

Generalized multiplicative bases for one-sided bimodule problems

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ABSTRACT. We consider a class of normal bimodule problems satisfying some structure, triangularity and finiteness conditions. For a bimodule problem from this class we construct explicitly an analogue of multiplicative basis which we call quasi multiplicative.

Introduction

A classification of bimodule problems of finite and tame representation type and their indecomposable representations and description of their representation categories belongs to important problems of representation theory [6, 7, 8]. A useful tool for a solution of the finiteness problem is so called “covering method” ([5, 3]), which is especially effective when the basis of associative algebra ([4]) or bimodule problem ([9]) is multiplicative. We give a generalization of the notion of a scalarly multiplicative basis from [9] and apply it for a wider class of bimodule problems. For a faithful bimodule problem from our class we construct explicitly the quasi multiplicative basis using mainly geometrical techniques.

1. Preliminaries

Let \mathbb{k} be algebraically closed field. Unless otherwise stated, all the categories we consider are the categories over \mathbb{k} , all morphism spaces are finite dimensional, and all functors are \mathbb{k} -linear. A category \mathbf{K} is called *local*,

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provided for every $X \in \text{Ob } \mathbf{K}$ the endomorphism algebra $\mathbf{K}(X, X)$ is local, and *regular*, if, in addition, every invertible morphism is automorphism. A category \mathbf{K} is called *fully additive* or *Krull-Schmidt category* if \mathbf{K} is a category with finite direct sums and every idempotent from \mathbf{K} splits, i. e. it has kernel and cokernel. A full subcategory $\mathbf{K}_0 \subset \mathbf{K}$ will be called *an additive skeleton* of \mathbf{K} , provided \mathbf{K}_0 is regular and every $X \in \text{Ob } \mathbf{K}$ is isomorphic to a finite direct sum of objects from \mathbf{K}_0 .

For a local category \mathbf{K} and for every $X \in \text{Ob } \mathbf{K}$ there exists the decomposition $\mathbf{K}(X, X) = \mathbb{k}\mathbf{1}_X \oplus \text{Rad } X$, where $\text{Rad } X$ is the Jacobson radical of the algebra $\mathbf{K}(X, X)$. If \mathbf{K} is regular, then we denote by $\text{Rad } \mathbf{K}$ the *radical* of \mathbf{K} , i. e. an ideal in \mathbf{K} such that $\text{Rad } \mathbf{K}(X, Y) = \mathbf{K}(X, Y)$ for $X \neq Y$, and $\text{Rad } \mathbf{K}(X, X) = \text{Rad } X$, $X, Y \in \text{Ob } \mathbf{K}$.

Let \mathbf{V} be a \mathbf{K} -bimodule ([1]). A category \mathbf{K} (a bimodule \mathbf{V}) is called *locally finite dimensional*, if for any $X \in \text{Ob } \mathbf{K}$ the spaces $\bigoplus_{Y \in \text{Ob } \mathbf{K}} \mathbf{K}(X, Y)$ and $\bigoplus_{Y \in \text{Ob } \mathbf{K}} \mathbf{K}(Y, X)$ ($\bigoplus_{Y \in \text{Ob } \mathbf{K}} \mathbf{V}(X, Y)$ and $\bigoplus_{Y \in \text{Ob } \mathbf{K}} \mathbf{V}(Y, X)$) are finite dimensional, and *finite dimensional*, provided the spaces $\bigoplus_{X, Y \in \text{Ob } \mathbf{K}} \mathbf{K}(X, Y)$ ($\bigoplus_{X, Y \in \text{Ob } \mathbf{K}} \mathbf{V}(X, Y)$) are finite dimensional.

Given a category \mathbf{K} , we denote by $\text{add } \mathbf{K}$ an *additive hull* of \mathbf{K} , i. e. a minimal fully additive category which contains \mathbf{K} . For a \mathbf{K} -bimodule \mathbf{V} , we denote by $\text{add } \mathbf{V}$ the corresponding $\text{add } \mathbf{K}$ -bimodule.

A pair $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ consisting of a category \mathbf{K} and a \mathbf{K} -bimodule \mathbf{V} is called a *bimodule problem* over \mathbf{K} or shortly bimodule problem. A bimodule problem \mathcal{A} will be called *normal*, provided the category \mathbf{K} is regular, and both \mathbf{K} and \mathbf{V} are locally finite dimensional. All the bimodule problems we will consider are assumed to be normal. Given some $S \subset \text{Ob } \mathbf{K}$ denote by \mathbf{K}_S the full subcategory of \mathbf{K} with $\text{Ob } \mathbf{K}_S = S$, and by \mathbf{V}_S the subbimodule $\mathbf{V}_S = \mathbf{K}_S \mathbf{V} \mathbf{K}_S$. A bimodule problem $\mathcal{A}_S = (\mathbf{K}_S, \mathbf{V}_S)$ is called the *restriction* of \mathcal{A} to S .

For a bimodule problem $\mathcal{A} = (\mathbf{K}, \mathbf{V})$, a *representation* M of \mathcal{A} is a pair $M = (M_{\mathbf{K}}, M_{\mathbf{V}})$, where $M_{\mathbf{K}} \in \text{Ob } \text{add } \mathbf{V} = \text{Ob } \text{add } \mathbf{K}$ and $M_{\mathbf{V}} \in \text{add } \mathbf{V}(M_{\mathbf{K}}, M_{\mathbf{K}})$. If M, N are two representations of \mathcal{A} , then a *morphism* f from M to N is a morphism $f \in \text{add } \mathbf{K}(M_{\mathbf{K}}, N_{\mathbf{K}})$ such that $N_{\mathbf{V}} \cdot f - f \cdot M_{\mathbf{V}} = 0$. The composition of morphisms and the unit morphisms in the representation category $\text{rep } \mathcal{A}$ and in the category $\text{add } \mathbf{K}$ coincide.

Given two bimodule problems $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ and $\mathcal{A}' = (\mathbf{K}', \mathbf{V}')$, a *morphism* of bimodule problems $\theta : \mathcal{A} \rightarrow \mathcal{A}'$ is a pair $\theta = (\theta_0, \theta_1)$, where $\theta_0 : \mathbf{K} \rightarrow \mathbf{K}'$ is a \mathbb{k} -functor, $\theta_1 : \mathbf{V} \rightarrow \mathbf{V}'$ is a \mathbf{K} -bimodule morphism with the \mathbf{K} -bimodule structure on \mathbf{V}' induced by θ_0 ([1]).

Let $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ be a normal bimodule problem. Bigraph $\Sigma (= \Sigma_{\mathcal{A}}) = (\Sigma_0, \Sigma_1)$ is called a *basis* of the bimodule problem \mathcal{A} , if $\Sigma_0 = \text{Ob } \mathbf{K}$,

$\Sigma_1^0(X, Y)$ is a basis of $\mathbf{V}(X, Y)$, and $\Sigma_1^1(X, Y)$ is a basis of $\text{Rad } \mathbf{K}(X, Y)$, $X, Y \in \text{Ob } \mathbf{K}$. For all $x, y \in \Sigma_1$ such that the product xy is not specified, we assume $xy = 0$. A bimodule problem is called connected, if its bigraph is connected.

Let \mathbf{V} be a \mathbf{K} -bimodule. We say that $x \in \text{Rad } \mathbf{K}(X, Y)$ *annihilates the bimodule* \mathbf{V} , if $xa = 0, bx = 0$ for any $Z \in \text{Ob } \mathbf{K}, a \in \mathbf{V}(Z, X), b \in \mathbf{V}(Y, Z)$. The ideal of the category \mathbf{K} consisting of all elements annihilating the bimodule \mathbf{V} is called the *annihilator* of \mathbf{V} and is denoted by $\text{Ann}_{\mathbf{K}}(\mathbf{V})$. A bimodule \mathbf{V} is called *faithful* provided $\text{Ann}_{\mathbf{K}}(\mathbf{V}) = 0$. We call a bimodule problem $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ *faithful*, if the bimodule \mathbf{V} is faithful. For a bimodule problem \mathcal{A} , a *faithful part* of \mathcal{A} is defined as the faithful bimodule problem $\mathcal{A}_{\text{red}}, \mathcal{A}_{\text{red}} = (\mathbf{K}_{\text{red}}, \mathbf{V})$, where $\mathbf{K}_{\text{red}} = \mathbf{K} / \text{Ann}_{\mathbf{K}} \mathbf{V}$. Remark that a restriction of faithful bimodule problem may not be faithful. Faithful part of restriction of a bimodule problem is called *faithful restriction*.

Let $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ be a bimodule problem, and $\mathbf{V}' \subset \mathbf{V}$ be a subbimodule of \mathbf{V} such that $\mathbf{V}' \neq 0, \mathbf{V}' \neq \mathbf{V}$. Denote by \succ the minimal relation of (strict) partial order on the set of bimodule problems such that $\mathcal{A} \succ \mathcal{A}'$ and $\mathcal{A} \succ \mathcal{A}''$, where $\mathcal{A}' \simeq (\mathbf{K}, \mathbf{V}')$, $\mathcal{A}'' \simeq (\mathbf{K}, \mathbf{V}/\mathbf{V}')$, and $\mathcal{A} \succ \mathcal{A}_S$ for any proper subset $S \subset \text{Ob } \mathbf{K}$. Similarly we denote by \sim the minimal equivalence such that for every ideal $\mathcal{I} \subset \mathbf{K}, \mathcal{I}\mathbf{V} = \mathbf{V}\mathcal{I} = 0$, holds $\mathcal{A} \sim \mathcal{A}_{\mathcal{I}} = (\mathbf{K}/\mathcal{I}, \mathbf{V})$, and if $\mathcal{A} \simeq \mathcal{B}$ then $\mathcal{A} \sim \mathcal{B}$. The transitive closure of \succ and \sim defines a preorder on the set of bimodule problems, which defines the strict order, denoted again by \succ . The relations \succ and \sim are obviously defined on the set of isoclasses of bimodule problems. If for bimodule problems \mathcal{A}, \mathcal{B} holds $\mathcal{A} \succ \mathcal{B}$ then we say \mathcal{B} is a *subproblem* of \mathcal{A} .

Let $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ be a normal bimodule problem, $\mathbf{R} = \text{Rad } \mathbf{K}$, Σ be a basis of \mathcal{A} . Radical \mathbf{R} is called *nilpotent* if $\mathbf{R}^n = 0$ for some $n \in \mathbb{N}$. The integer N is called the *nilpotence degree* of \mathcal{A} if $\mathbf{R}^N = 0$, but $\mathbf{R}^{N-1} \neq 0$. Denote by $\mathbf{V}_i = \mathbf{R}^{i-1}\mathbf{V}, i = 1, \dots, N$. We have two filtrations

$$\mathbf{R} \supset \mathbf{R}^2 \supset \dots \supset \mathbf{R}^{N-1} \supset 0, \quad \mathbf{V}_1 \supset \mathbf{V}_2 \supset \dots \supset \mathbf{V}_N \supset 0. \quad (1)$$

Remark that all inclusions in (1) are strict, and for \mathcal{A} faithful $\mathbf{V}_i \neq 0$ for all $i = 1, \dots, N$.

The map $h : \mathbf{R} \cup \mathbf{V} \rightarrow \mathbb{N}$ such that $h(x) = \max\{i \in \mathbb{N} \mid x \in \mathbf{R}^i \cup \mathbf{V}_i\}$ is called the *height* of an element. Let $h(0) = \infty$. Then $h(xy) \geq h(x) + h(y)$ and $h(x+y) \geq \min\{h(x), h(y)\}$ for $x, y \in \mathbf{R} \cup \mathbf{V}$. Let $\Sigma_1^{k(i)} = \Sigma_1^k \cap h^{-1}(i), i = 1, \dots, N, k = 0, 1$. Clearly, the set $\{\Sigma_1^{k(i)}, i = 1, \dots, N\}$ is a partition of $\Sigma_1^k, k = 0, 1$.

Definition 1. *The basis Σ of bimodule problem \mathcal{A} we call *triangled* (with respect to the filtration (1)), if $\bigcup_{l=i}^{N-1} \Sigma_1^{1(l)}$ is a basis of \mathbf{R}^i and $\bigcup_{l=i}^N \Sigma_1^{0(l)}$ is a basis of $\mathbf{V}_i, i = 1, \dots, N$.*

Lemma 1 ([6]). *Every normal finite dimensional bimodule problem \mathcal{A} with the nilpotent radical has a triangled basis.*

Remark 1. For a triangled basis Σ of a normal bimodule problem $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$, the following properties hold:

1) $\Sigma_1^{1(i)}$ is a basis of $\mathbb{R}^i/\mathbb{R}^{i+1}$ modulo \mathbb{R}^{i+1} , $i = 1, \dots, N-1$, $\Sigma_1^{0(i)}$ is a basis of $\mathbb{V}_i/\mathbb{V}_{i+1}$ modulo \mathbb{V}_{i+1} , $i = 1, \dots, N$;

2) for $x \in \mathbb{R} \cup \mathbb{V}$ the equality $x = \sum_{y \in \Sigma_1} \lambda_y y$, $\lambda_y \in \mathbb{k}$, implies $h(y) \geq h(x)$

for any $y \in \Sigma_1$ with $\lambda_y \neq 0$.

