Algebra and Discrete Mathematics
Volume 12 **(2011)**. Number 2. pp. 31 – 37

RESEARCH ARTICLE

Minimax isomorphism algorithm and primitive posets

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Communicated by V. V. Kirichenko

Abstract. The notion of minimax equivalence of posets, and a close notion of minimax isomorphism, introduced by the author are widely used in the study of quadratic Tits forms (in particular, for the description of P-critical and P-supercritical posets). In this paper, for an important special case, we modify an algorithm of classifying all posets minimax isomorphic to a given one (described earlier by the author together with M. V. Stepochkina) by introducing the concept of weak isomorphism.

Introduction

M. M. Kleiner [1] proved that a poset S has finite representation type if and only if it does not contain as a full subposet any of the following ones, which are called *critical posets*: $(1,1,1,1), (2,2,2), (1,3,3), (1,2,5)$ and $(N,4)$; now they are often called the critical posets of Kleiner. On the other hand, Ju. A. Drozd [2] shows that a poset is of finite representation type if and only if its quadratic Tits form is weakly positive (i.e. is positive definite only on the set of vectors with non-negative coordinates). Hence the critical set of Kleiner are critical with respect to the weakly positivity of the Tits form; and there are no other such posets. In [3] the author together with M. V. Stepochkina proved that a poset is critical with respect to the positivity of the Tits form (P-critical) if and only if it is minimax equivalent to a critical poset of Kleiner, and described all such posets (this equivalence was introduced by the author in [4]).

2000 Mathematics Subject Classification: *16G20.*
Key words and phrases: *critical poset, quadratic Tits form, minimax equivalence, weak isomorphism.*

A similar situation holds for tame posets. L. A. Nazarova [5] proved that a poset is tame if and only if it does not contain subsets of the form $(1,1,1,1,1)$, $(1,1,1,2)$, $(2,2,3)$, $(1,3,4)$, $(1,2,6)$ and $(N,5)$; it is equivalent to the weakly non-negativity of the quadratic Tits form. These posets are called *supercritical*. Consequently the supercritical sets are critical with respect to the weakly non-negativity of the Tits form (and there are no other such posets). The author together with M. V. Stepochkina [6] proved that a poset is critical with respect to the non-negativity of the Tits form (P-supercritical or NP-critical) if and only if it is minimax equivalent to a supercritical poset; all such critical sets are described in the paper [7].

The minimax equivalence and a close notion of minimax isomorphism were studied in detail in [3] (see also [6]), and, in particular, an algorithm was proposed to find all posets minimax equivalent (minimax isomorphic) to a given one. In this paper, for an important special case, we modify this algorithm by introducing the concept of weak isomorphism.

1. Minimax equivalence and minimax isomorphism

Throughout the paper, we consider only finite posets (including the empty one) and identify singletons with the elements themselves. By a subposet we always mean a full one.

Here we shall follow the paper [4].

Let P be a poset. For a minimal (resp. maximal) element a of P, denote by $Q = P_a^{\uparrow}$ (resp. $Q = P_a^{\downarrow}$) the following poset: $Q = P$ as usual sets, $Q \setminus a = P \setminus a$ as posets, the element a is maximal (resp. minimal) in Q, and a is comparable with x in Q if and only if they are incomparable in P. A poset T is called *minimax equivalent* or (min, max)-*equivalent* to a poset S, if there are posets $S_1, \dots, S_p$ ($p \geq 0$) such that, if one puts $S = S_0$ and $T = S_{p+1}$, then, for every $i = 0, 1, \dots, p$, either $S_{i+1} = (S_i)_{x_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{y_i}^{\downarrow}$. We shall write $S_{ab}^{\uparrow\uparrow}$ instead of $(S_a^{\uparrow})_b^{\uparrow}$, $S_{ab}^{\uparrow\downarrow}$ instead of $(S_a^{\uparrow})_b^{\downarrow}$, etc. It is easy to show, using the equalities $S_{aa}^{\uparrow\downarrow} = S_{aa}^{\downarrow\uparrow} = S$, that minimax equivalence is really an equivalence relation (see Corollary 2 [3]).

The notion of minimax equivalence can be naturally continued to the notion of *minimax isomorphism*: posets S and S' are minimax isomorphic if there exists a poset T, which is minimax equivalent to S and is isomorphic to S'.

