Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 12 **(2011)**. Number 1. pp. 53 – 68

Characterization of finite simple semigroup digraphs

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Communicated by B. V. Novikov

Abstract. This paper characterizes directed graphs which are Cayley graphs of finite simple semigroups, i.e. of a subspecies of completely regular semigroups. Moreover we investigate the structure of Cayley graphs of finite simple semigroups with a one-element connection set. We introduce the conditions for which they are isomorphic and connected.

Introduction

One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups G which possess a generating system A such that the Cayley graph $Cay(G, A)$ is planar, see for example [15]. In [10] Cayley graphs which represent groupoids, quasigroups, loops or groups are described. The result for groups originates from [15] and is meanwhile folklore, see for example [2]. After this it is natural to investigate Cayley graphs for semigroups which are unions of groups, so-called completely regular semigroups, see for example [12]. In [1] Cayley graphs which represent completely regular semigroups which are right (left) groups are characterized. We now investigate Cayley graphs which represent finite simple semigroups. Recent studies in different directions investigate transitivity of Cayley graph of groups and semigroups [7] and of right and left groups and of Clifford semigroups [11]. Other related results can be found,

2000 Mathematics Subject Classification: *05C25, 08B15, 20M19, 20M30.*
Key words and phrases: *Cayley graph, digraph, completely regular, completely simple semigroups, Rees matrix semigroup..*

for example, in [5], [7] and [9]. Relations to network theory together with many theoretical results are presented in [4]. The concept of Cayley graph of a groupoid has also been considered in relation to automata theory in a book by A.V. Kelarev [6].

In this paper , we characterize finite simple semigroup digraphs. We also describe the structure of Cayley graphs of finite simple semigroup with a one-element connection set. Moreover we introduce the conditions for which they are isomorphic and connected.

1. Basic definitions and results

All sets in this paper are assume to be finite. A semigroup S is called *simple* if it has no proper ideals. An element a of a semigroup S is *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called *completely regular* if all its elements are completely regular. A completely regular semigroup S is called *completely simple* if it is simple.

Since all sets in this paper are finite, simple semigroups are completely simple semigroups. In this case a simple semigroup is always union of groups.

Suppose that G is a group, I and Λ are nonempty sets, and P is a $\Lambda \times I$ matrix over a group G.

The *Rees matrix semigroup* $\mathcal{M}(G, I, \Lambda, P)$ *with sandwich matrix* P consists of all triples (g, i, λ), where $i \in I, \lambda \in \Lambda$, and $g \in G$ with multiplication defined by the rule $(g_1, i_1, \lambda_1)(g_2, i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2, i_1, \lambda_2)$.

For an element g of group G we denote by $|g|$ the *order of* g.

Theorem 1 ([13]). *A semigroup S is completely simple if and only if S is isomorphic to a Rees matrix semigroup.*

In the sequel we will mainly use the term Rees matrix semigroup instead of completely simple semigroup.

For an element of completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ we denote by p_1, p_2 and p_3 the natural projections of S onto G, onto I and onto Λ, respectively.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism*. If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism*. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is calleded a *digraph automorphism*.

Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The disjoint union of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := \left(\dot{\cup}_{i=1}^{n} V_i, \dot{\cup}_{i=1}^{n} E_i\right)$. Let $(V, E_1), (V, E_2), \ldots, (V, E_n)$ be digraphs. The edge sum of $(V, E_1), (V, E_2), \ldots, (V, E_n)$ is defined as $\bigoplus_{i=1}^{n}(V, E_i) := (V, \cup_{i=1}^{n} E_n)$ see for example [10].

Let S be a semigroup (group) and $A \subseteq S$. We define the *Cayley graph* $Cay(S, A)$ as follows: S is the vertex set and (u, v), $u, v \in S$, is an arc in $Cay(S, A)$ if there exists an element $a \in A$ such that $v = ua$. The set A is called the *connection set* of $Cay(S, A)$.

A digraph (V, E) is called a *semigroup (group) digraph* or *digraph of a semigroup (group)* if there exists a semigroup (group) S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $Cay(S, A)$.

For terms in graph theory not defined here see for example [2].

Theorem 2. ([2], [13], [14]) *A digraph (V, E) is a Cayley graph of a group G if and only if $Aut(V, E)$ contains a subgroup Δ which is isomorphic to G and for any two vertices $u, v \in V$ there exists $\sigma \in \Delta$ such that $\sigma(u) = v$.*

By the definition of completely simple semigroup, we have the following lemma.

Lemma 1. *Let G be a group, $S = \mathcal{M}(G, I, \Lambda, P)$ a completely simple semigroup, $A \subseteq S$, and let $(g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2) \in S$. Then $((g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2))$ is an arc in $Cay(S, A)$ if and only if there exists $a = (g, l, \lambda_2) \in A$ such that $g_2 = g_1 p_{\lambda_1 l} g$ and $i_1 = i_2$.*

A subdigraph F of a digraph G is called a strong subdigraph of G if and only if whenever u and v are vertices of F and (u, v) is an arc in G, then (u, v) is an arc in F as well.

2. Cayley graphs of finite simple semigroups

In view of Lemma 3.2 in [8], we know that a completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ is a right group if and only if $|I| = 1$.

In [1] Cayley graphs which represent right groups are characterized. Then we obtain the following proposition.

Proposition 1. *Let G be a group, $S = \mathcal{M}(G, \{i\}, \Lambda, P)$.*
Take $a = (g, i, \beta) \in S$. Then

(1) *the Cayley graph $Cay(S, \{a\})$ contains a strong group subdigraph $Cay(G, \{p_{\beta i} g\})$;*

(2) *$((g_1, i, \lambda_1), (g_2, i, \lambda_2))$ is an arc in $Cay(S, \{a\})$ if and only if $g_2 = g_1 p_{\lambda_1 i} g$ and $\lambda_2 = \beta$.*

If $S = \mathcal{M}(G, I, \Lambda, P)$ is a completely simple semigroup, let p_i denote the projection of S on to its i^{th} coordinate. In the next theorem, we characterize a Cayley graph of a finite simple semigroup.