Definition 2. *A normal bimodule problem \mathcal{A} with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$ we call admitted if the set $\text{Ob } \mathbb{K}$ can be decomposed to a disjoint union $\text{Ob } \mathbb{K} = \text{Ob } \mathbb{K}^+ \cup \text{Ob } \mathbb{K}^-$ such that inequality $\mathbb{V}(X, Y) \neq 0$ implies $X \in \text{Ob } \mathbb{K}^-$, $Y \in \text{Ob } \mathbb{K}^+$, and $\mathbb{R}(X, Y) \neq 0$ implies $X, Y \in \text{Ob } \mathbb{K}^+$. The property of a bimodule problem \mathcal{A} to be admitted depends only on the bigraph $\Sigma_{\mathcal{A}}$, therefore we will use the notation $\Sigma_0^+ = \text{Ob } \mathbb{K}^+$ and $\Sigma_0^- = \text{Ob } \mathbb{K}^-$.*

Let $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ be an admitted bimodule problem with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$ and a triangled basis Σ .

Remark 2. There are the decompositions

$$\mathbb{V}_i = \bigoplus_{E \in \Sigma_0^-, A \in \Sigma_0^+} \mathbb{V}_i(E, A), \quad \mathbb{R}^i = \bigoplus_{A, B \in \Sigma_0^+} \mathbb{R}^i(A, B), \quad i = 1, \dots, N$$

of \mathbb{k} -vector spaces with the multiplications

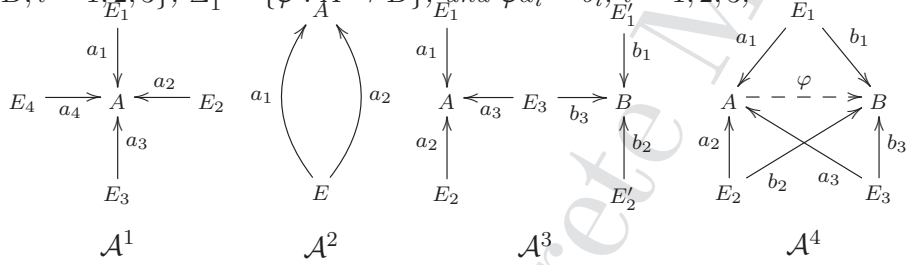
$$\begin{aligned} \mathbb{R}^i(A, B) \times \mathbb{V}_j(E, A) &\rightarrow \mathbb{V}_{i+j}(E, B), & A, B \in \Sigma_0^+, E \in \Sigma_0^-, \\ \mathbb{R}^i(B, C) \times \mathbb{R}^j(A, B) &\rightarrow \mathbb{R}^{i+j}(A, C), & A, B, C \in \Sigma_0^+. \end{aligned}$$

Definition 3. *For $a, b \in \mathbb{V}$, we say that $a <_R b$ (or simply $a < b$), if $b \in \mathbb{R}a$. In other words, $a < b$ if and only if $ra = b$ for some $r \in \mathbb{R}(A, B)$, in this case we write $a <_r b$. Obviously, the order $<_R$ on \mathbb{V} is non-reflexive and transitive. Two elements $a, b \in \mathbb{V}$ are called comparable is either $a <_R b$ or $b <_R a$. For $A \in \Sigma_0^+$ let $\text{ord } A = \sum_{E \in \Sigma_0^-} \dim_{\mathbb{k}} \mathbb{V}(E, A) = \sum_{E \in \Sigma_0^-} |\Sigma_1(E, A)|$. It is clear, that $h(a) < h(b)$ for any $a, b \in \Sigma_1^+$, $a < b$, i. e. h is monotonous with respect to $<_R$.*

Lemma 2. *The admitted bimodule problems $\mathcal{A}^k = (\mathbb{K}^k, \mathbb{V}^k)$, $k = 1, \dots, 4$, given respectively by the following bases (see bigraphs below)*

1) $\Sigma_0^+ = \{A\}$, $\Sigma_0^- = \{E_1, E_2, E_3, E_4\}$, $\Sigma_1^0 = \{a_i : E_i \rightarrow A, i = 1, \dots, 4\}$, $\Sigma_1^+ = \emptyset$;

- 2) $\Sigma_0^+ = \{A\}$, $\Sigma_0^- = \{E\}$, $\Sigma_1^0 = \{a_1, a_2 : E \rightarrow A\}$, $\Sigma_1^1 = \emptyset$;
 3) $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, E'_1, E_2, E'_2, E_3 = E'_3\}$, $\Sigma_1^0 = \{a_i : E_i \rightarrow A, b_i : E'_i \rightarrow B, i = 1, 2, 3\}$, $\Sigma_1^1 = \emptyset$;
 4) $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, E_2, E_3\}$, $\Sigma_1^0 = \{a_i : E_i \rightarrow A, b_i : E_i \rightarrow B, i = 1, 2, 3\}$, $\Sigma_1^1 = \{\varphi : A \rightarrow B\}$, and $\varphi a_i = b_i, i = 1, 2, 3$;



are of strictly unbounded representation type (see [6]).

Definition 4. Define the class \mathcal{C} of admitted bimodule problems $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$ and a triangled basis Σ such that for any $E \in \Sigma_0^-$, $A, B \in \Sigma_0^+$, $A \neq B$:

- 1) $\text{ord } A \leq 3$;
- 2) any $a_1, a_2 \in \Sigma_1^0(E, A)$ are comparable;
- 3) if $\text{ord } A = \text{ord } B = 3$, then any $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$ are comparable;
- 4) if $\varphi \in \mathbb{R}(A, B)$, then $\sum_{E \in \Sigma_0^-} \dim_{\mathbb{k}} \varphi \mathbb{V}(E, A) < 3$.

If one of the conditions 1)–4) does not hold, then, using Lemma 2, it is easy to check that the bimodule problem is of strictly unbounded representation type.

Let $\mathcal{A} \in \mathcal{C}$. For any $x \in \mathbb{R} \cup \mathbb{V}$ there is a basis decomposition $x = \sum_{y \in \Sigma_1} \lambda_y y$, where almost all $\lambda_y \in \mathbb{k}$ equals 0. Denote by $\text{con}_y x = \lambda_y$ the content of y in x . Two nonzero elements $x, y \in \mathbb{R} \cup \mathbb{V}$ are called *collinear* if $\mathbb{k}^* x = \mathbb{k}^* y$, in this case we write $x \parallel y$. Given $a, b \in \Sigma_1^0$ we shall write $a \underset{\Sigma}{\leq} b$ and $a \underset{\xi}{\leq} b$ if there exists $\xi \in \Sigma_1^1$ such that $b \parallel \xi a$.

Definition 5. Given $A, B \in \text{Ob } \Sigma_0^+$, denote by $\mathcal{A}^{(A, B)} = (\mathbb{K}^{(A, B)}, \mathbb{V}^{(A, B)})$ the restriction of the bimodule problem \mathcal{A} to the set

$$S_{A, B} = \{A, B\} \cup \{E \in \Sigma_0^- \mid \mathbb{V}(E, A) \neq 0 \text{ or } \mathbb{V}(E, B) \neq 0\}.$$

Denote by $\Sigma^{(A, B)} \subset \Sigma$ the basis of $\mathcal{A}^{(A, B)}$ which is the restriction of Σ to $\mathcal{A}^{(A, B)}$. We will write $\mathcal{A}^{(A)}$ instead of $\mathcal{A}^{(A, A)}$ in the case $A = B$. The bimodule problem $\mathcal{A}^{(A, B)}$ inherits the triangled structure from \mathcal{A} (and may have the proper one).

Remark 3. Let $A, B \in \Sigma_0^+$. If \mathcal{A} is faithful, then bimodule problem $\mathcal{A}^{(A,B)}$ is faithful as well. Moreover, $R(A, B) = \text{Rad}(\mathbb{K}^{(A,B)})(A, B)$. This fact follows from the equality

$$\text{Ann}_{\mathbb{K}}(\mathbb{V}) = \bigcup_{A, B \in \text{Ob } \Sigma_0^+} \text{Ann}_{\mathbb{K}^{(A,B)}}(\mathbb{V}^{(A,B)}).$$

2. Quasi multiplicative basis. Main result

Let $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ be a bimodule problem from \mathcal{C} with nilpotent radical $R = \text{Rad } \mathbb{K}$ and a triangled basis Σ .

A change of basis $\Sigma_1 = \Sigma_1^0 \cup \Sigma_1^1$ consists of a family of changes of bases in all $\mathbb{V}(E, A)$ (the change of Σ_1^0) and in all $R(A, B)$ (the change of Σ_1^1), $A, B \in \Sigma_0^+$, $E \in \Sigma_0^-$. These new bases gives the new basis Σ' of \mathcal{A} . The change of basis from Σ to Σ' we call *triangled*, provided both Σ and Σ' are triangled.

Definition 6. Let $x, y \in \Sigma_1^i(X, Y)$, $i = 0, 1$. For $\lambda_x \in \mathbb{k}$, $\lambda_y \in \mathbb{k}^*$, the change of basis from Σ to Σ' such that $y' = \lambda_y y + \lambda_x x$, and $z' = z$ for all $z \in \Sigma_1 \setminus \{y\}$ we call elementary. An elementary change is called correct, if $h(x) \geq h(y)$. Denote by $\mathfrak{C}_\lambda(x, y)$ and $\mathfrak{C}_\lambda(y)$ elementary changes $y' = y + \lambda x$, $\lambda \in \mathbb{k}$, and $y' = \lambda y$, $\lambda \in \mathbb{k}^*$, respectively.

The change of basis from Σ to Σ' is called standard if it is the superposition of correct elementary changes. We use only standard changes of basis. Usually we do not modify the notations of basis and its elements after change, and write Σ and y instead of Σ' and y' respectively.

Definition 7. For $A, B \in \Sigma_0^+$, $E \in \Sigma_0^-$, $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$ let

$$\begin{aligned} \mathbb{S}(a, b) &= \{\xi \in \Sigma_1^1(A, B) \mid \text{con}_b(\xi a) \neq 0\}, \\ \mathbb{C}(a, b) &= \{\xi \in \Sigma_1^1(A, B) \mid \xi a \parallel b\} \subset \mathbb{S}(a, b). \end{aligned} \tag{2}$$

A pair (a, b) is called adjusted if $\mathbb{S}(a, b) = \mathbb{C}(a, b)$. For any $\varphi \in \Sigma_1^1$, denote $\mathbb{P}_\varphi = \{(a, b) \in \Sigma_1^0 \times \Sigma_1^0 \mid \varphi \in \mathbb{S}(a, b)\}$. A $\varphi \in \Sigma_1^1$ is called single provided $\mathbb{P}_\varphi = \{(a, b)\}$ and the pair (a, b) is adjusted, and joint if $\mathbb{P}_\varphi = \{(a_1, b_1), (a_2, b_2)\}$ with $a_1 \neq a_2$, $b_1 \neq b_2$, and the pairs (a_1, b_1) , (a_2, b_2) are adjusted. Obviously, if \mathcal{A} is faithful, $\mathbb{P}_\varphi \neq \emptyset$ for any $\varphi \in \Sigma_1^1$.

Remark 4. Let $A, B \in \Sigma_0^+$, $A \neq B$, $\varphi \in \Sigma_1^1(A, B)$. If three adjusted pairs (a_i, b_i) , $i = 1, 2, 3$, lies in \mathbb{P}_φ , then $\mathcal{A} \notin \mathcal{C}$ by Definition 4, item 4).

Lemma 3. Let $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$, and $a < b$.

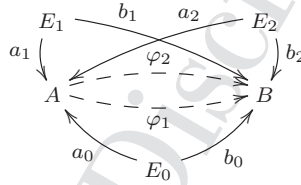
1) If $\varphi, \psi \in \mathbb{S}(a, b)$ and $h(\varphi) \geq h(\psi)$, then there is a correct elementary change of basis $\mathfrak{C}_\lambda(\varphi, \psi)$, $\lambda \in \mathbb{k}^*$, such that $\mathbb{S}'(a, b) = \mathbb{S}(a, b) \setminus \{\psi\}$.

2) There is a standard change of basis from Σ_1 to Σ'_1 on $\Sigma_1^1(A, B)$, such that $|S'(a, b)| = 1$.

