

## On the one-side equivalence of matrices with given canonical diagonal form

Volodymyr Shchedryk

Communicated by M. Ya. Komarnytskyj

**ABSTRACT.** The simpler form of a matrix with canonical diagonal form  $diag(1, \dots, 1, \varphi, \dots, \varphi)$  obtained by the one-side transformation is determined.

Let  $R$  be an adequate ring [1] i.e. a commutative domain in which every finitely generated ideal is principal, and which further satisfies the following condition: for any  $a, c \in R$  with  $a \neq 0$ , one can write  $a = rs$  with  $(r, c) = 1$  and  $(s', c) \neq 1$  for any non unit divisor  $s'$  of  $s$ . Let  $A$  be an  $n \times n$  matrix over  $R$ . It is known [1] that there exist invertible matrices  $P, Q$ , such that

$$PAQ = diag(\varphi_1, \dots, \varphi_n) = \Phi. \quad (1)$$

The matrix  $\Phi$  is called the canonical diagonal form of the matrix  $A$ ,  $\varphi_i | \varphi_{i+1}$ ,  $i = 1, \dots, n-1$ . In solving of some matrix problems especially factorization of matrices [2,3], in description of all the Abelian subgroups [4], there emerges the necessity of finding all the non-associated matrices with canonical diagonal form given beforehand. Usual Hermite normal form does not approach to our purposes because it evaluates in the rough way and gives a possibility to describe non-associated matrices with set-up determinant only. That is why there emerges the necessity of building such form of matrix with respect to one sided transformation, giving a glance to which is enough to make a decision as for its canonical diagonal form. The equality (1) gives us a possibility to write matrix  $A$  in the following way  $A = P^{-1}\Phi Q^{-1}$ . Making changes in its right part we will have a new form

---

**2000 Mathematics Subject Classification:** 15A21.

**Key words and phrases:** adequate ring, canonical diagonal form, Hermite normal form, one-side equivalence of matrices, invariants, primitive matrices.

$P^{-1}\Phi$ . But this type of matrices is not a normal form of the matrix  $A$  as for the right side changes because the matrix  $P$  determined ambiguously.

By [2] the set  $\mathbf{P}_A$  of all invertible matrices which satisfies equation (1) has the form  $\mathbf{P}_A = G_\Phi P$ , where

$$G_\Phi = \{H \in GL_n(R) \mid H\Phi = \Phi H_1, H_1 \in GL_n(R)\}.$$

This set is a multiplicative group and if  $\det \Phi \neq 0$  consists of all invertible matrices of the form

$$H = \left\| \begin{array}{cccccc} h_{11} & h_{12} & \dots & h_{1,n-1} & h_{1n} \\ \frac{\varphi_2}{\varphi_1} h_{21} & h_{22} & \dots & h_{2,n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1} h_{n1} & \frac{\varphi_n}{\varphi_2} h_{n2} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n,n-1} & h_{nn} \end{array} \right\|.$$

Thus,  $\mathbf{P}_A$  is a left conjugacy class  $GL_n(R)$  with respect to the group  $G_\Phi$ . Therefore, in order that the matrix  $P^{-1}\Phi$  be a normal form of the matrix  $A$ , with respect to the transformation from the right, it is necessary either to choose a representative in the class  $G_\Phi P$  or, what is the same, indicate the normal form of the invertible matrices with respect to the action of the group  $G_\Phi$ . The present paper is devoted to the investigation of this question.

Let  $\Phi = E_t \oplus \varphi E_{n-t}$ ,  $\Phi_* = \varphi E_t \oplus E_{n-t}$ ,  $\varphi \neq 0$ ,  $1 \leq t < n$ , where  $E_t$  is the identity  $t \times t$  matrix. In this case, the group  $G_\Phi$  consists of all invertible matrices of the form

$$\left\| \begin{array}{cc} H_{11} & H_{12} \\ \varphi H_{21} & H_{22} \end{array} \right\|,$$

where  $H_{11}$  is a  $t \times t$  matrix. A matrix is called primitive if the greatest common divisor of minor of maximal order is equal to 1. The matrix  $A$  is called left associate to the matrix  $B$  if  $A = UB$ , where  $U \in GL_n(R)$ . This fact will be denoted  $A \overset{l}{\sim} B$ .