Theorem 3. *A digraph (V, E) is a Cayley graph of a finite simple semigroup if and only if the following conditions hold*

(1) *(V, E) is the disjoint union of n isomorphic subdigraphs $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ for some $n \in \mathbb{N}$;*

(2) *(V_i, E_i) has n subdigraphs $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), ..., (V_{in}, E_{in})$ such that $(V_i, E_i) = \bigoplus_{j=1}^{n} (V_{ij}, E_{ij})$ with $V_i = V_{ij}$ for every $j \in \{1, 2, .., n\}$;*

(3) *(V_{ij}, E_{ij}) contains m disjoint strong subdigraphs $(V_{ij}^1, E_{ij}^1), (V_{ij}^2, E_{ij}^2), ..., (V_{ij}^m, E_{ij}^m)$ such that $V_{ij} = \bigcup_{\alpha=1}^{m} V_{ij}^{\alpha}$;*

(4) *there exists a group G and a family of digraph isomorphisms $\{f_{ij}^{\alpha}\}_{\alpha=1}^{m}$ such that $f_{ij}^{\alpha} : (V_{ij}^{\alpha}, E_{ij}^{\alpha}) \to Cay(G, a_{ij}^{\alpha} A_{ij}^{\alpha})$ for some $a_{ij}^{\alpha} \in G$, $A_{ij}^{\alpha} \subseteq G$ with $A_{kj}^{\alpha} = A_{tj}^{\alpha}$, $a_{kj}^{\alpha} = a_{tj}^{\alpha}$ for all $k, t \in \{1, 2, ..., n\}$;*

(5) *for each $u \in V_{ij}^{\alpha}, v \in V_{ij}^{\beta}$, $(u, v) \in E$ if and only if $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u) a_{ij}^{\alpha} a$ for some $a \in A_{ij}^{\beta}$.*

Proof. $(\Rightarrow)$ Let (V, E) be a Cayley graph of a finite simple semigroup. Then there exists a finite simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ where G is a group, $I = \{1, 2, ..., n\}$, $\Lambda = \{1, 2, ..., m\}$, and P is a $\Lambda \times I$ matrix over a group G, such that $(V, E) \cong Cay(S, A)$ for some $A \subseteq S$. Hence we will prove that $(1), (2), (3), (4)$ and (5) are true for $Cay(S, A)$.

(1) For each $i \in I$, set $V_i := G \times \{i\} \times \Lambda$, and $E_i := E(Cay(S, A)) \cap (V_i \times V_i)$. Hence (V_i, E_i) is a strong subdigraph of $Cay(S, A)$ and $Cay(S, A) = \bigcup_{i=1}^{n} (V_i, E_i)$. We show that $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ are isomorphic. Let $p, q \in I$, $p \neq q$, define a map ϕ from (V_p, E_p) to (V_q, E_q) by $\phi((g, p, r)) = (g, q, r)$. Since $|V_p| = |V_q|$, ϕ is a well defined bijection. To prove that ϕ and ϕ^{-1} are digraph homomorphisms. For $(g, p, r), (g', p, r') \in V_p$, take $((g, p, r), (g', p, r')) \in E_p$. Since $E_p \subseteq E(Cay(S, A))$, $((g, p, r), (g', p, r'))$ is an arc in $Cay(S, A)$. By Lemma 1, there exists $(a, l, r'') \in A$ such that $g' = gp_{rl}a$, $r' = r''$, and thus $(g', q, r') = (gp_{rl}a, q, r'') = (g, q, r)\ (a, l, r'')$. Then $((g, q, r), (g', q, r'))$ is an arc in $Cay(S, A)$. It follows that $((g, q, r),\ (g', q, r')) \in E_q$. This shows that ϕ is a digraph homomorphism. Similarly, ϕ^{-1} is a digraph homomorphism. Hence ϕ is a digraph isomorphism. Now we prove that $Cay(S, A)$ is the disjoint union of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$. By definition of V_i,

$S = \dot{\bigcup} V_i$. Since $E_i \subseteq E\big(Cay(S,A)\big)$, $\dot{\bigcup} E_i \subseteq E\big(Cay(S,A)\big)$. Let $\big((g,j,r),(\acute{g},k,\acute{r})\big) \in E(Cay(S,A)$. By Lemma 1, $j = k$, and thus $\big((g,j,r),(\acute{g},k,\acute{r})\big) \in E_k$. Then $\big((g,j,r),(\acute{g},k,\acute{r})\big) \in \dot{\bigcup} E_i$. Hence $E\big(Cay(S,A)\big) \subseteq \dot{\bigcup} E_i$, and so $E\big(Cay(S,A)\big) = \dot{\bigcup} E_i$. Therefore $Cay(S,A) = \dot{\bigcup}(V_i, E_i)$.

(2) Let $S_{ij} = \mathcal{M}(G,\{i\},\Lambda,P_j)$ where $i,j \in \{1,2,..,n\}$,

$$P_j = \begin{bmatrix} p_{1j} \\ p_{2j} \\ . \\ . \\ . \\ p_{mj} \end{bmatrix},$$

and P_j is the j^{th} column of P, and let $A_{ij} =: \{(g,i,\beta)|(g,j,\beta) \in A\}$. Take $(V_{ij},E_{ij}) =: Cay(S_{ij},A_{ij})$. Hence $V_{ij} = \mathcal{M}(G,\{i\},\Lambda,P_j) = V_i$. To prove that $(V_i,E_i) = \bigoplus_{j=1}^{n}(V_{ij},E_{ij})$. Let $((g,i,\alpha),(g',i,\beta)) \in E_i$. Therefore $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S,A)$. By Lemma 1, there exists $(g'',l,\gamma) \in A$ such that $\gamma = \beta$ and $g' = gp_{\alpha l}g''$. Since $(g'',l,\beta) \in A$, $(g'',i,\beta) \in A_{il}$. Thus $((g,i,\alpha),\ (g',i,\beta))$ is an arc in $Cay(S_{il},A_{il})$ as $(g,i,\alpha)(g'',i,\beta) = (gp_{\alpha l}g'',i,\beta) = (g',i,\beta)$. Therefore $((g,i,\alpha)\,,(g',i,\beta)) \in E_{il} \subseteq \bigcup_{j=1}^{n} E_{ij}$, and so $E_i \subseteq \bigcup_{j=1}^{n} E_{ij}$. Let $((g,i,\alpha),\ (g',i,\beta)) \in \bigcup_{j=1}^{n} E_{ij}$. Therefore $((g,i,\alpha),(g',i,\beta)) \in E_{il}$ for some $l \in \{1,2,..,n\}$. It follows that $((g,i,\alpha),(g',\ i,\beta))$ is an arc in $Cay(S_{il},A_{il})$. Then there exists $(g'',i,\gamma) \in A_{il}$ such that $\gamma = \beta$ and $g' = gp_{\alpha l}g''$ by Lemma 1. Hence $(g'',l,\beta) \in A$ because $(g'',i,\beta) \in A_{il}$. Then $(g',i,\beta) = (gp_{\alpha l}g'',i,\beta) = (g,i,\alpha)(g'',l,\beta)$, $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S,A)$. Therefore $((g,i,\alpha),\ (g',i,\beta)) \in E_i$, and thus $\bigcup_{j=1}^{n} E_{ij} \subseteq E_i$. Hence $E_i = \bigcup_{j=1}^{n} E_{ij}$. This show that $(V_i,E_i) = \bigoplus_{j=1}^{n}(V_{ij},E_{ij})$.