Proof. Since $h(\varphi) \geq h(\psi)$, then the elementary change of basis $\psi' = \psi - \frac{\text{con}_b(\psi a)}{\text{con}_b(\varphi a)}\varphi$ is correct and leads to the condition $S'(a, b) = S(a, b) \setminus \{\psi\}$. The second item follows from the first by induction algorithm. \square

We call elements $\varphi_1, \varphi_2 \in \Sigma_1^1(A, B)$, $A, B \in \Sigma_0^+$, $A \neq B$, joint parallel if $\text{ord}(A) = \text{ord}(B) = 3$, and there are $E_0, E_1, E_2 \in \Sigma_0^-$, $a_i \in \Sigma_1^0(E_i, A)$, $b_i \in \Sigma_1^0(E_i, B)$, $i = 0, 1, 2$, such that the following hold:

- 1) $C(a_0, b_0) = \{\varphi_1, \varphi_2\}$, $C(a_i, b_i) = \{\varphi_i\}$, $i = 1, 2$;
- 2) $P_{\varphi_i} = \{(a_0, b_0), (a_i, b_i)\}$, $i = 1, 2$ (see diagram below).

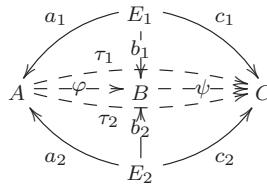


Here some of vertices E_0, E_1, E_2 may be equal, but the arrows a_0, a_1, a_2 (b_0, b_1, b_2) are pairwise different.

Given $a_1, \dots, a_t \in \Sigma_1$, define $\mathbb{k}^* \langle a_1, \dots, a_t \rangle = \left\{ \sum_{i=1}^t \lambda_i a_i \mid \lambda_i \in \mathbb{k}^* \right\}$.

Definition 8. We say that the multiplication rule holds on \mathcal{A} if given any $\varphi, \psi \in \Sigma_1^1$ with $\psi \neq 0$, one of the following conditions holds:

- 1) there is $\tau \in \Sigma_1^1$ s.t. $\psi \varphi \parallel \tau$;
- 2) φ, ψ are joint, and there are single $\tau_1, \tau_2 \in \Sigma_1^1$ such that $\psi \varphi \in \mathbb{k}^* \langle \tau_1, \tau_2 \rangle$, and there are $E_1, E_2 \in \Sigma_0^-$, with, possibly, $E_1 = E_2$, $A, B, C \in \Sigma_0^+$, where two of the vertices A, B, C may be equal, $a_i \in \Sigma_1^0(E_i, A)$, $b_i \in \Sigma_1^0(E_i, B)$, $c_i \in \Sigma_1^0(E_i, C)$, such that $\varphi a_i \parallel b_i$, $\psi b_i \parallel c_i$, $i = 1, 2$, and $\tau_j a_i \parallel \delta_{ij} c_i$, $i, j = 1, 2$ where δ_{ij} is the Kronecker delta:



Definition 9. The triangled basis Σ of a bimodule problem $\mathcal{A} \in \mathcal{C}$ is called quasi multiplicative if the following properties hold:

- 1) Any pair $(a, b) \in \Sigma_1^0 \times \Sigma_1^0$ with $S(a, b) \neq \emptyset$ is adjusted.
- 2) Any $\varphi \in \Sigma_1^1$ with $P_\varphi \neq \emptyset$ is either single or joint.

3) For any $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$, the inequality $|\mathbf{C}(a, b)| \leq 2$ holds. If $\mathbf{C}(a, b) = \{\varphi_1, \varphi_2\}$, then φ_1, φ_2 are joint parallel.

4) The multiplication rule holds on \mathcal{A} .

It is an approximation of the notion of multiplicative basis ([4], [9]).

Theorem 1 (Main result). *Let \mathcal{A} be a faithful connected finite dimensional bimodule problem from class \mathcal{C} with nilpotent radical. Then there exists a standard change of triangled basis to a quasi multiplicative one.*

The rest of this paper is devoted to the proof of Theorem 1.

3. Bimodule problems with $|\Sigma_0^+| = 1$

Proposition 1. *Let $\mathcal{A} = (K, V)$ be a faithful connected bimodule problem from \mathcal{C} with nilpotent radical $R = \text{Rad } K$ and a triangled basis Σ such that $\Sigma_0^+ = \{A\}$. Then there exists a standard change of basis such that Σ becomes multiplicative and one the following conditions hold:*

- 1) $\Sigma_1^1(A, A) = \emptyset$;
- 2) $|\Sigma_1^1(A, A)| = 1$, $|\Sigma_0^-| \leq 2$, and there exists $E \in \Sigma_0^-$ such that $\Sigma_1^0(E, A) = \{a_1, a_2\}$, $\varphi_{12}a_1 = a_2$ for $\varphi_{12} \in \Sigma_1^1(A, A)$;
- 3) $2 \leq |\Sigma_1^1(A, A)| \leq 3$, $\Sigma_0^- = \{E\}$, $\Sigma_1^0(E, A) = \{a_1, a_2, a_3\}$, and $\Sigma_1^1(A, A) = \{\varphi_{12}, \varphi_{23}, \varphi_{13}\}$ with, possibly, $\varphi_{12} = \varphi_{23}$, where $\varphi_{13} = \varphi_{23}\varphi_{12}$, $\varphi_{12}a_1 = a_2$, $\varphi_{23}a_2 = a_3$, $\varphi_{13}a_1 = a_3$ (all other products are zero).

Proof. By Definition 4, $\text{ord } A \leq 3$. Thus $\dim_{\mathbb{k}} V \leq 3$ and $|\Sigma_0^-| \leq 3$. If $R = 0$, then, obviously, $\Sigma_1^1(A, A) = \emptyset$. Assume $R \neq 0$.

Consider the case $|\Sigma_0^-| = 1$. Here $\dim_{\mathbb{k}} V_i/V_{i+1} \leq 1$, $i = 1, 2$, $R^3V = V_4 = 0$ and so $R^3 = 0$. Let N be nilpotence degree of \mathcal{A} . Then $2 \leq N \leq 3$. We have $V_1 \supseteq V_2 \supseteq V_3$ and $\dim_{\mathbb{k}} V_i/V_{i+1} = 1$, $i = 1, 2$. Let $a_i \in \Sigma_1^{0(i)}$, $i = 1, \dots, N$.

Let $N = 2$. If $\varphi \in \Sigma_1^1$, then $\varphi a_1 = \lambda_\varphi a_2$ for some $\lambda_\varphi \in \mathbb{k}^*$ and $\varphi a_2 = 0$. If there is another $\psi \in \Sigma_1^1$, then $\lambda_\psi \varphi - \lambda_\varphi \psi = 0$ due to faithfulness of \mathcal{A} . Hence, $|\Sigma_1^1| = 1$. Applying correct elementary change $\mathfrak{C}_{\lambda_\varphi^{-1}}(\varphi)$, we obtain $\Sigma_1^1 = \{\varphi_{12}\}$ such that $\varphi_{12}a_1 = a_2$.

Let $N = 3$. Then for any $\varphi \in \Sigma_1^{1(2)}$, we have $\varphi a_1 \parallel a_3$, $\varphi a_k = 0$, $k = 2, 3$. Hence, as above, $\Sigma_1^{1(2)} = \{\varphi_{13}\}$ and $\varphi_{13}a_1 = a_3$, $\varphi_{13}a_2 = 0$. Now if $\varphi \in S(a_1, a_3) \setminus \{\varphi_{13}\}$, then there exists correct elementary change $\mathfrak{C}_\lambda(\varphi_{13}, \varphi)$ such that $\text{con}_{a_3}(\varphi a_1) = 0$, i. e.

$$S(a_1, a_3) = \mathbf{C}(a_1, a_3) = \{\varphi_{13}\}. \quad (3)$$

Further, there is $\varphi \in \Sigma_1^{1(1)}$ such that $\varphi a_2 = \lambda_\varphi a_3$, $\lambda_\varphi \in \mathbb{k}^*$. For another $\psi \in \Sigma_1^{1(1)}$ with $\psi a_2 = \lambda_\psi a_3$, $\lambda_\psi \in \mathbb{k}^*$, we have $(\lambda_\psi \varphi - \lambda_\varphi \psi)a_2 = 0$.

Applying correct elementary changes $\mathfrak{C}_{-\lambda_\psi/\lambda_\varphi}(\varphi, \psi)$ and $\mathfrak{C}_{\lambda_\varphi^{-1}}(\varphi)$ (that does not change $\mathbf{C}(a_1, a_3)$), we obtain that $\Sigma_1^{1(1)}$ contains φ_{23} such that $\varphi_{23}a_2 = a_3$ and $\varphi a_2 = 0$ for any $\varphi \in \Sigma_1^{1(1)} \setminus \{\varphi_{23}\}$, i. e.

$$\mathbf{S}(a_2, a_3) = \mathbf{C}(a_2, a_3) = \{\varphi_{23}\}. \quad (4)$$

Now it is obvious that $\Sigma_1^1 \setminus \{\varphi_{13}, \varphi_{23}\} \subset \mathbf{C}(a_1, a_2) \subset \Sigma_1^{1(1)}$. For $\varphi, \psi \in \mathbf{C}(a_1, a_2)$, $\varphi \neq \varphi_{23}$, $\mathfrak{C}_\lambda(\varphi, \psi)$ does not change (3) and (4). Then, similarly, $\mathbf{S}(a_1, a_2) = \mathbf{C}(a_1, a_2) = \{\varphi_{12}\}$. If $\varphi_{12} \neq \varphi_{23}$, then there is some $\mathfrak{C}_\lambda(\varphi_{12})$ such that $\varphi_{12}a_1 = a_2$, $\varphi_{12}a_2 = 0$. In this case $\varphi_{23}a_1 = 0$. Otherwise, $\varphi_{23}a_1 = \lambda a_2$. Applying correct elementary changes $\mathfrak{C}_\lambda(\varphi_{13})$ and $\mathfrak{C}_{\lambda^{-1}}(a_1)$, we obtain $\varphi_{23}a_1 = a_2$, $\varphi_{23}a_2 = a_3$ and $\varphi_{13}a_1 = a_3$. In both cases $\varphi_{23}\varphi_{12}a_1 = a_3 = \varphi_{13}a_1$, and hence $\varphi_{23}\varphi_{12} = \varphi_{13}$.

It remains to consider the case $|\Sigma_0^-| > 1$. If $|\Sigma_0^-| = |\Sigma_1^0|$, then $\mathbf{R} = 0$. So $\Sigma_0^- = \{E_1, E_2\}$, and $|\Sigma_1^0| = 3$. We have (up to renumbering) $\Sigma_1^0(E_1, A) = \{a_1, a_2\}$, $\Sigma_1^0(E_2, A) = \{a\}$. Since a_1 and a_2 are comparable by Definition 4 and \mathcal{A} is faithful, then $\Sigma_1^1(A, A) = \{\varphi\}$, and $a_2 \parallel \varphi a_1$. Therefore, we can obtain $\varphi a_1 = a_2$. \square

4. Bimodule problems with $|\Sigma_0^+| = 2$

Proposition 2. *Let $\mathcal{A} = (\mathbf{K}, \mathbf{V})$ be a faithful connected bimodule problem from \mathcal{C} with nilpotent radical $\mathbf{R} = \text{Rad } \mathbf{K}$ and a triangled basis Σ such that $\Sigma_0^+ = \{A, B\}$, $A \neq B$, and $\Sigma_0^- = \{E\}$. Then there exists a standard change \mathfrak{C} of basis such that Σ becomes quasi multiplicative. If $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are endowed with quasi multiplicative basis, then \mathfrak{C} leaves these bases unchangeable.*

We give the proof by series of lemmas under conditions of Proposition 2. First of all we note that since $\mathcal{A} \in \mathcal{C}$ is a faithful connected bimodule problem, then the bimodule problems $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are faithful connected as well by Remark 3. We assume that bases of $\mathcal{A}^{(A)}$, $\mathcal{A}^{(B)}$ are quasi multiplicative (they exist by Proposition 1). So, it is sufficient to change the basic elements from $\Sigma_1^1(A, B) \cup \Sigma_1^1(B, A)$ without changing the rest of them from Σ_1 in order to make Σ quasi multiplicative.

By Proposition 1, we can assert that $\Sigma_1^0(E, A) = \{a_1, \dots, a_{q_A}\}$, where $1 \leq q_A = \dim_{\mathbf{k}} \mathbf{V}(E, A) \leq 3$ due to Definition 4, $\Sigma_1^1(A, A) = \{\alpha_{ij} \mid 1 \leq i < j \leq q_A\}$ where, possibly, $\alpha_{12} = \alpha_{23}$ for the case $q_A = 3$, and $\mathbf{C}(a_i, a_j) = \{\alpha_{ij}\}$, $\alpha_{13} = \alpha_{23}\alpha_{12}$ (if $q_A = 3$), $\alpha_{ij}a_i = a_j$, $i < j$, and all other products equals zero.