**Lemma 1.** *Let*

$$B = \left\| \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} \right\|$$

*be a primitive  $n \times (n - k + 1)$  matrix,  $t < k < n$ . The matrices  $B_1, B_3$  is  $t \times (n - k + 1)$ ,  $(n - k + 1) \times (n - k + 1)$  matrices, respectively. Let*

$$\Phi_* B \overset{l}{\sim} \left\| \begin{array}{c} 0 \\ 0 \\ B_3 \end{array} \right\|. \tag{2}$$

Then there exists a matrix  $H \in G_\Phi$  such that

$$HB = \left\| \begin{array}{c} B_1 \\ 0 \\ B_3 \end{array} \right\|.$$

*Proof.* Consider the matrix equation

$$XB_3 = \left\| \begin{array}{c} \varphi B_1 \\ B_2 \end{array} \right\|. \tag{3}$$

The matrix

$$\Phi_*B = \left\| \begin{array}{c} \varphi B_1 \\ B_2 \\ B_3 \end{array} \right\|$$

is extended matrix of equation (3). From (2) it follows that the invariant factors of the matrices  $\Phi_*B, B_3$  are equal. By Theorem 2 from [3, p. 218] equation (3) has the solution  $X = U = \left\| \begin{array}{c} U_1 \\ U_2 \end{array} \right\|$ , where  $U_1$  is a  $t \times (n - k + 1)$  matrix and  $U_2$  is a  $(k - t - 1) \times (n - k + 1)$  matrix. Then

$$\begin{aligned} & \left\| \begin{array}{ccc} E_t & 0 & -U_1 \\ 0 & E_{k-t-1} & -U_2 \\ 0 & 0 & E_{n-k+1} \end{array} \right\| \left\| \begin{array}{c} \varphi B_1 \\ B_2 \\ B_3 \end{array} \right\| = \\ & = \left\| \begin{array}{cc} E_{k-1} & -U \\ 0 & E_{n-k+1} \end{array} \right\| \left\| \begin{array}{c} \varphi B_1 \\ B_2 \\ B_3 \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \\ B_3 \end{array} \right\|. \end{aligned}$$

This implies that

$$\underbrace{\left\| \begin{array}{ccc} E_t & 0 & 0 \\ 0 & E_{k-t-1} & -U_2 \\ 0 & 0 & E_{n-k+1} \end{array} \right\|}_H \left\| \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} \right\| = \left\| \begin{array}{c} B_1 \\ 0 \\ B_3 \end{array} \right\|.$$

Observing that  $H \in G_\Phi$ , we conclude the proof of the lemma. □

**Lemma 2.** Let  $A$  be an  $n \times m$  matrix and  $H \in G_\Phi$ . Then

$$\Phi_*HA \stackrel{l}{\sim} \Phi_*A.$$

*Proof.* Since

$$H = \left\| \begin{array}{cc} H_{11} & H_{12} \\ \varphi H_{21} & H_{22} \end{array} \right\|,$$

where  $H_{11}$  is a  $t \times t$  matrix we have

$$\Phi_* H = \left\| \begin{array}{cc} \varphi H_{11} & \varphi H_{12} \\ \varphi H_{21} & H_{22} \end{array} \right\| = \left\| \begin{array}{cc} H_{11} & \varphi H_{12} \\ H_{21} & H_{22} \end{array} \right\| \Phi_* = H_1 \Phi_*$$

The matrix  $\Phi_*$  is nonsingular, so that  $\det H = \det H_1$  i.e. the matrix  $H_1$  is invertible. Consequently,

$$\Phi_* H A = H_1 \Phi_* A \stackrel{l}{\sim} \Phi_* A.$$

□

**Lemma 3.** *Let*

$$B_k = \left\| \begin{array}{cccccc} b_{11} & b_{12} & b_{13} & \dots & b_{1,n-k} & b_{1,n-k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{t1} & b_{t2} & b_{t3} & \dots & b_{t,n-k} & b_{t,n-k+1} \\ \hline b_{t+1,1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{k1} & 0 & 0 & \dots & 0 & 0 \\ \hline b_{k+1,1} & \beta_{k+1} & 0 & & 0 & 0 \\ b_{k+2,1} & b_{k+2,2} & \beta_{k+2} & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & & \beta_{n-1} & 0 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n,n-k} & \beta_n \end{array} \right\| = \left\| \begin{array}{cc} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|$$

be a primitive  $n \times (n - k + 1)$  matrix,  $t < k < n$ , and

$$\Phi_* B_k \stackrel{l}{\sim} \left\| \begin{array}{cc} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|, \tag{4}$$

where

$$D_{21} = \left\| 0 \dots 0 \beta_k \right\|^T.$$

Then there exists a matrix  $H \in G_\Phi$  such that

$$H B_k = \left\| \begin{array}{cc} B'_{11} & B'_{12} \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|. \tag{5}$$