(3) Set $V_{ij}^{\alpha} := \mathcal{M}(G,\{i\},\{\alpha\},P_j^{\alpha})$, $E_{ij}^{\alpha} := E(Cay(S_{ij},A_{ij})) \cap (V_{ij}^{\alpha} \times V_{ij}^{\alpha})$ where $P_j^{\alpha} = [p_{\alpha j}]$ is an 1×1 matrix. Therefore $V_{ij}^{\alpha} \subseteq V_{ij}$ and thus $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ is a strong subdigraph of (V_{ij},E_{ij}). Let $\alpha,\beta \in \Lambda$ and $\alpha \neq \beta$. To prove that $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ and $(V_{ij}^{\beta},E_{ij}^{\beta})$ are disjoint. Since $V_{ij}^{\alpha} \cap V_{ij}^{\beta} = \emptyset$, by the definition of E_{ij}^{α} and E_{ij}^{β}, $E_{ij}^{\alpha} \cap E_{ij}^{\beta} = \emptyset$. Therefore $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ and $(V_{ij}^{\beta},E_{ij}^{\beta})$ are disjoint subdigraphs of (V_{ij},E_{ij}). Hence $\bigcup_{\alpha=1}^{m} V_{ij}^{\alpha} = \bigcup_{\alpha=1}^{m} \mathcal{M}(G,\{i\},\{\alpha\},[p_{\alpha j}]) = \mathcal{M}(G,\{i\},\Lambda,P_j) = V_i$.

(4) Let $A_{ij}^{\alpha} := \{g|(g,i,\alpha) \in A_{ij}\}$. To prove that $(V_{ij}^{\alpha},E_{ij}^{\alpha}) \cong Cay(G,a_{ij}^{\alpha}A_{ij}^{\alpha})$ where $a_{ij}^{\alpha} = p_{\alpha j}$. Let $f_{ij}^{\alpha} : (V_{ij}^{\alpha},E_{ij}^{\alpha}) \to$

$Cay(G, p_{\alpha j}A_{ij}^{\alpha})$ be the projection of V_{ij}^{α} on to its first coordinate, i.e. $f_{ij}^{\alpha} = p_1$. We first show that f_{ij}^{α} and $f_{ij}^{\alpha-1}$ are digraph homomorphisms. For (g, i, α), $(g', i, \alpha) \in V_{ij}^{\alpha}$, take $((g, i, \alpha), (g', i, \alpha)) \in E_{ij}^{\alpha}$. By the definition of E_{ij}^{α}, $((g, i, \alpha), (g', i, \alpha))$ is an arc in $Cay(S_{ij}, A_{ij})$. Then there exists $(g'', i, \gamma) \in A_{ij}$ such that $g' = gp_{\alpha j}g''$ and $\alpha = \gamma$ by Lemma 1. Thus there is $g'' \in A_{ij}^{\alpha}$. It follows that $(f_{ij}^{\alpha}(g, i, \alpha), f_{ij}^{\alpha}(g', i, \alpha)) = (g, g')$ is an arc in $Cay(G, p_{\alpha j}A_{ij}^{\alpha})$. Hence f_{ij}^{α} is a digraph homomorphism. For $g, g' \in G$, let (g, g') be an arc in $Cay(G, p_{\alpha j}A_{ij}^{\alpha})$. Therefore $g' = gp_{\alpha j}g''$ for some $g'' \in A_{ij}^{\alpha}$. By the definition of A_{ij}^{α}, there is $(g'', i, \alpha) \in A_{ij}$ and so $(g', i, \alpha) = (gp_{\alpha j}g'', i, \alpha) = (g, i, \alpha)(g'', i, \alpha)$. Therefore $((g, i, \alpha), (g', i, \alpha))$ is an arc in $Cay(S_{ij}, A_{ij})$. This shows that $f_{ij}^{\alpha-1}$ is a digraph homomorphism. Let $k, t \in \{1, 2, ..., n\}$. We show that $A_{kj}^{\alpha} = A_{tj}^{\alpha}$. Take $g \in A_{kj}^{\alpha}$. Then $(g, k, \alpha) \in A_{kj}$ and $(g, j, \alpha) \in A$. By the definition of A_{tj}, $(g, t, \alpha) \in A_{tj}$, and thus $g \in A_{tj}^{\alpha}$. This shows that $A_{kj}^{\alpha} \subseteq A_{tj}^{\alpha}$. Similarly $A_{kj}^{\alpha} \supseteq A_{tj}^{\alpha}$. Thus $A_{kj}^{\alpha} = A_{tj}^{\alpha}$ for all $k, t \in \{1, 2, ..., n\}$. Since $a_{kj}^{\alpha} = p_{\alpha j}$ and $a_{tj}^{\alpha} = p_{\alpha j}$, $a_{kj}^{\alpha} = a_{tj}^{\alpha}$ for all $k, t \in \{1, 2, ..., n\}$.