Similarly, $\Sigma_1^0(E, B) = \{b_1, \dots, b_{q_B}\}$, where $1 \leq q_B = \dim_{\mathbf{k}} \mathbf{V}(E, B) \leq 3$, $\Sigma_1^1(B, B) = \{\beta_{ij} \mid 1 \leq i < j \leq q_B\}$ (possibly, $\beta_{12} = \beta_{23}$) and $\mathbf{C}(b_i, b_j) =$

$\{\beta_{ij}\}$, $\beta_{13} = \beta_{23}\beta_{12}$ (if $q_B = 3$), $\beta_{ij}b_i = b_j$, $i < j$, and all other products are zero.

Let h , h_A and h_B be heights of elements of bimodule problems \mathcal{A} , $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ respectively.

Remark 5. Any nonzero $a \in \mathbb{V}(E, A)$ is uniquely decomposed to a sum $a = \sum_{k=i}^{q_A} \lambda_k a_k$ with $i = h_A(a)$, $\lambda_i \in \mathbb{k}^*$, $1 \leq i \leq q_A$. For $a, a' \in \mathbb{V}(E, A)$, $a <_{\mathbb{R}} a'$ if and only if $h_A(a) < h_A(a')$ (see Proposition 1).

Lemma 4. Let $a \in \mathbb{V}(E, A)$, $b \in \mathbb{V}(E, B)$, and $a < b$. Then $a_u <_{\mathbb{R}} b_v$ for all u, v such that $1 \leq u \leq h_A(a)$ and $h_B(b) \leq v \leq q_B$. Besides, $\mathbb{S}(a_i, b_j) \neq \emptyset$ if and only if $a_i <_{\mathbb{R}} b_j$.

Proof. Denote $i = h_A(a)$, $j = h_B(b)$. Due to transitivity of $<$ it is sufficient to show that $a_i <_{\mathbb{R}} b_j$. By Remark 5, we can assume $a = a_i + a'$, where $a' = 0$ or $h_A(a') > i$, and $r'a_i = a'$ for some $r' \in \mathbb{R}$. Similarly, $b = b_j + b'$, where $b' = 0$ or $h_B(b') > j$. Since $ra = b$ for some $r \in \mathbb{R}$, then $a_i <_s b$ for $s = r + rr'$, and so $a_i <_{\mathbb{R}} b$. While $b <_{\mathbb{R}} b'$, then $s'a_i = b'$ for some $s' \in \mathbb{R}$. So $a_i <_{s-s'} b_j$, and hence $a_i <_{\mathbb{R}} b_j$.

To prove second statement consider $\xi \in \mathbb{S}(a_i, b_j)$. Then $h_B(\xi a_i) \leq h_B(b_j) = j$. Since $a_i <_{\mathbb{R}} \xi a_i$, then $a_i <_{\mathbb{R}} b_j$ as proved above. Conversely, while $ra_i = b_j$ with $r = \sum_{\xi \in \Sigma_1^1} \lambda_\xi \xi$, then $\mathbb{S}(a_i, b_j) \neq \emptyset$. \square

4.1. The case $q_A = q_B = 3$

We have $\Sigma_1^0 = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ with $a_1 <_{\alpha_{12}} a_2 <_{\alpha_{23}} a_3$, $\alpha_{13} = \alpha_{23}\alpha_{12}$, and $b_1 <_{\beta_{12}} b_2 <_{\beta_{23}} b_3$, $\beta_{13} = \beta_{23}\beta_{12}$. Any $a_k \in \Sigma_1^0(E, A)$ and $b_l \in \Sigma_1^0(E, B)$, $1 \leq k, l \leq 3$, are comparable by Definition 4. Then, due to the triangularity, there is the linear order on the set Σ_1^0 . Without loss of generality we can assume a_1 the smallest element. For $x, y \in \Sigma_1^0$, let $n(x, y) = |\{z \in \Sigma_1^0 \mid x < z < y\}|$. If $\alpha_{12} = \alpha_{23} = \alpha$ then $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, and $\alpha^3 = 0$.

Lemma 5. If $q_A = q_B = 3$, $\alpha_{12} = \alpha_{23} = \alpha$, and $\beta_{12} = \beta_{23} = \beta$, then $n(a_1, a_2) = n(a_2, a_3) = 0$ and $n(b_1, b_2) = n(b_2, b_3) = 0$. Therefore the following case occurs: $a_1 <_{\alpha} a_2 <_{\alpha} a_3 < b_1 <_{\beta} b_2 <_{\beta} b_3$.

Proof. If there exists $b \in \mathbb{V}(E, B)$ such that $a_i < b < a_{i+1}$, $i = 1, 2$, then due to Lemma 4, there is $b_j \in \Sigma_1^0(E, B)$ such that $a_i < b_j < a_{i+1}$, in particular, $n(a_i, a_{i+1}) > 0$.

Assume $n(a_2, a_3) > 1$ (the cases $n(a_1, a_2)$, $n(b_1, b_2)$, $n(b_2, b_3) > 1$ can be considered similarly). Then $a_2 <_r b_j <_\beta b_{j+1} <_s a_3$ for some $r \in R(A, B)$, $s \in R(B, A)$, $j \in \{1, 2\}$, and we have $s\beta r = \alpha + \lambda\alpha^2 \in R(A, A)$. We can assume $\lambda = 0$. Indeed, otherwise $s\beta r\alpha = \alpha^2$ and $s\beta r' = \alpha$ for $r' = r - \lambda r\alpha \in R(A, B)$. Since $C(a_2, a_3) = S(a_2, a_3) = \{\alpha\}$, then there are $b', b'' \in V(E, B)$ such that $a_1 <_r b' <_\beta b'' <_s a_2$. But this is impossible while $q_B = 3$.

Hence $n(a_1, a_2) = n(a_2, a_3) = 1$, and $a_1 <_r b_1 <_s a_2$ with $r \in R(A, B)$, $s \in R(B, A)$. As above, we can assume $sr = \alpha$, $a_2 < b_2 < a_3$. Then $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$.

Note that $h(\varphi) = 1$ for any $\varphi \in \bigcup_{i=1,2,3} S(a_i, b_i)$. By Lemma 3, there is standard change of basis such that $S(a_3, b_3) = C(a_3, b_3) = \{\varphi_3\}$ and $S(a_2, b_2) = C(a_2, b_2) = \{\varphi_2\}$ where, possibly, $\varphi_2 = \varphi_3$. Let $\varphi_i a_i = \lambda_i b_i$, $\lambda_i \in \mathbb{k}^*$, $i = 2, 3$. If there is $\varphi \in S(a_1, b_1)$, $\varphi \neq \varphi_2, \varphi_3$, then we can do standard change of basis such that $S(a_1, b_1) = C(a_1, b_1) = \{\varphi_1\}$, $\varphi_1 \neq \varphi_2, \varphi_3$. Otherwise, we can assume $S(a_1, b_1) \subset \{\varphi_2, \varphi_3\}$. Let $\varphi_3 \in S(a_1, b_1)$ and $\varphi_3 a_1 = \lambda_1 b_1 + \lambda'_2 b_2 + \lambda'_3 b_3$, $\lambda_1 \neq 0$. If $\varphi_3 \neq \varphi_2$, then standard change $\varphi'_3 = \varphi_3 - \frac{\lambda'_2}{\lambda_2} \varphi_2 \alpha - \frac{\lambda'_3}{\lambda_3} \varphi_3 \alpha^2$ leads to equalities $\varphi'_3 a_1 = \lambda_1 b_1$, $\varphi'_3 a_2 = \varphi_3 a_2 = 0$, $\varphi'_3 a_3 = \lambda_3 b_3$. If $\varphi_2 \in S(a_1, b_1)$, then similarly we obtain $\varphi'_2 a_1 = \mu_1 b_1$, $\varphi'_2 a_2 = \varphi_2 a_2 = \lambda_2 b_2$, $\varphi'_2 a_3 = 0$, $\mu_1 \in \mathbb{k}^*$. Hence $S(a_i, b_i) = C(a_i, b_i)$, $i = 1, 2, 3$, and we have one of the cases:

- 1) $C(a_i, b_i) = \{\varphi_i\}$, $i = 1, 2, 3$, where two of $\varphi_1, \varphi_2, \varphi_3$ may be equal;
- 2) $C(a_i, b_i) = \{\varphi_i\}$, $i = 2, 3$, $C(a_1, b_1) = \{\varphi_2, \varphi_3\}$.

In the case $\varphi_2 = \varphi_3$ we have $\varphi_1 = \varphi_3$ as well, and similarly $S(a_1, b_1) = C(a_1, b_1) = \{\varphi_1\}$. But this case contradicts $\mathcal{A} \in \mathcal{C}$ (see Remark 4).

Similarly, we can find $\psi_1, \psi_2 \in \Sigma_1^1$ such that $S(b_1, a_2) = C(b_1, a_2) = \{\psi_1\}$, $S(b_2, a_3) = C(b_2, a_3) = \{\psi_2\}$, where possibly $\psi_1 = \psi_2$, and finally

$$a_1 <_{\varphi_1} b_1 <_{\psi_1} a_2 <_{\varphi_2} b_2 <_{\psi_2} a_3 <_{\varphi_3} b_3 \quad \text{or} \quad a_1 <_{\varphi_2, \varphi_3} b_1 <_{\psi_1} a_2 <_{\varphi_2} b_2 <_{\psi_2} a_3 <_{\varphi_3} b_3.$$

Let $\varphi \in C(a_1, b_1)$, then $\psi_1 \varphi \parallel \alpha$. Since $\alpha a_2 = a_3$ then $\varphi \in C(a_2, b_2)$, and therefore $\varphi = \varphi_2$. Furthermore, $\varphi_2 \psi_1 \parallel \beta$, then $\psi_1 = \psi_2$ and $\varphi_2 a_3 \neq 0$, hence $\varphi_2 = \varphi_3$. But this contradicts Remark 4. \square

4.2. Comparable pairs

Definition 10. Denote by $\Pi = \Pi_{A,B}$ the set

$$\Pi = \{(a_i, b_j) \in \Sigma_1^0(E, A) \times \Sigma_1^0(E, B) \mid a_i \leq_{\mathbb{R}} b_j\}.$$

Let us define partial order on Π : given different pairs $(a_i, b_j), (a_u, b_v) \in \Pi$, we say that $(a_i, b_j) \prec (a_u, b_v)$ if $a_u \leq a_i < b_j \leq b_v$. We say that two

pairs $(a_i, b_j), (a_u, b_v) \in \Pi$ are comparable, provided $(a_i, b_j) \prec (a_u, b_v)$ or $(a_u, b_v) \prec (a_i, b_j)$. We call the map $\Pi \ni (a_i, b_j) \xrightarrow{h_m} \max_{\xi \in \mathcal{S}(a_i, b_j)} h(\xi) \in \mathbb{N}$ the maximal height, and denote by $\mathcal{S}_m(a_i, b_j) = \{\xi \in \mathcal{S}(a_i, b_j) \mid h(\xi) = h_m(a_i, b_j)\}$. For $(a_i, b_j) \in \Pi$, $\mathcal{S}_m(a_i, b_j) \neq \emptyset$.

Lemma 6. *If $(a_i, b_j) \in \Pi$, $\xi \in \mathcal{S}_m(a_i, b_j)$, then $h_B(\xi a_i) = h_B(b_j) = j$. In particular, $h(\xi a_i) = h(b_j)$, and $h(b_j) - h(a_i) \geq h_m(a_i, b_j)$.*

Proof. If $\xi a_i \parallel b_l + b'$ for $l < j$ and $h_B(b') > l$, then $\beta_{lj} \xi a_i \parallel b_j + b''$ and $h_B(b'') > j$. Hence there exists $\eta \in \mathcal{S}(a_i, b_j)$ such that $h(\eta) \geq h(\beta_{lj} \xi) > h(\xi) = h_m(a_i, b_j)$, and we get contradiction. Then $h(b_j) = h(\xi a_i) \geq h(\xi) + h(a_i)$, hence $h(b_j) - h(a_i) \geq h_m(a_i, b_j)$. \square

Lemma 7. *If $(a_i, b_j), (a_u, b_v) \in \Pi$ and $h_m(a_i, b_j) = h_m(a_u, b_v)$ then the pairs $(a_i, b_j), (a_u, b_v)$ are incomparable.*

Proof. If $\text{con}_{b_j}(ra_i) \neq 0$, $r \in \mathcal{R}(A, B)$, then $h_m(a_i, b_j) \geq h(r)$. Indeed, if $r = \sum_{\xi \in \Sigma_1^+(A, B)} \lambda_\xi \xi$, then $h(\xi) \geq h(r)$ whenever $\lambda_\xi \neq 0$. Since $\text{con}_{b_j}(ra_i) \neq 0$, then there exists $\xi \in \mathcal{S}(a_i, b_j)$ such that $\lambda_\xi \neq 0$, and therefore $h_m(a_i, b_j) = \max_{\eta \in \mathcal{S}(a_i, b_j)} h(\eta) \geq h(\xi) \geq h(r)$.

