supseteq S(K, L)$, then $S^*_{kl} \neq \emptyset$ for arbitrary k and l , and by Lemma 3, S^* is equal to some subsemigroup $S(H; g_2, \dots, g_n)$. But $S^*_{11} = (G)_{11}$, so $H = G$ and $S(H; g_2, \dots, g_n) = B(n, G)$. \square

We denote by G^0 the group G with a zero adjoint.

Theorem 7. *For a finite group G*

$$l(B(n, G)) = n \cdot l(G^0) + \frac{n(n-1)}{2}|G| + n - 1. \quad (9)$$

Proof. We use induction on the parameter n . Since $B(1, G) \simeq G^0$, then for $n = 1$ equality (9) is true.

Now, assume that equality (9) is proved for all semigroups $B(m, G)$, where $m < n$. From Theorem 6 it follows that $l(B(n, G))$ is equal to $1 + l(S(K, L))$ for some partition $K \cup L = \{1, 2, \dots, n\}$, or to $1 + l(S(H; g_2, \dots, g_n))$ for some maximal subgroup $H < G$.

Let us determine $l(S(K, L))$ for $|K| = p$, $|L| = q$, $p + q = n$. To this end, consider three subsets of $S(K, L)$:

$$S_K = \{0\} \cup \bigcup_{i,j \in K} (G)_{ij}, \quad S_L = \{0\} \cup \bigcup_{i,j \in L} (G)_{ij}, \quad I = \{0\} \cup \bigcup_{i \in L, j \in K} (G)_{ij}.$$

It is obvious that each of these subsets is a subsemigroup; in particular, $S_K \simeq B(p, G)$, $S_L \simeq B(q, G)$, and I is a subsemigroup with zero multiplication. Moreover, $S(K, L) = S_K \cup S_L \cup I$. It is easy to check that I is an ideal in $S(K, L)$, and the Rees quotient $S(K, L)/I$ is isomorphic to $S_K \cup S_L$. In turn, in the semigroup $S_K \cup S_L$ both subsemigroup S_K and S_L are ideals with the following property: Rees quotient modulo S_K is isomorphic to S_L and, vice versa, Rees quotient modulo S_L is isomorphic to S_K . Therefore, by Lemma 1

$$l(S(K, L)) = l(I) + l(S_K \cup S_L) = l(I) + l(S_K) + l(S_L). \quad (10)$$

Since for a semigroup T with zero multiplication we have $l(T) = |T| - 1$, by the inductive hypothesis we get:

$$\begin{aligned} l(S(K, L)) &= pq|G| + \left(p \cdot l(G^0) + \frac{p(p-1)}{2}|G| + p - 1 \right) + \\ &+ \left(q \cdot l(G^0) + \frac{q(q-1)}{2}|G| + q - 1 \right) = n \cdot l(G^0) + \frac{n(n-1)}{2}|G| + n - 2. \end{aligned}$$

In particular, $l(S(K, L))$ does not depend on the choice of K and L .

To complete the proof of the Theorem we need to show that

$$l(B(n, G)) = l(S(K, L)) + 1. \quad (11)$$

We will do this by induction on the order of the group G . Equality (11) is obvious if $|G| = 1$, because in this case every maximal subsemigroup

of $B(n, G)$ is isomorphic to $S(K, L)$. Assume now that (11) holds for all groups of order t . Let $t < |G| \leq 2t$ and H be a maximal subgroup of G , then $|H| \leq |G|/2 \leq t$. Since $S(H; g_2, \dots, g_n) \simeq B(n, H)$, then by the inductive hypothesis

$$l(S(H; g_2, \dots, g_n)) = n \cdot l(H^0) + \frac{n(n-1)}{2}|H| + n - 1,$$

what clearly is less than $l(S(K, L))$. This completes the proof of the equality (11). \square

4. Length of \mathcal{IS}_n

Theorem 8. *The length of the inverse symmetric semigroup \mathcal{IS}_n is equal to*

$$l(\mathcal{IS}_n) = \sum_{k=1}^n \left[\binom{n}{k} \left(\left\lceil \frac{3k}{2} \right\rceil - b(k) + 1 \right) + \frac{\binom{n}{k} (\binom{n}{k} - 1)}{2} \cdot k! - 1 \right],$$

where we denote by $\lceil x \rceil$ the least integer, which is not less than x , and by $b(k)$ the number of nonzero digits in the binary expansion of k .

Proof. It is known that semigroup \mathcal{IS}_n has $n + 1$ ideals [2, Chapter 4], which form the chain

$$\{0\} = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_n = \mathcal{IS}_n.$$

For every k , $1 \leq k \leq n$, the Rees quotient I_k/I_{k-1} is isomorphic to the Brandt semigroup $B\left(\binom{n}{k}, S_k\right)$, where S_k is the symmetric group of degree k . Thus, from Lemma 1 and Theorem 7 it follows that the length of semigroup \mathcal{IS}_n is equal to

$$\begin{aligned} l(\mathcal{IS}_n) &= \sum_{k=1}^n l(I_k/I_{k-1}) = \sum_{k=1}^n l\left(B\left(\binom{n}{k}, S_k\right)\right) \\ &= \sum_{k=1}^n \left[\binom{n}{k} \cdot l(S_k^0) + \frac{\binom{n}{k} (\binom{n}{k} - 1)}{2} |S_k| + \binom{n}{k} - 1 \right]. \end{aligned} \quad (12)$$

In the paper [4] it is proved that $l(S_k) = \lceil \frac{3k}{2} \rceil - b(k) - 1$. Therefore, $l(S_k^0) = l(S_k) + 1 = \lceil \frac{3k}{2} \rceil - b(k)$. Moreover, $|S_k| = k!$. Putting these values into (12), we get the statement of the Theorem. \square

References

-
- [1] *Lipscomb S.* Symmetric Inverse Semigroups. Math. Surveys and Monographs, **46**, AMS, Providence, RI, 1996.
- [2] *Ganyushkin O., Mazorchuk V.* Classical Finite Transformation semigroups. An Introduction. Algebra and Applications, **9**, Springer-Verlag, London, 2009.
- [3] *Shevrin L.N., Ovsyannikov A.Ya.* Semigroups and their subsemigroup lattices. Kluwer Academic Publishers, 1996.
- [4] *Cameron P.I., Solomon R.G., Turul A.* Chains of Subgroups in Symmetric Groups // J. Algebra. — 1989. — Vol. 127. — P. 340–352

CONTACT INFORMATION

O. Ganyushkin

Department of Mechanics and Mathematics,
Kyiv Taras Shevchenko University, 64,
Volodymyrska st. 01033, Kyiv, UKRAINE
E-Mail: ganiushk@univ.kiev.ua

I. Livinsky

Department of Mathematics, University of Man-
itoba, 186 Dysart Road, R3T2N2, Winnipeg,
CANADA
E-Mail: umlivini@cc.umanitoba.ca

Received by the editors: 26.10.2011
and in final form 17.12.2011.