(5) For each $u = (g, i, \alpha) \in V_{ij}^{\alpha}$, and $v = (g', i, \beta) \in V_{ij}^{\beta}$. We prove that $((g, i, \alpha), (g', i, \beta)) \in E$ if and only if $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ for some $a \in A_{ij}^{\beta}$.
($\Rightarrow$) Let $((g, i, \alpha), (g', i, \beta)) \in E$. Then $((g, i, \alpha), (g', i, \beta))$ is an arc in $Cay(S, A)$. By Lemma 1, there exists $(a, j, \xi) \in A$ such that $g' = gp_{\alpha j}a$ and $\beta = \xi$. Then we have that $(a, j, \beta) = (a, j, \xi) \in A$. By the definition of A_{ij}, there exists $(a, i, \beta) \in A_{ij}$, and hence $a \in A_{ij}^{\beta}$. Therefore $f_{ij}^{\beta}(v) = g' = gp_{\alpha j}a = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ where $a_{ij}^{\alpha} = p_{\alpha j}$.
($\Leftarrow$) Let $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ for some $a \in A_{ij}^{\beta}$. Therefore $g' = f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a = ga_{ij}^{\alpha}a$ and there exists $(a, i, \beta) \in A_{ij}$. By the definition of A_{ij}, there is $(a, j, \beta) \in A$. Since $a_{ij}^{\alpha} = p_{\alpha j}$, $(g', i, \beta) = (ga_{ij}^{\alpha}a, i, \beta) = (gp_{\alpha j}a, i, \beta) = (g, i, \alpha)(a, j, \beta)$, and thus $((g, i, \alpha), (g', i, \beta))$ is an arc in $Cay(S, A)$.

($\Leftarrow$) Choose $k \in \{1, 2, ..., n\}$, by (1), (2), and (3), we get $V = \bigcup_{i=1}^{n}\bigcup_{\alpha=1}^{m} V_{ik}^{\alpha}$ is the disjoint union. Let $S = \mathcal{M}(G, I, \Lambda, P)$ where $I = \{1, 2, ..., n\}$, $\Lambda = \{1, 2, ..., m\}$,

$$P = \begin{bmatrix} a_{k1}^{1} & a_{k2}^{1} & \cdots & a_{kn}^{1} \\ a_{k1}^{2} & a_{k2}^{2} & \cdots & a_{kn}^{2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}^{m} & a_{k2}^{m} & \cdots & a_{kn}^{m} \end{bmatrix},$$

and let $A := \bigcup_{\alpha=1}^{m}\bigcup_{j=1}^{n}(A_{jk}^{\alpha} \times \{k\} \times \{\alpha\})$. We show that $(V, E) \cong$

$Cay(S,A)$. Define a map f from (V,E) to $Cay(S,A)$ by $f(v)=(f^{\alpha}_{ik}(v),i,\alpha)$ for any $v\in V^{\alpha}_{ik}, i\in\{1,2,...,n\},$ and $\alpha\in\{1,2,...,m\}$. Let $u,v\in V$ and $u=v$. Then $u=v\in V^{\beta}_{rk}$ for some $r\in\{1,2,...,n\}$ and $\beta\in\{1,2,...,m\}$. Hence $f^{\beta}_{rk}(u)=f^{\beta}_{rk}(v)$ and $(f^{\beta}_{rk}(u),r,\beta)=(f^{\beta}_{rk}(v),r,\beta)$. Therefore f is well defined. Let $u,v\in V$ and $f(u)=f(v)$. Then $u\in V^{\beta}_{lk}$ and $v\in V^{\delta}_{tk}$ for some $l,t\in\{1,2,...,n\}$ and $\beta,\delta\in\{1,2,...,m\}$, and so $(f^{\beta}_{lk}(u),l,\beta)=f(u)=f(v)=(f^{\delta}_{tk}(v),t,\delta)$. Hence $f^{\beta}_{lk}(u)=f^{\delta}_{tk}(v)$, $l=t$, and $\beta=\delta$. Then $u,v\in V^{\beta}_{lk}$ and $f^{\beta}_{lk}(u)=f^{\beta}_{lk}(v)$. Since f^{β}_{lk} is an isomorphism, $u=v$. This shows that f is an injection. By (4), $|G\times\{i\}\times\{\alpha\}|=|G|=|V^{\alpha}_{ik}|$ for all $i\in\{1,2,...,n\}$ and $\alpha\in\{1,2,...,m\}$. Thus $|S=\mathcal{M}(G,I,\Lambda,P)|=|\bigcup_{i=1}^{n}\bigcup_{\alpha=1}^{m}(G\times\{i\}\times\{\alpha\})|=|\bigcup_{i=1}^{n}\bigcup_{\alpha=1}^{m}V^{\alpha}_{ik}|=|V|$. Hence f is a surjection. Now we must prove that f and f^{-1} are digraph homomorphisms. Let $u,v\in V$ and $(u,v)\in E$. By (1), $u,v\in V_t$ for some $t\in\{1,2,...,n\}$. Then $u\in V^{\beta}_{tk}$ and $v\in V^{\delta}_{tk}$ for some $\beta,\delta\in\{1,2,...,m\}$. From (5), $f^{\delta}_{tk}(v)=f^{\beta}_{tk}(u)a^{\beta}_{tk}a$ for some $a\in A^{\delta}_{tk}$. Hence $(a,k,\delta)\in(A^{\delta}_{tk}\times\{k\}\times\{\delta\})\subseteq A$. By (4), $a^{\beta}_{tk}=a^{\beta}_{kk}$. Since a^{β}_{kk} is the entry in the β^{th} row and k^{ht} column of P, $a^{\beta}_{tk}=a^{\beta}_{kk}=p_{\beta k}$. Therefore $f(v)=(f^{\delta}_{tk}(v),t,\delta)=(f^{\beta}_{tk}(u)a^{\beta}_{tk}a,t,\delta)=(f^{\beta}_{tk}(u)p_{\beta k}a,t,\delta)=(f^{\beta}_{tk}(u),t,\beta)(a,k,\delta)=f(u)(a,k,\delta)$. Then we get that $(f(u),f(v))$ is an arc in $Cay((G\times L_n\times R_m),A)$. This shows that f is a digraph homomorphism. Let $g,g'\in G$, $j,t\in\{1,2,...,n\}$, $\beta,\delta\in\{1,2,...,m\}$, and let $((g,j,\beta),(g',t,\delta))$ be an arc in $Cay(S,A)$. Then there exists $(a,k,\xi)\in A$ such that $g'=gp_{\beta k}a$, $t=j$, and $\delta=\xi$. By the definition of A, $a\in A^{\xi}_{qk}=A^{\delta}_{qk}$ for some $q\in\{1,2,...,n\}$. From (4), $A^{\delta}_{qk}=A^{\delta}_{jk}$ and so $a\in A^{\delta}_{jk}$ Since $g,g'\in G$, there exists $u\in V^{\beta}_{jk}$ and $v\in V^{\delta}_{jk}$ such that $f^{\beta}_{jk}(u)=g$ and $f^{\delta}_{jk}(v)=g'$ by (4). Therefore $f^{\delta}_{jk}(v)=f^{\beta}_{jk}(u)p_{\beta k}a$. By the definition of P and (4), $p_{\beta k}=a^{\beta}_{kk}=a^{\beta}_{jk}$. Hence $(f^{-1}(g,l_j,r_\beta),f^{-1}(g',l_t,r_\delta))=(f^{-1}((f^{\beta}_{jk}(u),j,\beta)),f^{-1}((f^{\delta}_{jk}(v),j,\delta)))=(u,v)\in E$ by (5). Thus f^{-1} is a digraph homomorphism. $\square$