For $(a_i, b_j), (a_u, b_v) \in \Pi$, the inequality $h_m(a_i, b_j) < h_m(a_u, b_v)$ holds whenever $(a_i, b_j) \prec (a_u, b_v)$. Let $(a_i, b_j) \prec (a_u, b_v)$, $\varphi \in \mathcal{S}_m(a_i, b_j)$. If $i = u$ and $j < v$, then $\text{con}_{b_v}(ra_i) \neq 0$ for $r = \beta_{jv} \varphi$. If $u < i$, $j = v$, then $\text{con}_{b_j}(ra_u) \neq 0$ for $r = \varphi \alpha_{ui}$. If $u < i$, $j < v$, then $\text{con}_{b_v}(ra_u) \neq 0$ for $r = \beta_{jv} \varphi \alpha_{ui}$. For all the cases, $h_m(a_u, b_v) \geq h(r) > h(\varphi) = h_m(a_i, b_j)$. \square

Further, writing $(a_i, b_j), (a_u, b_v) \in \Pi$, we assume $(a_i, b_j) \neq (a_u, b_v)$.

Remark 6. If $(a_i, b_j), (a_u, b_v) \in \Pi$ are incomparable, then either $i < u$, $j < v$ or $i > u$, $j > v$.

Proof. The pairs $(a_i, b_j), (a_i, b_v)$, $j \neq v$, are comparable as well as the pairs $(a_i, b_j), (a_u, b_j)$, $i \neq u$. Hence $i \neq u$, $j \neq v$. Given $i < u$, assume that $j > v$, then $a_i < a_u < b_v < b_j$, and hence $(a_u, b_v) \prec (a_i, b_j)$. \square

Definition 11. For any subset $\mathfrak{X} \subset \Pi$, put $\mathcal{C}(\mathfrak{X}) = \bigcup_{(a_i, b_j) \in \mathfrak{X}} \mathcal{C}(a_i, b_j)$.

Given $\varphi \in \Sigma_1^+(A, B)$, let $\mathfrak{X}_\varphi = \{(a_i, b_j) \in \mathfrak{X} \mid \varphi \in \mathcal{C}(a_i, b_j)\}$. Obviously, $\mathfrak{X}_\varphi \neq \emptyset$ if and only if $\varphi \in \mathcal{C}(\mathfrak{X})$. A subset $\mathfrak{X} \subset \Pi$ is named upper closed if for any $(a_i, b_j) \in \mathfrak{X}$ and $(a_u, b_v) \in \Pi$ there hold:

- if $h_m(a_u, b_v) > h_m(a_i, b_j)$, then $(a_u, b_v) \in \mathfrak{X}$;
- if $h_m(a_u, b_v) = h_m(a_i, b_j)$ and $u > i$, then $(a_u, b_v) \in \mathfrak{X}$.

The pair $(a_i, b_j) \in \Pi \setminus \mathfrak{X}$ is called boundary for the upper closed set $\mathfrak{X} \subset \Pi$ if $\mathfrak{X} \cup \{(a_i, b_j)\}$ is upper closed too. A upper closed subset $\mathfrak{X} \subset \Pi$ is said to be canonical if the following conditions hold:

- 1) if $(a_i, b_j) \in \mathfrak{X}$ then we have $|\mathbb{S}(a_i, b_j)| = |\mathbb{C}(a_i, b_j)| = 1$;
- 2) $1 \leq |\mathfrak{X}_\varphi| \leq 2$ for each $\varphi \in \mathbb{C}(\mathfrak{X})$;
- 3) if $\mathfrak{X}_\varphi = \{(a_i, b_j), (a_u, b_v)\}$, then $a_i \neq a_u$, $b_j \neq b_v$, and $h_m(a_i, b_j) = h_m(a_u, b_v) = 1$.

Remark that $\mathfrak{X} = \emptyset$ is canonical.

A standard basis change is called *careful* for the canonical subset $\mathfrak{X} \subset \Pi$ if \mathfrak{X} is a canonical subset with respect to the new basis as well.

Lemma 8. *Let $\mathfrak{X} \subset \Pi$ be a canonical subset and let $(a_i, b_j) \in \Pi \setminus \mathfrak{X}$ be a boundary pair. Then $\mathbb{S}_m(a_i, b_j) \subset \mathbb{C}(a_i, b_j)$.*

Proof. If $\text{con}_{b_k}(\xi a_i) \neq 0$, $\xi \in \mathbb{S}(a_i, b_j)$, $k > j$, then $\xi \in \mathbb{S}(a_i, b_k)$, and so $(a_i, b_k) \in \mathfrak{X}$ since $(a_i, b_k) \succ (a_i, b_j)$. We have $\mathbb{S}(a_i, b_k) = \mathbb{C}(a_i, b_k) = \{\xi\}$ by the definition of \mathfrak{X} , hence $\xi a_i \parallel b_k$ that contradicts to inclusion $\xi \in \mathbb{S}(a_i, b_j)$. Therefore $\text{con}_{b_k}(\xi a_i) = 0$ for all $\xi \in \mathbb{S}(a_i, b_j)$ and $k > j$.

Let $\varphi \in \mathbb{S}_m(a_i, b_j)$. If $\varphi a_i \not\parallel b_j$, then we conclude $\varphi a_i = \mu_k b_k + \dots + \mu_j b_j$ with $k < j$ and $\mu_k \neq 0$, $\mu_j \neq 0$. Therefore $(\beta_{kj} \varphi) a_i = \mu_k b_j + \dots + \mu_j (\beta_{kj} b_j)$. There exists ξ in decomposition $\beta_{kj} \varphi = \sum_{\xi \in \Sigma_1^1(A, B)} \lambda_\xi \xi$ such that $\lambda_\xi \neq 0$ and $\text{con}_{b_j}(\xi a_i) \neq 0$, so $\xi \in \mathbb{S}(a_i, b_j)$. But $h(\xi) \geq h(\beta_{kj} \varphi) > h(\varphi) = h_m(a_i, b_j)$, and we obtain contradiction. \square

Lemma 9. *If $q_A = q_B = 3$, $\alpha_{12} = \alpha_{23} = \alpha$, and $\beta_{12} = \beta_{23} = \beta$, then $\Pi_{A, B} = \{(a_i, b_j) \mid i, j = 1, 2, 3\}$ is canonical, each $\varphi \in \Sigma_1^1(A, B)$ is single, and multiplication rule holds.*

Proof. By Lemma 5, $a_1 <_\alpha a_2 <_\alpha a_3 <_\tau b_1 <_\beta b_2 <_\beta b_3$. Hence, $\Pi_{A, B} = \{(a_i, b_j) \mid i, j = 1, 2, 3\}$ is upper closed. By Lemma 3 there is standard basis change that leads to condition $\mathbb{S}(a_3, b_1) = \mathbb{C}(a_3, b_1) = \{\tau\}$. Obviously, $h(\tau) = 1$.

Let $\tau a_2 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$, then standard basis change $\tau' = \tau - \lambda_1 \tau a - \lambda_2 \beta \tau a - \lambda_3 \beta^2 \tau a$ implies $\tau' a_2 = 0$, $\tau' a_3 = b_1$. Similarly, we can obtain $\tau a_1 = 0$, $\tau a_2 = 0$, $\tau a_3 = b_1$.

Denote $\varphi_{ij} = \beta^{j-1} \tau \alpha^{3-i}$, $i, j = 1, 2, 3$. Then $\varphi_{ij} = \varphi_{kl}$ implies $i = k$, $j = l$. Indeed, if $k < i$, then $0 = \beta^{l-1} \tau \alpha^{3-k} a_i = \beta^{j-1} \tau \alpha^{3-i} a_i = b_j$. So $i = k$. Similarly, $j = l$. It is easy to check that $\varphi_{ij} a_k = \delta_{ik} b_j$.

If there is $\varphi \in \mathbb{S}(a_i, b_j) \setminus \{\varphi_{ij}\}$, then $h(\varphi) \leq h(\varphi_{ij})$ and $\varphi \neq \varphi_{kl}$ for all k, l . So basis change $\mathfrak{C}_\lambda(\varphi_{ij}, \varphi)$ with $\lambda = \text{con}_{b_j}(\varphi a_i)$ is correct and leads to condition $\varphi \notin \mathbb{S}(a_i, b_j)$. Thus, $\mathbb{C}(a_i, b_j) = \mathbb{S}(a_i, b_j) = \{\varphi_{ij}\}$ and $\Sigma_1^1(A, B) = \{\varphi_{ij}, i, j = 1, 2, 3\}$. Multiplication rule holds obviously. \square

4.3. Induction step

We exclude the case of Lemma 9 from further consideration in this subsection. Let us fix a canonical subset $\mathfrak{X} \subset \Pi$ and a boundary pair $(a_i, b_j) \in \Pi \setminus \mathfrak{X}$.

Lemma 10. *If there exists $\varphi \in \mathcal{S}_m(a_i, b_j) \setminus \mathcal{C}(\mathfrak{X})$, then there is a careful basis change such that $\mathcal{S}(a_i, b_j) = \mathcal{C}(a_i, b_j) = \{\varphi\}$, $\varphi \notin \mathcal{C}(\mathfrak{X})$.*

Proof. By Lemma 8, $\varphi \in \mathcal{C}(a_i, b_j)$. For a $\psi \in \mathcal{S}(a_i, b_j) \setminus \{\varphi\}$, we have $h(\psi) \leq h(\varphi)$. Hence, there exists $\lambda \in \mathbb{k}^*$ such that the elementary basis change $\psi' = \psi + \lambda\varphi$ is correct and leads to the condition $\mathcal{S}(a_i, b_j) = \mathcal{S}(a_i, b_j) \setminus \{\psi\}$ by Lemma 3.

Let us show that this change is careful. If $\psi \notin \mathcal{C}(\mathfrak{X})$ then it is clear. Otherwise, there is the pair $(a_u, b_v) \in \mathfrak{X}$ such that $\mathcal{S}(a_u, b_v) = \mathcal{C}(a_u, b_v) = \{\psi\}$. If $\varphi a_u \neq 0$, then $\varphi \in \mathcal{S}(a_u, b_w)$ for some $v \leq w \leq q_B$ since $h(\varphi a_u) \geq h(\psi a_u) = h(b_v)$. Then $(a_u, b_w) \in \mathfrak{X}$ while $(a_u, b_w) \succ (a_u, b_v)$, and therefore $\varphi \in \mathcal{C}(\mathfrak{X})$. We have a contradiction. So $\varphi a_u = 0$, and $\psi' a_u = \psi a_u$, hence this change of basis is careful. \square

Corollary 1. *If $\mathcal{S}_m(a_i, b_j) \not\subset \mathcal{C}(\mathfrak{X})$ then $\mathfrak{X} \cup \{(a_i, b_j)\}$ is a canonical subset as well.*

Lemma 11. *Let $\mathcal{Y} = \{(a_{i_k}, b_{j_k}), k = 1, \dots, n\} \subset \Pi$ be any set of pairwise incomparable pairs in Π . Then $n \leq 3$, and if $n = 3$, then $q_A = q_B = 3$, $\mathcal{Y} = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$.*

Proof. By Remark 6 $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ up to renumbering. Then $n \leq q_A \leq 3$. If $n = 3$, then $i_1 = j_1 = 1$, $i_2 = j_2 = 2$, $i_3 = j_3 = 3$. \square

Lemma 12. *If $\mathcal{S}_m(a_i, b_j) \subset \mathcal{C}(\mathfrak{X})$, then $h_m(a_i, b_j) = 1$.*

Proof. Assume $h = h_m(a_i, b_j) > 1$. Denote $\mathcal{Y} = \bigcup_{\varphi \in \mathcal{S}_m(a_i, b_j)} \mathfrak{X}_\varphi$. By definition 11, $|\mathcal{Y}| \geq |\mathcal{S}_m(a_i, b_j)|$, and $h_m(a_u, b_v) = h$ for any $(a_u, b_v) \in \mathcal{Y}$, and hence the pairs from $\mathcal{Y} \cup \{(a_i, b_j)\}$ are pairwise incomparable by Lemma 7. By Lemma 11, $|\mathcal{Y}| \leq 2$.

Assume $\mathcal{S}_m(a_i, b_j) = \{\varphi, \psi\}$, and $|\mathfrak{X}_\varphi| = |\mathfrak{X}_\psi| = 1$, $\mathcal{Y} = \mathfrak{X}_\varphi \cup \mathfrak{X}_\psi$. Then, by Lemma 11, $q_A = q_B = 3$, and $\mathcal{Y} \cup \{(a_i, b_j)\} = \{(a_k, b_k) \mid k = 1, 2, 3\}$, besides, $(a_i, b_j) = (a_1, b_1)$ and $\Pi = \mathfrak{X} \cup \{(a_1, b_1)\}$ while \mathfrak{X} is upper closed. Let $\mathfrak{X}_\varphi = \{(a_2, b_2)\}$, $\mathfrak{X}_\psi = \{(a_3, b_3)\}$.