*Proof.* Consider the equation

$$XB_{32} = \varphi B_{12}. \quad (6)$$

The equality

$$\Phi_* B_k = \left\| \begin{array}{cc} \varphi B_{11} & \varphi B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|,$$

is valid. From (4) we conclude that

$$\left\| \begin{array}{c} \varphi B_{12} \\ 0 \\ B_{32} \end{array} \right\| \sim \left\| \begin{array}{c} 0 \\ 0 \\ B_{32} \end{array} \right\|.$$

This implies that the invariant factors of the matrices  $B_{32}$ ,  $\left\| \begin{array}{c} \varphi B_{12} \\ B_{32} \end{array} \right\|$  are equal. By Theorem 2 from [3, p. 218], equation (6) has the solution  $X = U_{13}$ . Thus, the equality

$$\left\| \begin{array}{ccc} E_t & 0 & -U_{13} \\ 0 & E_{k-t} & 0 \\ 0 & 0 & E_{n-k} \end{array} \right\| \left\| \begin{array}{cc} \varphi B_{11} & \varphi B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\| = \left\| \begin{array}{cc} B'_{11} & 0 \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|,$$

holds, where

$$B'_{11} = \left\| b'_{11} \dots b'_{t1} \right\|^T.$$

By Lemma 2,

$$\left\| \begin{array}{cc} B'_{11} & B'_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\| \sim \left\| \begin{array}{cc} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|,$$

so that

$$(b'_{11}, \dots, b'_{t1}, b_{t+1.1}, \dots, b_{k1}) = \beta_k.$$

According to property 6 from [5], there exist  $v_1, \dots, v_k$  such that

$$v_1 b'_{11} + \dots + v_t b'_{t1} + v_{t+1} b_{t+1.1} + \dots + v_k b_{k1} = \beta_k,$$

and

$$(v_k, \varphi) = 1.$$

Let us complement the primitive row  $\left\| v_1 \dots v_k \right\|$  to an invertible matrix  $V_k$  in which this row is the last. Consider the invertible matrix

$$\left\| \begin{array}{cc} V_k & 0 \\ 0 & E_{n-k} \end{array} \right\| \left\| \begin{array}{ccc} E_t & 0 & -U_{13} \\ 0 & E_{k-t} & 0 \\ 0 & 0 & E_{n-k} \end{array} \right\| = \left\| \begin{array}{cc} V_k & U \\ 0 & E_{n-k} \end{array} \right\| = V.$$

Taking into account that  $\| v_1 \dots v_k u_{k+1} \dots u_n \|$  is the  $k$ -th row of this matrix, we obtain

$$\| v_1 \dots v_k u_{k+1} \dots u_n \| \Phi_* B_k = \| \beta_k \ 0 \ \dots \ 0 \|,$$

i.e.,

$$\begin{aligned} \| \varphi v_1 \dots \varphi v_t \ v_{t+1} \dots v_k \ u_{k+1} \dots u_n \| B_k &= \\ &= \| \beta_k \ 0 \ \dots \ 0 \|. \end{aligned}$$

Since

$$(v_1, \dots, v_k) = 1, (v_k, \varphi) = 1,$$

we have

$$(\varphi v_1, \dots, \varphi v_t, v_{t+1}, \dots, v_k) = 1.$$

It means that the matrix

$$F_k = \left\| \begin{array}{cccccccc} \varphi v_1 & \dots & \varphi v_t & v_{t+1} & \dots & v_k & u_{k+1} & \dots & u_n \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{array} \right\|$$

is primitive. By property 2 from [5], the matrix  $F_k$  can be complemented to an invertible matrix  $H_k = \left\| \begin{array}{c} * \\ F_k \end{array} \right\|$  which belongs to  $G_\Phi$ . Then

$$H_k B_k = \left\| \begin{array}{cccc} b'_{11} & b'_{12} & \dots & b'_{1,n-k+1} \\ \dots & \dots & \dots & \dots \\ b'_{t1} & b'_{t2} & \dots & b'_{t,n-k+1} \\ \hline b'_{t+1.1} & b'_{t+1.2} & \dots & b'_{t+1,n-k+1} \\ \dots & \dots & \dots & \dots \\ b'_{k-1.1} & b'_{k-1.2} & \dots & b'_{k-1,n-k+1} \\ \hline \beta_k & 0 & & 0 \\ b_{k+1.1} & \beta_{k+1} & & 0 \\ & & \dots & \\ b_{n1} & b_{n2} & & \beta_n \end{array} \right\| = \left\| \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \right\|.$$