Example 1. Consider the finite simple semigroup $S=\mathcal{M}(\mathbb{Z}_3,I,\Lambda,P),\mathbb{Z}_3=\{\bar{0},\bar{1},\bar{2}\}$ with $I=\{1,2\}$, $\Lambda=\{1,2\}$,

$$P=\begin{bmatrix}\bar{0}&\bar{0}\\\bar{0}&\bar{1}\end{bmatrix},\text{ and thus } P_1=\begin{bmatrix}\bar{0}\\\bar{0}\end{bmatrix},P_2=\begin{bmatrix}\bar{0}\\\bar{1}\end{bmatrix},$$

and let $a_1=(\bar{0},1,1)$, $a_2=(\bar{1},2,1)$, $a_3=(\bar{0},2,2)$. Then we give the Cayley graphs $Cay(S,A)$ for all the three different one-element connection sets A, as indicated in the pictures.

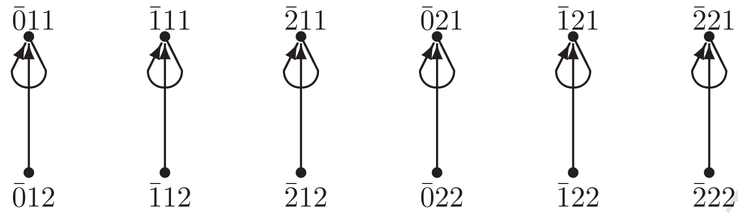

Fig. 1. $Cay(S, \{a_1\})$

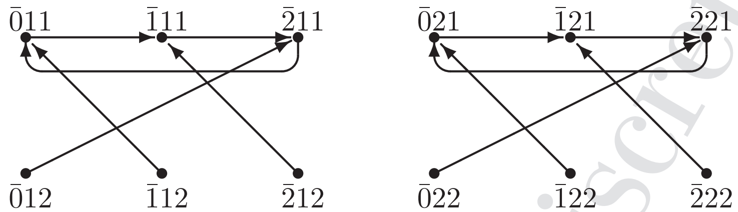

Fig. 2. $Cay(S, \{a_2\})$

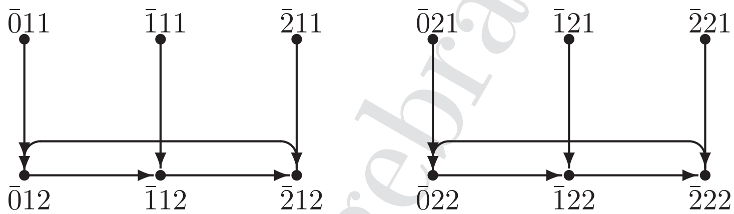

Fig. 3. $Cay(S, \{a_3\})$

So we have
$Cay(S, \{a_1\}) =$
$Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\}) \bigcup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$,
$Cay(S, \{a_2\}) =$
$Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1)\}) \bigcup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1)\})$,
$Cay(S, \{a_3\}) =$
$Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{0}, 1, 2)\}) \bigcup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{0}, 2, 2)\})$.
Therefore $Cay(S, \{a_1, a_2, a_3\}) =$
$\big[Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\}) \ \bigoplus Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2),$
$\{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})\big]$
$\bigcup \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\}) \ \bigoplus Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2),$
$\{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})\big]$.
We see that $Cay(S, \{a_1, a_2, a_3\}) = (V_1, E_1) \bigcup (V_2, E_2)$ where
$(V_1, E_1) = \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\})$

$$\bigoplus Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})\big] \text{ and}$$
$$(V_2, E_2) = \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$$
$$\bigoplus Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})\big].$$

Let $(V_{11}, E_{11}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\})$,
$(V_{12}, E_{12}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})$,
$(V_{21}, E_{21}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$,
and $(V_{22}, E_{22}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})$.
Then we get that $(V_{11}, E_{11}), (V_{12}, E_{12}), (V_{21}, E_{21})$ and (V_{12}, E_{12}) are right group digraphs, and so $(V_1, E_1) = (V_{11}, E_{11}) \bigoplus (V_{12}, E_{12})$ and $(V_2, E_2) = (V_{21}, E_{21}) \bigoplus (V_{12}, E_{12})$.

In the next lemmas, we describe the structure of Cayley graphs of a completely simple semigroup with a one-element connection set, which will be used in Theorem 4 and Theorem 5. By the proof of Theorem 3 (1-2), we have the following lemma.