Since $h_m(a_k, b_k) = h > 1$, $k = 1, 2, 3$, and there is a basic element of each height, then either $a_1 < a_2 < b_1 < a_3 < b_2 < b_3$ or $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$. In the first case $h_m(a_1, b_1) = h_m(a_3, b_3) = 2$ but $h_m(a_2, b_2) = 3$, and we get a contradiction.

In the second case $h_m(a_k, b_k) = 3$, $k = 1, 2, 3$. Let $\tau \in \mathbb{S}(a_3, b_1)$. Then $h(\tau\alpha_{13}) \geq h(\tau) + h(\alpha_{13}) \geq 3$, and $\tau\alpha_{13} = \sum_{\xi \in \Xi} \lambda_\xi \xi$, where $\Xi = \{\xi \in \Sigma_1^1(A, B) \mid h(\xi) \geq 3\}$.

Since each pair (a_k, b_k) is adjusted, $\text{con}_{b_k}(\xi a_k) = 0$ for all $\xi \in \Xi \setminus \{\varphi, \psi\}$. Then $0 = \text{con}_{b_2}(\tau\alpha_{13}a_2) = \lambda_\varphi$ while $\text{con}_{b_2}(\psi a_2) = 0$. Similarly, $\lambda_\psi = 0$, in contradiction with condition $\text{con}_{b_1}(\tau\alpha_{13}a_1) \neq 0$.

Therefore, $\mathbb{S}_m(a_i, b_j) = \{\varphi\} \subset \mathbb{C}(a_i, b_j)$, $h(\varphi) = h$. Since $\varphi \in \mathbb{C}(\mathfrak{X})$, there exists $(a_u, b_v) \in \mathfrak{X}$ such that $\mathbb{C}(a_u, b_v) = \{\varphi\}$, the pairs (a_i, b_j) and (a_u, b_v) are incomparable, and $i < u$, $j < v$ by Remark 6 in view of \mathfrak{X} is upper closed.

Since $h > 1$, then φ is a linear combination of the summands belonging to the set

$$\Gamma = \{\xi\alpha, \beta\xi, \beta\xi\alpha \mid \alpha \in \Sigma_1^1(A, A), \beta \in \Sigma_1^1(B, B), \xi \in \Sigma_1^1(A, B)\} \cap \mathbb{R}^h.$$

Since $\varphi a_i \parallel b_j$ (resp., $\varphi a_u \parallel b_v$) then there is $\gamma \in \Gamma$ such that $\text{con}_{b_j}(\gamma a_i) \neq 0$ (resp., $\text{con}_{b_v}(\gamma a_u) \neq 0$). Denote $\Gamma' = \{\gamma \in \Gamma \mid \text{con}_{b_j}(\gamma a_i) \neq 0\}$, $\Gamma'' = \{\gamma \in \Gamma \mid \text{con}_{b_v}(\gamma a_u) \neq 0\}$. Then $\Gamma' = \Gamma''$. Indeed, if $\gamma = \sum_{\substack{\psi \in \Sigma_1^1(A, B) \\ h(\psi) \geq h}} \lambda_\psi \psi \in \Gamma'$,

then $\text{con}_{b_j}(\psi a_i) = 0$ and $\text{con}_{b_v}(\psi a_u) = 0$ for any $\psi \neq \varphi$ by the construction. Therefore $\lambda_\varphi \neq 0$, $\text{con}_{b_v}(\gamma a_u) = \lambda_\varphi \text{con}_{b_v}(\varphi a_u) \neq 0$ and $\gamma \in \Gamma''$. The proof of inclusion $\Gamma'' \subset \Gamma'$ is similar. Remark that $\gamma a_i \neq 0$ and $\gamma a_u \neq 0$ for any $\gamma \in \Gamma'$.

Let $\gamma \in \Gamma'$. If $\gamma = \xi\alpha$ or $\gamma = \beta\xi\alpha$, then $\alpha a_i \neq 0$ and $\alpha a_u \neq 0$. So α is joint arrow while $i < u$, and $q_A = 3$. If $\gamma = \beta\xi$ or $\gamma = \beta\xi\alpha$, then $\gamma a_i \parallel b_j + b'$, where $h(b') > h(b_j)$, $j \geq 2$, and $b_v \parallel b_3$. Thus $b_j, b_v \in \beta\mathbb{V}(E, B)$. But $j < v$, hence β is a joint arrow, and $q_B = 3$.

If $\gamma = \xi\alpha \in \Gamma'$, then $\alpha = \alpha_{12} = \alpha_{23}$, so $i = 1$, $u = 2$, and we have $a_1 <_\alpha a_2 <_\xi b'_j$, $a_2 <_\alpha a_3 <_\xi b'_v$. Here $h(b'_j - b_j) > h(b_j)$ and $h(b'_v - b_v) > h(b_v)$. There is $r \in \mathbb{R}(B, B)$ such that $b'_j <_r b'_v$ by Remark 5. We obtain $a_2 <_\xi b'_j <_r b'_v$, and we can find $\beta \in \Sigma_1^1(B, B)$ such that $\text{con}_{b_v}(\beta\xi a_u) \neq 0$, hence $\beta\xi \in \Gamma'$ and β is a joint arrow. If $\gamma = \beta\xi$, then, similarly, α is a joint arrow.

If α, β are both joint, then $q_A = q_B = 3$, $\alpha = \alpha_{12} = \alpha_{23}$, $\beta = \beta_{12} = \beta_{23}$. This case of Lemma 9 is excluded from consideration. \square

Lemma 13. *Let $\mathbb{S}_m(a_i, b_j) \subset \mathbb{C}(\mathfrak{X})$. Then one of the following conditions holds (after some careful basis change):*

- 1) $\mathbb{S}(a_i, b_j) = \mathbb{C}(a_i, b_j) = \{\varphi\}$, $h(\varphi) = 1$, $\varphi \in \mathbb{C}(\mathfrak{X})$, and $|\mathfrak{X}_\varphi| = 1$;
- 2) $q_A = q_B = 3$, $i = j = 1$, $\Pi = \mathfrak{X} \cup \{(a_1, b_1)\}$ and $\mathbb{C}(a_k, b_k) = \{\varphi_k\}$, $\mathfrak{X}_{\varphi_k} = \{(a_k, b_k)\}$, $k = 2, 3$, $\mathbb{C}(a_1, b_1) = \{\varphi_2, \varphi_3\}$.

Proof. By Lemma 12, $h_m(a_i, b_j) = 1$. Clearly, $S_m(a_i, b_j) = S(a_i, b_j)$. By Lemma 8, $S(a_i, b_j) = C(a_i, b_j)$. Denote $\mathcal{Y} = \bigcup_{\varphi \in C(a_i, b_j)} \mathfrak{X}_\varphi$. By definition 11, $|\mathcal{Y}| \geq |C(a_i, b_j)|$, and if $(a_u, b_v) \in \mathcal{Y}$, then $h_m(a_u, b_v) = 1$. Hence, the pairs from $\mathcal{Y} \cup \{(a_i, b_j)\}$ are pairwise incomparable by Lemma 7, and so $|\mathcal{Y}| \leq 2$ by Lemma 11.

If $|\mathcal{Y}| = 1$ then $C(a_i, b_j) = \{\varphi\}$ and $|\mathfrak{X}_\varphi| = 1$, which implies 1). Assume $|\mathcal{Y}| = 2$. If $C(a_i, b_j) = \{\varphi\}$ then $\mathcal{Y} = \mathfrak{X}_\varphi$. Thus P_φ includes three adjusted pairs, which is impossible by Remark 4. Hence, $C(a_i, b_j) = \{\varphi, \psi\}$, and $\mathcal{Y} = \mathfrak{X}_\varphi \cup \mathfrak{X}_\psi$. There are $(a', b'), (a'', b'') \in \mathfrak{X}$ such that $a' < a''$, and $C(a', b') = \{\varphi\}$, $C(a'', b'') = \{\psi\}$. The pairs $(a_i, b_j), (a', b'), (a'', b'')$ are pairwise incomparable by Lemma 7, while $h(\varphi) = h(\psi) = 1$. Since \mathfrak{X} is upper closed, it is possible only if $i = j = 1$, then $a_1 <_{\varphi, \psi} b_1 <_{\varphi} a' <_{\psi} b' <_{\psi} a'' <_{\varphi} b''$, and $\Pi = \mathfrak{X} \cup \{(a_1, b_1)\}$. We obtain the case 2). \square

Lemma 14. *Let $\mathfrak{X} \subset \Pi$ be a canonical subset and let $(a_i, b_j) \in \Pi \setminus \mathfrak{X}$ be a boundary pair. Then there exists a careful basis change on $\Sigma_1^1(A, B)$ such that $\mathfrak{X} \cup \{(a_i, b_j)\}$ is a canonical subset in Π except of the case $q_A = q_B = 3$, $a_1 <_{\varphi_2, \varphi_3} b_1 <_{\varphi_2} a_2 <_{\varphi_3} b_2 < a_3 < b_3$, and $i = j = 1$, $\Pi = \mathfrak{X} \cup \{(a_1, b_1)\}$, $C(a_k, b_k) = \{\varphi_k\}$, $\mathfrak{X}_{\varphi_k} = \{(a_k, b_k)\}$, $k = 2, 3$, $C(a_1, b_1) = \{\varphi_2, \varphi_3\}$.*

Proof. By Lemma 12, if $\varphi \in C(\mathfrak{X} \cup \{(a_i, b_j)\})$ and $h(\varphi) > 1$, then φ is single. Then the proof follows from Lemmas 10 and 13. \square

Combining results of Lemmas 9, 14 and 12, we obtain the following

Corollary 2. *Under conditions of Proposition 2, one of the following cases occurs:*

- 1) Π is a canonical set;
- 2) $\Pi \setminus \{(a_1, b_1)\}$ is a canonical set; this is the case from Lemma 14.

In any case, for every $a_i \in \Sigma_1^0(E, A)$ and $\varphi \in \Sigma_1^1(A, B)$, either $\varphi a_i = 0$ or $\varphi a_i \parallel b_j$ for some $b_j \in \Sigma_1^0(E, B)$, and the height of each joint arrow in $\Sigma_1^1(A, B)$ equals 1.

Denote by $\Pi_{A,B}$ and $\Pi_{B,A}$ ¹ the sets, defined by definition 10. Assume both $\Pi_{A,B}$ and $\Pi_{B,A}$ satisfy Corollary 2. Then at most one of these set may be not canonical (since $a_1 < b_1$ and $b_1 < a_1$ are not simultaneous).

¹Defined as in Definition 10.

4.4. Multiplication rule on Σ_1^1

For any $\varphi, \psi \in \Sigma_1^1$, we say that the product $\psi\varphi$ belongs to Σ_1^1 , if $\psi\varphi \parallel \xi \in \Sigma_1^1$. Further considerations are provided under assertions of Corollary 2.

Lemma 15. *Under conditions of Proposition 2, multiplication rule holds for \mathcal{A} .*

Proof. Due to Corollary 2, either the set $\Pi_{A,B}$ is canonical, or we have the special case of Lemma 14 and $\Pi_{A,B} \setminus \{(a_1, b_1)\}$ is a canonical set. Similarly, $\Pi_{B,A}$ or $\Pi_{B,A} \setminus \{(b_1, a_1)\}$ is a canonical set. The special case occurs at most for one of the cases $\Pi_{A,B}$ and $\Pi_{B,A}$, so we can assume $\Pi_{B,A}$ to be canonical. Due to Proposition 1, the sets $\Pi_{A,A}$ and $\Pi_{B,B}$ are canonical as well.

Since $\text{Ann}_{\mathbb{R}} V = 0$ then $C(\Pi_{X,Y}) = \Sigma_1^1(X, Y)$ for all $X, Y \in \{A, B\}$. If X, Y or Z denotes one of the vertices A or B , then we write x_i, y_i or z_i instead of basic elements a_i or b_i respectively. Further we will omit the condition $1 \leq i$ for the indices.

Assume $r = \psi\varphi \neq 0$, $\varphi \in \Sigma_1^1(X, Y)$, $\psi \in \Sigma_1^1(Y, Z)$ where $X, Y, Z \in \{A, B\}$. Let

$$r = \sum_{\xi \in \Sigma_1^1(X, Z)} \lambda_{\xi} \xi, \quad \lambda_{\xi} \in \mathbb{k}. \quad (5)$$

Denote by Σ_r the set of all ξ from decomposition (5) such that $\lambda_{\xi} \neq 0$. Then $h(\xi) \geq h(r) > 1$ for any $\xi \in \Sigma_r$, and if $X \neq Z$, ξ is single by Corollary 2.

1) If $rx_i \neq 0$ for some $i \leq q_X$, then there exist $j \leq q_Y$ and $k \leq q_Z$ such that $\varphi x_i \parallel y_j$ and $\psi y_j \parallel z_k$ by construction, so $rx_i \parallel z_k$.