The main motivation for introducing the notion of minimax equivalence is the fact that the Tits forms of minimax equivalent posets are $\mathbb{Z}$-equivalent.

2. Main result

The definition of posets of the form $Q = P_a^{\uparrow}$ (resp. $Q = P_a^{\downarrow}$) can be extended to subposets. Namely, let S be a poset and A its lower (resp. upper) subposet, i.e. $x \in A$ whenever $x < y$ (resp. $x > y$) and $y \in A$. By $\overline{S} = S_A^{\uparrow}$ (resp. $\overline{S} = S_A^{\downarrow}$) we denote the following poset: $\overline{S} = S$ as usual sets, and $x < y$ in $\overline{S}$ if and only when either

a) $x < y$ in S and either $a, b \in A$, or $a, b \notin A$,

or b) x is incomparable with y in S and $y \in A, x \notin A$ (resp. $x \in A, y \notin A$).

In other words, the partial orders on A and $S \setminus A$ are the same as before but comparability and incomparability between elements of A and $S \setminus A$ are interchanged, and the new relations $z < t$ can only be "from $S \setminus A$ to A" (resp. "from A to $S \setminus A$)".

We shall write $S_{AB}^{\uparrow\uparrow}$ instead of $(S_A^{\uparrow})_B^{\uparrow}$.

A poset S is called a *sum of subposets* $A_1 \ldots, A_m$ if $A_i \cap A_j = \varnothing$ for any distinct i, j and $S = A_1 \cup \ldots \cup A_m$. If any two elements $a \in A_i$ and $b \in A_j$ are incomparable whenever $i \neq j$, this sum is called *direct*. In this case we write $S = A_1 + \ldots + A_m$, or $S = A_1 \coprod \ldots \coprod A_m$ if the sum is direct. The poset S is called *indecomposable* or *connected* if there is no direct sum decomposition $S = A_1 \coprod \ldots \coprod A_m$ with $m > 1$ and nonempty $A_1, \ldots, A_m$. Recall that a *primitive poset* is a direct sum of chains.

We shall say that an element of a poset S is a *node*, if it is comparable with all elements of S, and a *local node*, if it is a node of a direct summand of S. Obviously that each element of S is a node iff S is a chain, and is a local node iff S is a primitive poset.

We call *weak isomorphism of posets* a bijective map that preserves in both directions the comparability of elements and induces an isomorphism between their largest subposets without local nodes. In this case we say that the posets are *weakly isomorphic*. Note that two posets without local nodes or two primitive ones are weakly isomorphic iff they are isomorphic.

The aim of this paper is to prove the following theorem.

Theorem 1. *Let T be a poset and S a primitive poset. Then the following conditions are equivalent:*

1) *T is minimax isomorphic to S;*

2) *there exists a lower subposet X of S such that T is weakly isomorphic to $S_X^{\uparrow}$;*

2′) *there exists an upper subposet X' of S such that T is weakly isomorphic to $S_{X'}^{\downarrow}$.*

This theorem gives an algorithm for finding all posets minimax isomorphic to a given primitive poset, which modifies (in this particular case) a general algorithm described in [3].

Note that the primitive posets play an important role not only in the theory of quadratic Tits form but also in representation theory (this is due to the fact that most of the critical and supercritical posets are primitive); see, e.g., [8], [9].

3. Proof of Theorem 1

The notation $A < B$ for subposets of a poset P means that $a < b$ for any $a \in A, b \in B$. We assume that this inequality holds for $A = \varnothing$ or $B = \varnothing$; we shall also assume that $A < B$ always when we shall have inequalities $A < C$ and $C < B$ with $C = \varnothing$. For posets A and B, we define $[A < B]$ to be the poset $A \cup B$ with $A < B$; similarly, we define $[A_1 < \ldots < A_s]$ to be the poset $A_1 \cup \ldots \cup A_s$ with $A_1 < \ldots < A_s$.

2) $\Leftrightarrow$ 2$'$). The equivalence of conditions 2) and 2$'$) follows from the obvious equality $S^{\uparrow}_X = S^{\downarrow}_{X'}$ with $X' = S \setminus X$.