Lemma 2. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, $a = (g, j, \beta) \in S$, P_i the i^{th} column of P. Then $Cay(S, \{a\})$ is the disjoint union of n isomorphic strong subdigraphs $Cay(S_1, \{(g, 1, \beta)\})$, $Cay(S_2, \{(g, 2, \beta)\}), \ldots$, $Cay(S_n, \{(g, n, \beta)\})$ where $S_i = \mathcal{M}(G, \{i\}, \Lambda, P_j)$.*

Lemma 3. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, $G/\langle p_{\beta j} g\rangle = \{g_1\langle p_{\beta j} g\rangle, g_2\langle p_{\beta j} g\rangle, \ldots, g_t\langle p_{\beta j} g\rangle\}$ the set of all distinct left cosets of $\langle p_{\beta j} g\rangle$ in G, $a = (g, j, \beta) \in S$ and let $M_{ik} = \big(g_k\langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big) \cup \big(\bigcup_{\alpha \neq \beta} (g_k\langle p_{\beta j} g\rangle\ g^{-1} p_{\alpha j}^{-1} \times \{i\} \times \{\alpha\}\big)$, where $k \in \{1, 2, \ldots, t\}$ and $i \in I$. Then $M_{i1}, M_{i2}, \ldots, M_{it}$ are disjoint.*

Proof. Since $\{g_1\langle p_{\beta j} g\rangle, g_2\langle p_{\beta j} g\rangle, \ldots, g_t\langle p_{\beta j} g\rangle\}$ is the set of all distinct left cosets of $\langle p_{\beta j} g\rangle$ in G, we get that $M_{i1}, M_{i2}, \ldots, M_{it}$ are disjoint. □

Lemma 4. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, $a = (g, j, \beta) \in S$, (M_{ik}, E_{ik}) a strong subdigraph of $Cay(S, \{a\})$. Then $(M_{ik_1}, E_{ik_1}) \cong (M_{ik_2}, E_{ik_2})$ for all $k_1, k_2 \in \{1, 2, \ldots, t\}$.*

Proof. We define $f : (M_{ik_1}, E_{ik_1}) \to (M_{ik_2}, E_{ik_2})$ by

$$\big(g_{k_1}(p_{\beta j} g)^r, i, \beta\big) \mapsto \big(g_{k_2}(p_{\beta j} g)^r, i, \beta\big)$$
$$\big(g_{k_1}(p_{\beta j} g)^r g^{-1} p_{\alpha j}^{-1}, i, \alpha\big) \mapsto \big(g_{k_2}(p_{\beta j} g)^r g^{-1} p_{\alpha j}^{-1}, i, \alpha\big)\ for \alpha \neq \beta.$$

Since, for all $k \in \{1, 2, \ldots, t\}, g_k\langle p_{\beta j} g\rangle = \{g_k(p_{\beta j} g), g_k(p_{\beta j} g)^2, \ldots, g_k(p_{\beta j} g)^{|p_{\beta j} g|}\}$, f is a well defined bijection. Since we define the Cayley graph with right action, f and f^{-1} are homomorphisms. This means that f is a digraph isomorphism. Hence $(M_{ik_1}, E_{ik_1}) \cong (M_{ik_2}, E_{ik_2})$. □

Lemma 5. *Let* $S = \mathcal{M}(G, I, \Lambda, P)$ *be a finite simple semigroup,* $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, $a = (g, j, \beta) \in S$. *Then* $Cay(S_i, \{(g, i, \beta)\}) = \dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})$.

Proof. We first show that $S_i = \dot{\bigcup}_{k=1}^{t} M_{ik}$. Since $M_{ik} \subseteq S_i$ for all $k \in \{1, 2, ..., t\}$, $\dot{\bigcup}_{k=1}^{t} M_{ik} \subseteq S_i$. We will show that $S_i \subseteq \dot{\bigcup}_{k=1}^{t} M_{ik}$. Let $x = (g^{'}, i, \lambda) \in S_i$, we get $g^{'} \in G = \bigcup_{k=1}^{t} g_k \langle p_{\beta j} g\rangle$, and thus $g^{'} = g_w(p_{\beta j} g)^v$ for some $v \in \mathbb{N}$ and $w \in \{1, 2, \ldots, t\}$. We need to consider the following two cases.

(case 1) If $\lambda = \beta$, then $x = \big(g_w(p_{\beta j} g)^v, i, \beta\big) \in \big(g_w\langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big) \subseteq M_{iw} \subseteq \dot{\bigcup}_{k=1}^{t} M_{ik}$.

(case 2) If $\lambda \neq \beta$, then $xa = \big(g_w(p_{\beta j} g)^v, i, \lambda\big)(g, j, \beta) = \big(g_w(p_{\beta j} g)^v p_{\lambda j} g, i, \beta\big)$. Since $g_w(p_{\beta j} g)^v p_{\lambda j} g \in G = \bigcup_{k=1}^{t} g_k\langle p_{\beta j} g\rangle$, $g_w(p_{\beta j} g)^v p_{\lambda j} g \in g_u\langle p_{\beta j} g\rangle$ for some $u \in \{1, 2, \ldots, t\}$, we get that $g_w(p_{\beta j} g)^v p_{\lambda j} g = g_u(p_{\beta j} g)^{v^{'}}$ for some $\acute{v} \in \mathbb{N}$, and thus $g_w(p_{\beta j} g)^v = g_u(p_{\beta j} g)^{v^{'}} g^{-1} p_{\lambda j}^{-1}$. Therefore $x = \big(g_w(p_{\beta j} g)^v, i, \lambda\big) = \big(g_u(p_{\beta j} g)^d g^{-1} p_{\lambda j}^{-1}, i, \lambda\big) \in \big(g_u\langle p_{\beta j} g\rangle g^{-1} p_{\lambda j}^{-1} \times \{i\} \times \{\lambda\}\big) \subseteq M_{iu} \subseteq \dot{\bigcup}_{k=1}^{t} M_{ik}$.

Hence $S_i \subseteq \dot{\bigcup}_{k=1}^{t} M_{ik}$. Then we conclude that $S_i = \dot{\bigcup}_{k=1}^{t} M_{ik}$.