2) Let $r \in \mathbb{R}(A, A)$. If $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, then r belongs to $\Sigma_1^1(A, A)$. Indeed, if $r = \lambda\alpha + \mu\alpha^2$, $\lambda, \mu \in \mathbb{k}^*$, then $ra_1 = \lambda a_2 + \mu a_3$ in contradiction with item 1), and Lemma is proved. So, if $r \in \mathbb{R}(A, A)$, we assume all the arrows from $\Sigma_1^1(A, A)$ to be single below.

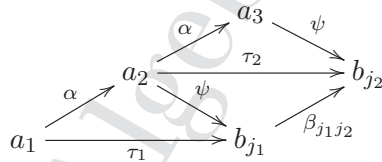
3) Now it is enough to consider the situation when all the arrows from Σ_r are single. Denote $\Sigma_r = \{\tau_1, \dots, \tau_p\}$, $p > 0$, and $P_{\tau_t} = \{(x_{i_t}, z_{k_t})\}$, $t = 1, \dots, p$. Then x_{i_1}, \dots, x_{i_p} are pairwise different arrows and $p \leq 3$. Indeed, if $x_{i_u} = x_{i_v}$ then $z_{k_u} \neq z_{k_v}$ while $\Pi_{X,Z} \setminus \{(x_1, z_1)\}$ is canonical, and the sum $rx_{i_u} = \sum_{v=1}^p \lambda_{\tau_v} \tau_v x_{i_u} = \sum_{i=1}^{q_Z} \mu_i z_i$ contains at least two nonzero summands in contradiction with item 1).

4) If $\tau_1, \tau_2 \in \Sigma_r$, $\tau_1 \neq \tau_2$, then φ, ψ are joint. Indeed, since $P_{\tau_t} = \{(x_{i_t}, z_{k_t})\}$, then $\varphi x_{i_t} \neq 0$ for $t = 1, 2$, and hence there exist y_{j_1}, y_{j_2} such that $(x_{i_t}, y_{j_t}) \in P_{\varphi}$. By item 3), $x_{i_1} \neq x_{i_2}$ and hence $P_{\varphi} = \{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})\}$,

and φ is joint. Obviously, both $(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})$ have the same maximal height. By Lemma 7, the pairs $(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})$ are incomparable, and hence $y_{j_1} \neq y_{j_2}$ by Remark 6. Since $\psi y_{j_t} \parallel r x_{i_t} \parallel z_{k_t}, t = 1, 2$, then ψ is joint as well, and $P_\psi = \{(y_{j_1}, z_{k_1}), (y_{j_2}, z_{k_2})\}$. The pairs $(y_{j_1}, z_{k_1}), (y_{j_2}, z_{k_2})$ are incomparable as above. In particular, $z_{k_1} \neq z_{k_2}$. Due to Remark 4, $|\Sigma_r| = p \leq 2$. In the case $p = 1$ the assertion of Lemma is obvious. Now assume $p = 2$.

5) The pairs $(x_{i_1}, z_{k_1}), (x_{i_2}, z_{k_2})$ are incomparable. Otherwise, up to renumbering, $x_{i_1} < x_{i_2} < z_{k_2} < z_{k_1}$. But $x_{i_1} < y_{j_1} < z_{k_1}$ and $x_{i_2} < y_{j_2} < z_{k_2}$. The elements y_{j_1} and y_{j_2} are comparable, hence either $x_{i_1} < y_{j_1} < y_{j_2} < z_{k_2} < z_{k_1}$, and $(y_{j_2}, z_{k_2}) \prec (y_{j_1}, z_{k_1})$, or $x_{i_1} < x_{i_2} < y_{j_2} < y_{j_1}$, and so $(x_{i_2}, y_{j_2}) \prec (x_{i_1}, y_{j_1})$. Both cases contradict to 4).

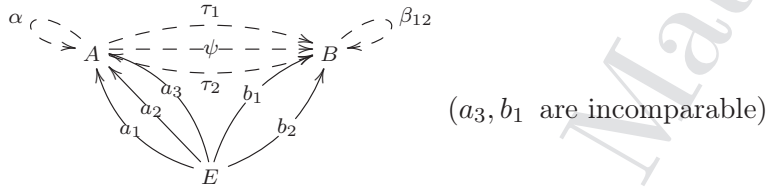
6) Let $r = \psi\varphi \in R(A, B)$, $\varphi = \alpha \in \Sigma_1^1(A, A)$. Then by 4), α and $\psi \in \Sigma_1^1(A, B)$ are joint. Hence $q_A = 3$, $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, $P_\alpha = \{(a_1, a_2), (a_2, a_3)\}$, $P_\psi = \{(a_2, b_{j_1}), (a_3, b_{j_2})\}$, $j_1 < j_2$. Using 4) and 5), we obtain $\psi\alpha = \lambda_1\tau_1 + \lambda_2\tau_2$, where $\lambda_1, \lambda_2 \in \mathbb{k}^*$ and $\tau_1, \tau_2 \in \Sigma_1^1(A, B)$ are single, and $P_{\tau_t} = \{(a_t, b_{j_t})\}$. By Corollary 2 and Definition 11, $C(a_t, b_{j_t}) = \{\tau_t\}$, $C(a_{t+1}, b_{j_t}) = \{\psi\}$, $t = 1, 2$. So we have the following partially ordered set:



where $\xrightarrow{\xi}$ means \prec , and $C(b_{j_1}, b_{j_2}) = \{\beta_{j_1j_2}\}$. If $|\Sigma_{\beta_{j_1j_2}\psi}| > 1$, then $\beta_{j_1j_2}$ is joint by 4), and the proof follows from Lemma 9. Otherwise $\beta_{j_1j_2}\psi \parallel \tau_2$.

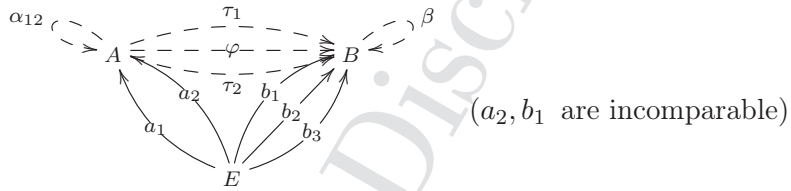
Assume that $b_{j_1} < a_3$ with $C(b_{j_1}, a_3) = \{\xi\}$. Then $a_1 < a_2 <_\psi b_{j_1} <_\xi a_3 < b_{j_2}$. We obtain $\xi\psi a_2 \parallel a_3$ and hence $\xi\psi \parallel \alpha + \mu\alpha^2$, $\mu \in \mathbb{k}$. In this case $\xi\psi a_1 \parallel a_2 + \mu a_3 \neq 0$ and $\psi a_1 \neq 0$ which contradicts to Remark 4. If $a_3 < b_{j_1}$ with $C(a_3, b_{j_1}) = \{\xi\}$, then $a_1 < a_2 <_\alpha a_3 <_\xi b_{j_1} < b_{j_2}$ and hence $\xi\alpha a_2 \parallel b_{j_1}$, and therefore $h(\psi) \geq h(\xi\alpha) > 1$ which contradicts to Definition 11, since ψ is joint. Thus, a_3 and b_{j_1} are incomparable. By 3) of Definition 4, $q_B = 2$. In particular, $j_1 = 1, j_2 = 2$.

Therefore, we have the following bigraph²:

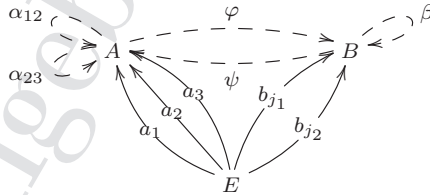


According to 2) of Definition 8 multiplication rule holds for $\psi\varphi$.

7) Let $r = \psi\varphi \in R(A, B)$, $\psi = \beta \in \Sigma_1^1(B, B)$. Then similarly $q_A = 2$, $q_B = 3$, $\Sigma_1^1(B, B) = \{\beta, \beta^2\}$, $\varphi, \tau_1, \tau_2 \in \Sigma_1^1(A, B)$, $\psi = \beta, \beta\varphi \in \mathbb{k}^*\langle \tau_1, \tau_2 \rangle$, $P_\varphi = \{(a_1, b_1), (a_2, b_2)\}$, $P_\beta = \{(b_1, b_2), (b_2, b_3)\}$, $P_{\tau_t} = \{(a_t, b_{t+1})\}$, $t = 1, 2$.



8) Let $r = \psi\varphi \in R(A, A)$. Then $q_A = 3$, $q_B \geq 2$, $\varphi \in \Sigma_1^1(A, B)$, $\psi \in \Sigma_1^1(B, A)$, $\psi\varphi = \mathbb{k}^*\langle \alpha_{12}, \alpha_{23} \rangle$, $\alpha_{12}, \alpha_{23} \in \Sigma_1^1(A, A)$, $\alpha_{12} \neq \alpha_{23}$, and $P_\varphi = \{(a_1, b_{j_1}), (a_2, b_{j_2})\}$, $P_\psi = \{(b_{j_1}, a_2), (b_{j_2}, a_3)\}$, $1 \leq j_1 < j_2 \leq q_B$;



If $q_B = 3$ and $b \in \Sigma_1^1(E, B) \setminus \{b_{j_1}, b_{j_2}\}$ then either $b < a_1$ or $a_3 < b$. Besides, $\beta_{12} \neq \beta_{23}$.

Indeed, by 2), $\alpha_{12} \neq \alpha_{23}$. By 3), 4), 5) we have $a_{i_1} <_\varphi b_{j_1} <_\psi a_{k_1}$, $a_{i_2} <_\varphi b_{j_2} <_\psi a_{k_2}$, and hence $a_1 <_\varphi b_{j_1} <_\psi a_2 <_\varphi b_{j_2} <_\psi a_3$. Then $ra_t \parallel a_{t+1}$, $\varphi a_t \parallel b_{j_t}$, $\psi b_{j_t} \parallel a_{t+1}$, $t = 1, 2$, and $r = \lambda\alpha_{12} + \mu\alpha_{23}$, $\lambda, \mu \in \mathbb{k}^*$.

If $q_B = 3$, then there exists $b_k \in \Sigma_1^0(E, B) \setminus \{b_{j_1}, b_{j_2}\}$. Since φ is joint, then $h(\varphi) = 1$ by Definition 11, and hence $b_k < a_1$ or $a_3 < b_k$. Assume $\beta_{12} = \beta_{23} = \beta$. If $b_1 < a_1 < b_2 < a_2 < b_3 < a_3$, then $\varphi\psi \parallel \beta + \mu\beta^2$, $\mu \in \mathbb{k}$, and hence $\psi b_1 \neq 0$ which is impossible by Remark 4. If $a_1 <_\varphi b_1 <_\psi a_2 <_\varphi$

²Note that the pictured subbigraphs can be non-full.

$b_2 < a_3 < b_3$, then $\varphi\psi\|\beta + \mu\beta^2$, $\mu \in \mathbb{k}$, and hence $\varphi a_3 \neq 0$ in contradiction with Remark 4. Finally, if we have the special case from Lemma 14, and $\varphi, \varphi' \in \mathcal{C}(a_1, b_1)$, then $\psi\varphi' \|\alpha_{12}$ since otherwise $\varphi' \in \mathcal{C}(a_2, b_2)$.

9) The case $r = \psi\varphi \in \mathcal{R}(B, B)$ can be obtained from 8) by swapping A and B . \square

The proof of this Lemma implies the following result.

Remark 7. If $q_A \leq 2$, $q_B \leq 2$, then all the arrows from Σ_1^1 are single and the product of two basic elements is basic up to scalar multiplier.

The proof of Proposition 2 is complete.

5. General case

Remark 8. Let $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ be a faithful connected bimodule problem from \mathcal{C} with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$ and a triangled basis Σ , and $A \in \Sigma_0^+$, $E \in \Sigma_0^-$. In this case $\mathbb{R}_{\{A, E\}} = \mathbb{R}(A, A)$. If $\mathbb{V}(E, A) \neq 0$ and $\text{Ann}_{\mathbb{R}(A, A)} \mathbb{V}(E, A) \neq 0$, then $\dim_{\mathbb{k}} \mathbb{V}(E, A) = 1$ and $\text{Ann}_{\mathbb{R}(A, A)} \mathbb{V}(E, A) = \mathbb{R}(A, A)$. Moreover, in this case $\text{ord } A = 3$, and there is $E_1 \in \Sigma_0^-$ such that $\dim_{\mathbb{k}} \mathbb{V}(E_1, A) = 2$, $\text{Ann}_{\mathbb{R}} \mathbb{V}(E_1, A) = 0$.