By Lemma 2

$$\Phi_* H_k B_k \stackrel{l}{\sim} \left\| \begin{array}{cc} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \\ A_3 \end{array} \right\|.$$

According to Lemma 1, the group  $G_\Phi$  contain a matrix  $H'_k$  such that

$$H'_k H_k \left\| \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \right\| = \left\| \begin{array}{c} A_1 \\ 0 \\ A_3 \end{array} \right\|,$$

which has form (5). The proof is complete.  $\square$

Let us denote by  $K(f)$  the set of representatives of the conjugate classes of  $R/Rf, f \in R$ .

**Theorem 1.** Let  $B = \|b_{ij}\|_1^n = \left\| \begin{matrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{matrix} \right\|$  be an invertible matrix, where  $B_{11}$  is a  $t \times t$  matrix and

$$\Phi_* B \sim \left\| \begin{matrix} \beta_1 & 0 & 0 \\ * & \beta_2 & 0 \\ & & \ddots \\ * & * & \beta_n \end{matrix} \right\| \tag{7}$$

is the left Hermite normal form of the matrix  $\Phi_* B$ . Then the group  $G_\Phi$  contains a matrix  $H$  such that

$$HB = \left\| \begin{matrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{matrix} \right\|, \tag{8}$$

where

$$C_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & 0 \\ c_{t+2,t+1} & \beta_{t+2} & 0 \\ \vdots & & \ddots \\ c_{n,t+1} & c_{n,t+2} & \beta_n \end{matrix} \right\|,$$

$c_{ij} \in K(\beta_j), i = t + 2, t + 3, \dots, n, j = t + 1, t + 2, \dots, n - 1$ . The elements  $c_{ij}$  are uniquely determined and do not depend on the choice of the matrix  $H$ .

*Proof.* Using (6) we obtain

$$\Phi_* \| b_{1n} \ b_{2n} \ \dots \ b_{nn} \| ^T \sim \| 0 \ \dots \ 0 \ \beta_n \| ^T.$$

By Theorem 2 from [6], there exists a matrix  $H_n \in G_\Phi$  such that

$$H_n \| b_{1n} \ b_{2n} \ \dots \ b_{nn} \| ^T = \| b'_{1n} \ \dots \ b'_{n-1,n} \ \beta_n \| ^T.$$

According to Lemma 2

$$\begin{aligned} & \Phi_* \| b'_{1n} \ \dots \ b'_{n-1,n} \ \beta_n \| ^T = \\ & = \| \varphi b'_{1n} \ \dots \ \varphi b'_{tn} \ b'_{t+1,n} \ \dots \ b'_{n-1,n} \ \beta_n \| ^T \sim \\ & \sim \| 0 \ \dots \ 0 \ \beta_n \| ^T. \end{aligned}$$

Therefore  $b'_{in} = \beta_n d_i$ ,  $i = t + 1, t + 2, \dots, n - 1$ . Then

$$\left( E_{n-t} \oplus \begin{pmatrix} 1 & 0 & 0 & -d_{t+1} \\ 0 & 1 & 0 & -d_{t+2} \\ & & \ddots & \vdots \\ 0 & 0 & 1 & -d_{n-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) H_n B = B_n.$$

Using Lemmas 2 and 3 in consequently to the last two columns of the matrix  $B_n$ , to the last three columns of the derived matrix and so fors we get  $H_{t+1} \in G_\Phi$  such that

$$H_{t+1} B = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where

$$D_{22} = \begin{pmatrix} \beta_{t+1} & 0 & 0 \\ d_{t+2,t+1} & \beta_{t+2} & 0 \\ \vdots & & \ddots \\ d_{n,t+1} & d_{n,t+2} & \beta_n \end{pmatrix}.$$

There exists a lower unitriangular matrix  $U$  such that

$$UD_{22} = \begin{pmatrix} \beta_{t+1} & 0 & 0 \\ c_{t+2,t+1} & \beta_{t+2} & 0 \\ \vdots & & \ddots \\ c_{n,t+1} & c_{n,t+2} & \beta_n \end{pmatrix}$$

is the left Hermite normal form of the matrix  $D_{22}$ , i.e.,  $c_{ij} \in K(\beta_j)$ ,  $i = t + 2, t + 3, \dots, n, j = t + 1, t + 2, \dots, n - 1$ . The the matrix  $(E_t \oplus U)H_{t+1}B$  has the form (8) and  $(E_t \oplus U)H_{t+1}B \in G_\Phi$ .