Since $(M_{i1}, E_{i1}), (M_{i2}, E_{i2}), \ldots, (M_{it}, E_{it})$ are strong subdigraphs of $Cay(S_i, \{(g, i, \beta)\})$, $E\big(\dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})\big) \subseteq E\big(Cay(S_i, \{(g, i, \beta)\})\big)$. Let $x = (u_1, i, \lambda_1)$, $y = (u_2, i, \lambda_2) \in S_i$ and (x, y) be an arc in $Cay(S_i, \{(g, i, \beta)\})$. Therefore $u_2 = u_1 p_{\lambda_1 j} g$ and $\lambda_2 = \beta$ by Proposition 1(2). Since $S_i = \dot{\bigcup}_{k=1}^{t} M_{ik}$, $x \in M_{ib_1}$ and $y \in M_{ib_2}$ for some $b_1, b_2 \in \{1, 2, \ldots, t\}$. Hence $y \in \big(g_{b_2}\langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big)$, and thus $y = (g_{b_2}(p_{\beta j} g)^{d^{'}}, i, \beta)$ for some $d^{'} \in \{1, 2, \ldots, |p_{\beta j}|\}$. Then $u_2 = g_{b_2}(p_{\beta j} g)^{d^{'}}$. We need only consider two cases:

(case1) If $\lambda_1 \neq \beta$, then $x \in \big(g_{b_1}\langle p_{\beta j} g\rangle g^{-1} p_{\lambda_1 j}^{-1} \times \{i\} \times \{\lambda_1\}\big)$. Hence $x = \big(g_{b_1}(p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1}, i, \lambda_1\big)$ for some $c \in \{1, 2, ..., |p_{\beta j} g|\}$, and thus $u_1 = g_{b_1}(p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1}$. Since $u_2 = u_1 p_{\lambda_1 j} g$,

$$\begin{aligned} g_{b_2}(p_{\beta j} g)^{d^{'}} &= g_{b_1}(p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1} p_{\lambda_1 j} g \\ &= g_{b_1}(p_{\beta j} g)^c. \end{aligned}$$

Then $b_1 = b_2$, and thus $x, y \in M_{ib_1}$. Hence $(x, y) \in E_{ib_1}$. We get that (x, y) is an arc in $\dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})$.

(case2) If $\lambda_1 = \beta$, then $x \in \big(g_{b_1}\langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big)$. Hence $x = \big(g_{b_1}(p_{\beta j}g)^{c'}, i, \beta\big)$ for some $c' \in \{1, 2, ..., |p_{\beta j}g|\}$, and so $u_1 = g_{b_1}(p_{\beta j}g)^{c'}$. Since $u_2 = u_1 p_{\lambda_1 j} g$,

$$\begin{aligned} g_{b_2}(p_{\beta j}g)^{d'} &= g_{b_1}(p_{\beta j}g)^{c'} p_{\lambda_1 j} g \\ &= g_{b_1}(p_{\beta j}g)^{c'} p_{\beta j} g \\ &= g_{b_1}(p_{\beta j}g)^{c'} p_{\beta j} g \\ &= g_{b_1}(p_{\beta j}g)^{c'+1}. \end{aligned}$$

Therefore $b_1 = b_2$ and thus $x, y \in M_{ib_1}$. Hence $(x, y) \in E_{ib_1}$. We get that (x, y) is an arc in $\dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})$.

Hence $E\big(\dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})\big) = E(Cay(S_i, \{(g, i, \beta)\}))$. We conclude that $Cay(S_i, \{(g, i, \beta)\}) = \dot{\bigcup}_{k=1}^{t}(M_{ik}, E_{ik})$. $\square$

Lemma 6. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, $a = (g, j, \beta) \in S$. Then $(M_{ik}, E_{ik}) \cong Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$ where $R_m = \{r_1, r_2, \ldots, r_m\}$ is a right zero semigroup.*

Proof. We define $f : (M_{ik}, E_{ik}) \to Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$ by

$$\begin{aligned} \big(g_k(p_{\beta j}g)^q, i, \beta\big) &\mapsto \big(g_k(p_{\beta j}g)^q, r_\beta\big) \\ \big(g_k(p_{\beta j}g)^q g^{-1} p_{\alpha j}^{-1}, i, \alpha\big) &\mapsto \big(g_k(p_{\beta j}g)^{q-1}, r_\alpha\big) for \alpha \neq \beta. \end{aligned}$$

Clearly, f is a well defined bijection.

We will show that f and f^{-1} are digraph homomorphisms. For $x, y \in M_{ik}$, let $(x, y) \in E_{ik}$. Then (x, y) is an arc in $Cay(S, \{a\})$, and thus $y = xa$. By Proposition 1(2), $p_3(y) = \beta$. Hence $y \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, and so $y = (g_k(p_{\beta j}g)^c, i, \beta)$ for some $c \in \{1, 2, \ldots, |p_{\beta j}g|\}$. We need only consider two cases :

(case 1) If $x \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, then $x = \big(g_k(p_{\beta j}g)^d, i, \beta\big)$ for some $d \in \{1, 2, \ldots, |p_{\beta j}g|\}$. Since $y = xa$, $\big(g_k(p_{\beta j}g)^c, i, \beta\big) = \big(g_k(p_{\beta j}g)^d, i, \beta\big)(g, j, \beta) = \big(g_k(p_{\beta j}g)^d p_{\beta j} g, i, \beta\big)$.
Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^d p_{\beta j} g$, and so

$$\begin{aligned} f(y) &= f\big(g_k(p_{\beta j}g)^c, i, \beta\big) \\ &= \big(g_k(p_{\beta j}g)^c, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^d p_{\beta j} g, r_\beta\big) \end{aligned}$$

$$\begin{aligned} &= \big(g_k(p_{\beta j}g)^d, r_\beta\big)\big(p_{\beta j}g, r_\beta\big) \\ &= f(x)(p_{\beta j}g, r_\beta). \end{aligned}$$

Therefore $\big(f(x), f(y)\big)$ is an arc in $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$.

(case 2) If $x \in \big(\bigcup_{\alpha\neq\beta}(g_k\langle p_{\beta j}g\rangle g^{-1}p_{\alpha j}^{-1} \times \{i\} \times \{\alpha\})\big)$, then $x = \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)$ for some $\alpha \neq \beta$ and $d' \in \{1, 2, \ldots, |p_{\beta j}g|\}$. Since $y = xa$,

$$\begin{aligned} \big(g_k(p_{\beta j}g)^c, i, \beta\big) &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)(g, j, \beta) \\ &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}p_{\alpha j}g, i, \beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'}, i, \beta\big). \end{aligned}$$

Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^{d'}$, and so

$$\begin{aligned} f(y) &= \big(g_k(p_{\beta j}g)^c, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'}, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'-1}p_{\beta j}g, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'-1}, r_\alpha\big)(p_{\beta j}g, r_\beta) \\ &= f(x)(p_{\beta j}g, r_\beta). \end{aligned}$$

Therefore $\big(f(x), fy)\big)$ is an arc in $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$.