Proposition 3. *Let $\mathcal{A} = (\mathbb{K}, \mathbb{V})$ be a faithful connected bimodule problem from \mathcal{C} with nilpotent radical $\mathbb{R} = \text{Rad } \mathbb{K}$ and a triangled basis Σ such that $|\Sigma_0^+| = 2$ (but not necessary $|\Sigma_0^-| = 1$). Then there exists a standard change \mathfrak{C} of basis from Σ to quasi multiplicative one. If $\mathcal{A}^{(A)}$ for $A \in \Sigma_0^+$ is endowed with quasi multiplicative basis, then \mathfrak{C} leaves these bases unchangeable.*

To prove Proposition 3, let $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, \dots, E_p\}$. Since \mathcal{A} is faithful, then the restrictions $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are faithful as well by Remark 3. Due to Proposition 1, we can assume that both $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ have quasi multiplicative bases.

Denote by \mathcal{A}_i and \mathcal{A}'_i the restriction and the faithful restriction of \mathcal{A} to the set $\{A, B, E_i\}$, $i = 1, \dots, p$, respectively. Then $\mathbb{V}'_i = \mathbb{V}_i$, and $\mathbb{R}'_i = \mathbb{R}_i / \text{Ann}_{\mathbb{R}_i} \mathbb{V}_i$. For the proof of proposition, it is enough to check, that we can choose a correct basis simultaneously for all \mathcal{A}'_i , $i = 1, \dots, p$.

Due to connectivity of \mathcal{A} , there exists i such that $\mathbb{V}(E_i, A) \neq 0$ and $\mathbb{V}(E_i, B) \neq 0$. Therefore, if $|\Sigma_0^-| > 1$, then $\dim \mathbb{V}(E_j, A) \leq 2$ and $\dim \mathbb{V}(E_j, B) \leq 2$ for all $j \neq i$. If $\mathbb{V}(E_j, A) = 0$ for some j , then denote $\tilde{\mathcal{A}} = (\tilde{\mathbb{K}}, \tilde{\mathbb{V}}) = \mathcal{A}_{\{A, B, \Sigma_0^- \setminus \{E_j\}\}}$. If $\dim \mathbb{V}(E_j, B) = 1$ then $\tilde{\mathcal{A}}$ is faithful as well as \mathcal{A} , and the proof for \mathcal{A} follows from the proof for $\tilde{\mathcal{A}}$. Otherwise, if $\dim \mathbb{V}(E_j, B) = 2$, then $\Sigma_1^1(B, B) = \{\beta_{12}\}$ by Proposition 1, and

$\text{Ann}_{\bar{\mathbb{K}}}\tilde{\mathcal{V}} = \langle \beta_{12} \rangle$. In this case the proof for \mathcal{A} follows from the proof for the faithful part of $\tilde{\mathcal{A}}$.

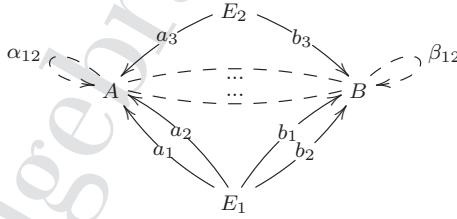
So we can assume that $\dim_{\mathbb{K}}\mathcal{V}(E_i, A) > 0$ and $\dim_{\mathbb{K}}\mathcal{V}(E_i, B) > 0$ for all $i = 1, \dots, p$. By Definition 4, $\sum_{i=1}^p \dim_{\mathbb{K}}\mathcal{V}(E_i, A) \leq 3$, $\sum_{i=1}^p \dim_{\mathbb{K}}\mathcal{V}(E_i, B) \leq 3$. Therefore $p \leq 3$. If $p = 1$, then the proof follows from Proposition 2. Hence $p = 2$ or $p = 3$.

Assume there exists $E_1 \in \Sigma_0^-$ such that $\dim_{\mathbb{K}}\mathcal{V}(E_1, A) = 3$ (resp., $\dim_{\mathbb{K}}\mathcal{V}(E_1, B) = 3$); then $p = 1$. In the case $\dim_{\mathbb{K}}\mathcal{V}(E_1, A) = 2$ (resp., $\dim_{\mathbb{K}}\mathcal{V}(E_1, B) = 2$) we have $p \leq 2$. Finally, $p = 3$ only for the case $\dim_{\mathbb{K}}\mathcal{V}(E_i, A) = \dim_{\mathbb{K}}\mathcal{V}(E_i, B) = 1$ for $i = 1, 2, 3$.

Consider the case $\Sigma_0^- = \{E_1, E_2\}$. By Proposition 2 \mathcal{A}'_1 is endowed with quasi multiplicative basis $\Sigma_1^1(\mathcal{A}'_1)$.

Lemma 16. *Assume $\Sigma_0^- = \{E_1, E_2\}$, $\dim_{\mathbb{K}}\mathcal{V}(E_2, A) = \dim_{\mathbb{K}}\mathcal{V}(E_2, B) = 1$. Then Proposition 3 holds.*

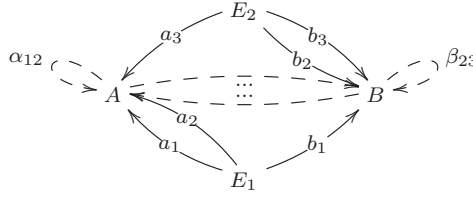
Proof. Assume \mathcal{A}'_1 satisfies Proposition 2. We have $1 \leq |\Sigma_1^0(E_1, A)| \leq 2$, $1 \leq |\Sigma_1^0(E_1, B)| \leq 2$. Consider the case $\Sigma_1^0(E_1, A) = \{a_1, a_2\}$, $a_1 < a_2$, $\Sigma_1^0(E_1, B) = \{b_1, b_2\}$, $b_1 < b_2$. By assumption, $\Sigma_1^0(E_2, A) = \{a_3\}$, and $\Sigma_1^0(E_2, B) = \{b_3\}$.



Note that $\mathcal{S}(a_3, b_3) = \mathcal{C}(a_3, b_3)$. Due to triangularity condition, either $\mathcal{C}(a_3, b_3) = \emptyset$ or $\mathcal{C}(b_3, a_3) = \emptyset$. Since \mathcal{A} is faithful, and $h(\varphi) = 1$ for any $\varphi \in \mathcal{C}(a_3, b_3)$, then $\mathcal{C}(a_3, b_3)$ contains at least one $\varphi \notin \Sigma_1^1(\mathcal{A}'_1)$. If $\varphi, \psi \in \mathcal{C}(a_3, b_3)$, $\varphi \notin \Sigma_1^1(\mathcal{A}'_1)$, $\psi \in \Sigma_1^1(\mathcal{A}'_1)$, then correct elementary change $\mathfrak{C}_\lambda(\varphi, \psi)$ leads to condition $\mathcal{C}(a_3, b_3) = \{\varphi\}$ without corruption of multiplication on $\Sigma_1^1(\mathcal{A}'_1)$. Finally, if $\mathcal{C}(a_3, b_3) \subset \Sigma_1^1(\mathcal{A}'_1)$, then $|\mathcal{C}(a_3, b_3)| \leq 2$ while $h_m(a_3, b_3) = 1$. If $\mathcal{C}(a_3, b_3) = \{\varphi\}$, then φ can not be joint in \mathcal{A}'_1 by Remark 4. Hence, φ is joint in \mathcal{A} . If $\mathcal{C}(a_3, b_3) = \{\varphi, \psi\}$, φ, ψ are joint parallel in \mathcal{A} . Obviously, multiplication rule holds in any case. The cases $|\Sigma_1^0(E_1, A)| = 1$ or $|\Sigma_1^0(E_1, B)| = 1$ are similar. \square

Lemma 17. *Assume $\Sigma_0^- = \{E_1, E_2\}$, $\Sigma_1^0(E_1, A) = \{a_1, a_2\}$, $a_1 < a_2$, $\Sigma_1^0(E_2, A) = \{a_3\}$, $\Sigma_1^0(E_1, B) = \{b_1\}$, $\Sigma_1^0(E_2, B) = \{b_2, b_3\}$, $b_2 < b_3$. Then Proposition 3 holds.*

Proof. Let us apply Proposition 2 to \mathcal{A}'_1 . We have

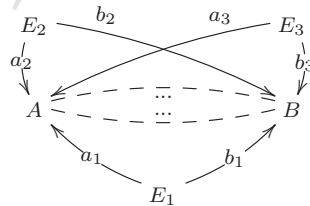


Denote $\Xi = \Sigma_1^1(\mathcal{A}'_1)(A, B)$. Since \mathcal{A}'_1 has quasi multiplicative basis $\Sigma_1^1(\mathcal{A}'_1)$, then $|\Xi \cup \Sigma_1^1(\mathcal{A}'_1)(B, A)| \leq 2$. If $\Xi = \{\varphi, \psi\}$, then $a_1 <_{\alpha_{12}} a_2 <_{\varphi} b_1$, $a_1 < b_1$ and hence $h(\varphi) = 1$, $\psi \parallel \varphi \alpha_{12}$. By the associativity condition, $\psi \notin \Sigma_1^1(\mathcal{A}'_2)(A, B)$. If $\Xi = \{\varphi\}$, then $h(\varphi) = 1$. Thus, if $\Xi \neq \emptyset$, then Ξ contains the unique φ of height 1. If $\varphi \notin \Sigma_1^1(\mathcal{A}'_2)$ or $\Xi = \emptyset$, then there is a standard change of $\Sigma_1^1(\mathcal{A}'_2)(A, B)$ such that each pair $(a_i, b_j) \in \Pi_{A,B}(\mathcal{A}'_2)$ is adjusted and every arrow in $\Sigma_1^1(A, B)$ is single.

Let $\varphi \in \Xi \cap \Sigma_1^1(\mathcal{A}'_2)$. Then $h(\varphi) = 1$. Consider $(a_i, b_j) \in \Pi_{A,B}(\mathcal{A}'_2)$. If $\varphi, \psi \in \mathcal{S}(a_i, b_j)$, then $\mathcal{C}_\lambda(\psi, \varphi)$ leads to condition $\varphi \notin \mathcal{S}(a_i, b_j)$. If $\varphi \notin \mathcal{S}(a_i, b_j)$, then $|\mathcal{S}(a_i, b_j)| = 1$ and $\mathcal{S}(a_i, b_j) = \mathcal{C}(a_i, b_j)$ after some standard basis change on $\Sigma_1^1(\mathcal{A}'_2)(A, B)$. If $\mathcal{S}(a_i, b_j) = \{\varphi\}$, then φ is joint. In any case, each arrow from $\Sigma_1^1(A, B) \setminus \{\varphi\}$ is single. All changes above does not corrupt quasi multiplicativity of $\Sigma_1^1(\mathcal{A}'_1)$. The basis on $\Sigma_1^1(\mathcal{A}'_2)(B, A)$ can be chosen similarly. It is clear that multiplication rule holds on \mathcal{A} . \square

Lemma 18. Let $\Sigma_0^- = \{E_1, \dots, E_p\}$, $2 \leq p \leq 3$, and $\Sigma_1^0(E_i, A) = \{a_i\}$, $\Sigma_1^0(E_i, B) = \{b_i\}$, $i = 1, \dots, p$. Then Proposition 3 holds.

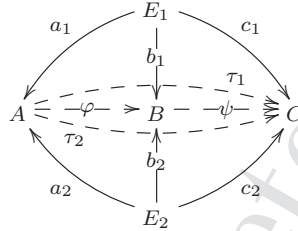
Proof. Here we have



As above, applying a standard basis change we obtain that each comparable pair is adjusted and every arrow from Σ_1^1 is either single or joint, which implies multiplication rule on \mathcal{A} . \square

The proof of Proposition 3 is completed.

Lemma 19. *Let $\mathcal{A} \in \mathcal{C}$ be a faithful bimodule problem with a triangled basis Σ , and $\Sigma_0^+ = \{A, B, C\}$. If $\varphi \in \Sigma_1^1(A, B)$, $\psi \in \Sigma_1^1(B, C)$, and $|\{A, B, C\}| = 3$, then $\psi\varphi$ belongs to Σ_1^1 except of the case:*



where $\psi\varphi \in \mathbb{k}^*(\tau_1, \tau_2)$, $\tau_1, \tau_2 \in \Sigma_1^1(A, C)$, and $P_\varphi = \{(a_1, b_1), (a_2, b_2)\}$; $P_\psi = \{(b_1, c_1), (b_2, c_2)\}$, $P_{\tau_t} = \{(a_t, c_t)\}$, $t = 1, 2$.

The proof is similar to the proof of Lemma 15.

The proof of Theorem 1. Since \mathcal{A} is a faithful, then the bimodule problems $\mathcal{A}^{(A)}$, $\mathcal{A}^{(A,B)}$ are faithful as well for any $A, B \in \Sigma_0^+$ by Remark 3. Applying consequently Proposition 1 to $\mathcal{A}^{(A)}$, Proposition 3 to $\mathcal{A}^{(A,B)}$, and Lemma 19 for all $A, B, C \in \Sigma_0^+$, we obtain the proof of Theorem 1. \square

Conclusion

The main result of the paper states the existence of quasi multiplicative basis for a faithful connected finite dimensional bimodule problem with nilpotent radical from considered class \mathcal{C} . The authors are going to apply the obtained results to construct the universal covering for such problem.

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