We will show the uniqueness of the elements  $c_{ij}$ . Let  $H_1 \in G_\Phi$  and

$$H_1 B = \begin{pmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{pmatrix},$$

where

$$C'_{22} = \begin{pmatrix} \beta_{t+1} & 0 & 0 \\ c'_{t+2,t+1} & \beta_{t+2} & 0 \\ \vdots & & \ddots \\ c'_{n,t+1} & c'_{n,t+2} & \beta_n \end{pmatrix},$$

$c'_{ij} \in K(\beta_j)$ ,  $i = t + 2, t + 3, \dots, n, j = t + 1, t + 2, \dots, n - 1$ . Concerning the proof of Lemma 3 we conclude that there exists an upper unitriangular



matrix  $U$  such that

$$U\Phi_* \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|. \quad (9)$$

Since

$$H \left\| \begin{array}{c} B_{12} \\ B_{22} \end{array} \right\| = \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\|$$

and

$$H_1 \left\| \begin{array}{c} B_{12} \\ B_{22} \end{array} \right\| = \left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\|,$$

we obtain

$$\left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\| = H_2 \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\|, H_2 = H_1 H^{-1} \in G_\Phi.$$

Rewrite (9) us

$$U\Phi_* H_2^{-1} H_2 \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|.$$

By Lemma 2

$$\Phi_* H_2^{-1} = H_3 \Phi_*, H_3 \in GL_n(R).$$

Thus,

$$UH_3 \Phi_* \left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|.$$

Let

$$(UH_3)^{-1} = \left\| \begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array} \right\|,$$

where  $V_{11}$  is a  $t \times t$  matrix. Then

$$\left\| \begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array} \right\| \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} \varphi C'_{12} \\ C'_{22} \end{array} \right\|,$$

i.e.,  $V_{22}C_{22} = C'_{22}$ . Since  $|C_{22}| = |C'_{22}|$ , the matrix  $V_{22}$  is invertible. Hence, the matrices  $C_{22}$ ,  $C'_{22}$  are left associated. Therefore their left Hermite normal forms are equal. Remark that the matrices  $C_{22}$ ,  $C'_{22}$  are precisely the left Hermite form. This finished the proof.  $\square$

**Theorem 2.** Let  $A = P^{-1}\Phi Q^{-1}$ , where  $\Phi = E_t \oplus \varphi E_{n-t}$ . Then there exists an invertible matrix  $U$  such that

$$AU = V^{-1}\Phi,$$

where  $V$  is the matrix of the form (8).

*Proof.* By Theorem 1, there exists  $H \in G_\Phi$  such that the matrix  $HP = V$  has the form (8). Then

$$A = P^{-1}\Phi Q^{-1} = (HP)^{-1}(H\Phi)Q^{-1} = V^{-1}\Phi H_1 Q^{-1}.$$

Since the matrix  $H_1$  is invertible,  $U = QH_1^{-1}$  is desired matrix.  $\square$

### References

- [1] Helmer O. *The elementary divisor theorem for certain rings without chain condition*, Bull. Amer. Math. Soc., 1943, Vol.49, pp. 225 - 236.
- [2] Shchedryk V.P. *Structure and properties of matrix divisors over commutative elementary divisor domain* Math. studii, 1998, Vol.10(2), pp. 115-120.
- [3] Kazimirskij P.S. *Decomposition of matrix polynomials into factors* Kyiv, Naukova dumka, 1981. - 224 p.
- [4] Bhowmik G., Ramare O. *Algebra of matrix arithmetic* Journal of Algebra, 1998, Vol.210, pp.194 - 215.
- [5] Shchedryk V.P. *Some determinant properties of primitive matrices over Bezout B-domain* Algebra and Discrete Mathematics, 2005, №2, pp. 46-57.

### CONTACT INFORMATION

**Volodymyr  
Shchedryk**

Pidstryhach Institute for Applied Problems of  
mechanics and Mathematics NAS of Ukraine, 3b  
Naukova Str., L'viv, 79060  
*E-Mail:* shchedrykv@ukr.net

Received by the editors: 16.09.2011  
and in final form 21.12.2011.