2) $\Rightarrow$ 1)**.** We first introduce some notation. When a poset T is minimax equivalent to a poset S (see the corresponding definition in Section 1) we write $T = S^{\varepsilon_0\varepsilon_1\ldots\varepsilon_p}_{z_0 z_1\ldots z_p}$, where $(z_i, \varepsilon_i) = (x_i, \uparrow)$ if $S_{i+1} = (S_i)^{\uparrow}_{x_i}$, and $(z_i, \varepsilon_i) = (y_i, \downarrow)$ if $S_{i+1} = (S_i)^{\downarrow}_{y_i}$. In the case when each ε_i is equal to $\uparrow$ (resp. $\downarrow$), we also write $T = S^{\uparrow}_{\alpha}$ (resp. $T = S^{\downarrow}_{\alpha}$) with α to be the sequence $(x_0, x_1, \ldots, x_p)$ (resp. $(y_0, y_1, \ldots, y_p)$).

Now turn to the proof, assuming that the subposet X is proper (otherwise $S^{\uparrow}_X = S$ and then T is isomorphic to S).

Let first $T = S^{\uparrow}_X$. Denote by X_1 the set of all minimal elements in X and by X_i for $i > 1$ (inductively) the set of minimal elements in $X \setminus (\cup_{j=1}^{i-1} X_j)$ (it is obvious that $\cup_{i=1}^{r} X_i = X$, where r is the largest i such that $X_i \neq \varnothing$); we write $h(x) = i$ for an element $x \in X$ if $x \in X_i$. Fix a sequence $\beta = (x_1, x_2, \ldots, x_s)$ with pairwise distinct elements such that $h(x_1) \leq h(x_2) \leq \ldots \leq h(x_s)$. Obviously, the expression $S^{\uparrow}_{\beta}$ is correct, and it is easy to verify, using the definitions of posets of the form $S^{\uparrow}_a$ and $S^{\uparrow}_A$, that $S^{\uparrow}_{\beta} = S^{\uparrow}_X$ (see, in this regard, Proposition 6 [3]). So T is equal to $S^{\uparrow}_{\beta}$, i.e. is minimax equivalent to S.

From this we obviously have that T is minimax isomorphic to S if T is isomorphic to $S^{\uparrow}_X$ (for some X).

The last fact will often be used below without explicitly mentioning.

We shall need some lemmas.

Lemma 1. *The posets of the form $P = [A < B]$ and $Q = [B < A]$ are minimax equivalent.*

The assertion follows from the $P^{\uparrow\uparrow}_{AA} = Q$.

Corollary 1. *If $S = [L_1 < C < L_2]$ where L_1, L_2 are chains and C does not contain nodes, and T is weakly isomorphic to S, then T is minimax isomorphic to S.*

Lemma 2. *If $L \neq \varnothing$ is a chain then the posets $P = [B < a] \coprod L$ and $Q = [a < B] \coprod L$ are minimax isomorphic.*

Indeed, if c is the smallest element of L, then $Q^{\uparrow\uparrow}_{ac}$ is isomorphic to P and therefore P is minimax isomorphic to Q.

We continue the proof, already assuming that T is weakly isomorphic to $S^{\uparrow}_{X}$. By $m = w(S)$ we denote the width of S (the maximum number of pairwise incomparable elements of S); note that $w(\varnothing) = 0$. Put $M = \{1, \ldots, m\}$.

When $w(S) \leq 2$, then $S^{\uparrow}_{X}$ is a primitive poset of width $r \leq 2$ and therefore T and $S^{\uparrow}_{X}$ are isomorphic.

Let $w(S) = m > 2$ and let S be a direct sum of chains $L_1, \ldots, L_m$. Put

$$X_i = X \cap L_i, \;\; Y_i = L_i \setminus X_i \;\; (1 \leq i \leq m),$$

$$\overline{X}_1 = \cup_{i \in I} X_i, \text{ where } I = \{s \,|\, X_s = L_s\},$$

$$\overline{Y}_1 = \cup_{i \in J} Y_i, \text{ where } J = \{s \,|\, Y_s = L_s\},$$

$$\overline{X}_2 = \cup_{i \in M \setminus I} X_i, \;\; \overline{Y}_2 = \cup_{i \in M \setminus J} Y_i, \;\; Z = \overline{X}_2 \cup \overline{Y}_2.$$

Since $S^{\uparrow}_{X} = S^{\uparrow\uparrow}_{\overline{X}_2 \overline{X}_1}$, we have that $Z^{\uparrow}_{\overline{X}_2}$ is a subposet of $S^{\uparrow}_{X}$ and $S^{\uparrow}_{X}$ uniquely determined by the subposets $Z^{\uparrow}_{\overline{X}_2} (= \overline{Y}_2 \cup \overline{X}_2)$, $\overline{Y}_1$, $\overline{X}_1$ and the following additional relations: $\overline{Y}_1 < \overline{X}_1, \overline{Y}_1 < \overline{X}_2, \overline{Y}_2 < \overline{X}_1$.

From the equality $Z^{\uparrow}_{\overline{X}_2} = Z^{\uparrow \ldots \uparrow}_{X_{k_1} \ldots X_{k_t}}$ where $\{k_1, \ldots, k_t\} = M \setminus I$ it follows that $Z^{\uparrow}_{\overline{X}_2}$ is a connected poset without nodes if $w(Z) > 2$. Then from the above form of the poset $S^{\uparrow}_{X}$ it follows that it has the same property as $Z^{\uparrow}_{\overline{X}_2}$, and therefore T is isomorphic to $S^{\uparrow}_{X}$ (and this case we have considered above). If $w(Z) = 2$, then $Z^{\uparrow}_{\overline{X}_2}$ also is a direct sum of two chains, but in the case when $\overline{X}_1 \cup \overline{Y}_1 \neq \varnothing$, the poset $S^{\uparrow}_{X}$ is connected without nodes too.

It is easy to see (taking into account all the above and remembering that $w(S) > 2$ and the subposet X is proper) that it suffices to consider the following cases:

1) $w(Z) = 0, \;\; w(\overline{X}_1) = 1, \;\; w(\overline{Y}_1) > 1;$

2) $w(Z) = 0, \;\; w(\overline{X}_1) > 1, \;\; w(\overline{Y}_1) = 1;$

3) $w(Z) = 0, \;\; w(\overline{X}_1) > 1, \;\; w(\overline{Y}_1) > 1;$

4) $w(Z) = 1,\ w(\overline{X}_1) = 0,\ w(\overline{Y}_1) > 1$, or $w(\overline{X}_1) > 1,\ w(\overline{Y}_1) = 0$;

5) $w(Z) = 1,\ w(\overline{X}_1) \neq 0,\ w(\overline{Y}_1) \neq 0$.

Recall that poset T is weak isomorphic to the poset $S^{\uparrow}_{X}$ and we need to prove that T is minimax isomorphic to S. This is done by Corollary 1 in cases 1), 2), by Lemma 2 in case 4); $S^{\uparrow}_{X}$ is connected without nodes in cases 3), 5).

1) $\Rightarrow$ 2)**.** It is sufficient to consider the case when T is minimax equivalent to S. Then by the main results of [3] either $T = S^{\uparrow}_{Z}$ for a lower subposet $Z \neq S$ (possible $Z = \varnothing$), or $T = (S^{\uparrow}_{Y})^{\uparrow}_{Z}$ where Y is a proper lower subposet of S, Z is a nonempty lower subposet of Y and $Z < S \setminus Y$. So we have to consider the second case.

Since S is a primitive, it follows from $Z < S \setminus Y$ that Z and $S \setminus Y$ belong to the same maximal chain L of S, and therefore the poset $T = (S^{\uparrow}_{Y}{}^{\uparrow}_{Z} = (S^{\downarrow}_{S\setminus Y})^{\uparrow}_{Z}$ is weakly isomorphic to the poset $S^{\uparrow}_{X}$ with X to be the lower subchain of L which has length $|Z| + |S \setminus Y|$. Indeed,

$$T = S' \coprod L' \text{ where } S' = [S\setminus Y < S\setminus L < Z] \text{ and } L' = L\setminus(Z\cup(S\setminus Y)),$$

$$S^{\uparrow}_{X} = S'' \coprod L'' \ \text{ where } S'' = [S \setminus L < X] \ \text{ and } \ L'' = L \setminus X.$$

But since the chains L' and L'' (possibly empty) are of the same length, and the nonempty chains $[S \setminus Y < Z] = S' \setminus (S \setminus L)$ and $X = S'' \setminus (S \setminus L)$, consisting of local nodes of T and $S^{\uparrow}_{X}$ respectively, are also isomorphic, we have that T and $S^{\uparrow}_{X}$ are weakly isomorphic, as claimed.

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Received by the editors: 12.10.2011
and in final form 24.12.2011.