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism.
Hence $(M_{ik}, E_{ik}) \cong Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$. □

Example 2. Consider the finite simple semigroup $S = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}, \bar{1}\bar{1}\}$ with $I = \{1, 2\}$, $\Lambda = \{1, 2\}$,

$$P = \begin{bmatrix} \bar{0}\bar{0} & \bar{1}\bar{0} \\ \bar{0}\bar{1} & \bar{1}\bar{1} \end{bmatrix}, \text{ and thus } P_1 = \begin{bmatrix} \bar{0}\bar{0} \\ \bar{0}\bar{1} \end{bmatrix}, P_2 = \begin{bmatrix} \bar{1}\bar{0} \\ \bar{1}\bar{1} \end{bmatrix}.$$

Then we give the Cayley graph $Cay\big(S, \{(\bar{1}\bar{0}, 1, 2)\}\big)$

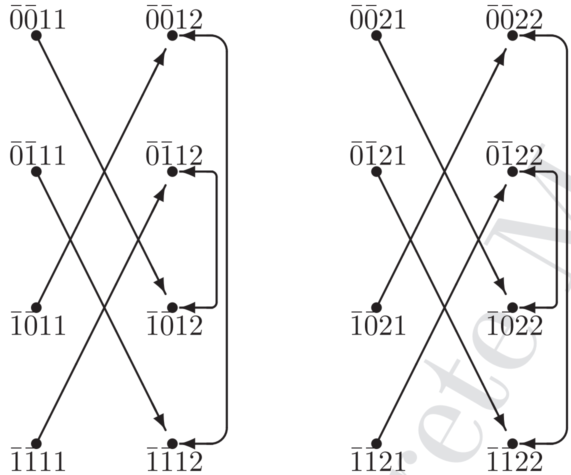

Fig. 4. $Cay\big(S, \{(\bar{1}\bar{0}, 1, 2)\}\big)$

In Fig. 4, we see that $Cay\big(S, \{(\bar{1}\bar{0}, 1, 2)\}\big) = Cay(S_1, \{(\bar{1}\bar{0}, 1, 2)\}) \dot{\bigcup} Cay(S_2, \{(\bar{1}\bar{0}, 2, 2)\})$ where $S_1 = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{1\}, \Lambda, P_1)$, $S_2 = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{2\}, \Lambda, P_2)$. Then it is the union of right group digraphs.

We have $\langle p_{21}\bar{1}\bar{0}\rangle = \{\bar{1}\bar{1}, \bar{0}\bar{0}\}, \mathbb{Z}_2 \times \mathbb{Z}_2/\langle p_{21}\bar{1}\bar{0}\rangle = \{g_1\langle p_{21}\bar{1}\bar{0}\rangle, g_2\langle p_{21}\bar{1}\bar{0}\rangle\}$ where $g_1 = \bar{0}\bar{0}$, $g_2 = \bar{0}\bar{1}$. Hence

$M_{11} = \{(\bar{1}\bar{1}, 1, 2), (\bar{0}\bar{0}, 1, 2), (\bar{0}\bar{1}, 1, 1), (\bar{1}\bar{0}, 1, 1)\}$,
$M_{12} = \{(\bar{1}\bar{0}, 1, 2), (\bar{0}\bar{1}, 1, 2), (\bar{0}\bar{0}, 1, 1), (\bar{1}\bar{1}, 1, 1)\}$,
$M_{21} = \{(\bar{1}\bar{1}, 2, 2), (\bar{0}\bar{0}, 2, 2), (\bar{0}\bar{1}, 2, 1), (\bar{1}\bar{0}, 2, 1)\}$,
$M_{22} = \{(\bar{0}\bar{1}, 2, 2), (\bar{1}\bar{0}, 2, 2), (\bar{0}\bar{0}, 2, 1), (\bar{1}\bar{1}, 2, 1)\}$.

We see that $(M_{11}, E_{11}) \cong (M_{12}, E_{12}) \cong (M_{21}, E_{21}) \cong (M_{22}, E_{22}) \cong Cay\big(\langle p_{21}\bar{1}\bar{0}\rangle \times R_2, \{(p_{21}\bar{1}\bar{0}, r_2)\}\big)$ and $Cay\big(S, \{a\}\big) = \dot{\bigcup}_{i=1}^{2}\dot{\bigcup}_{k=1}^{2}(M_{ik}, E_{ik})$ where $R_2 = \{r_1, r_2\}$ is a right zero semigroup.

By Lemma 2-6, we get that a Cayley graph of a finite simple semigroup $\mathcal{M}(G, I, \Lambda, P)$ with a one-element connection set $\{(g, j, \beta)\}$, is the disjoint union of $|I|t$ copies of $Cay\big(\langle p_{\beta j}g\rangle \times R_{|\Lambda|}, \{(p_{\beta j}g, r_\beta)\}\big)$ where $t = |G/\langle p_{\beta j}g\rangle|$. Then $|I|t$ is the number of connected components of $Cay(S, \{a\})$.

In the next theorem, we give the conditions for two Cayley graphs of finite simple semigroups $Cay(S, \{a\})$ and $Cay(S, \{b\})$ to be isomorphic.

Theorem 4. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}, a = (g, j, \beta), b = (g', i, \lambda) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $|p_{\beta j}g| = |p_{\lambda i}g'|$.*

Proof. ($\Longrightarrow$) Suppose that $Cay(S, \{a\}) \cong Cay(S, \{b\})$. By Lemma 6, we get that $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big) \cong$

$Cay\big(\langle p_{\lambda i}g^{'}\rangle \times R_m, \{(p_{\lambda i}g^{'}, r_\lambda)\}\big)$. Therefore $|\langle p_{\beta j}g\rangle \times R_m| = |\langle p_{\lambda i}g^{'}\rangle \times R_m|$, and thus $|\langle p_{\beta j}g\rangle| = |\langle p_{\lambda i}g^{'}\rangle|$. Hence $|p_{\beta j}g| = |p_{\lambda i}g^{'}|$.
($\Longleftarrow$) Assume that $|p_{\beta j}g| = |p_{\lambda i}g^{'}| = l$.
By Lemma 2-6, we only need to show that
$Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big) \cong Cay\big(\langle p_{\lambda i}g^{'}\rangle \times R_m, \{(p_{\lambda i}g^{'}, r_\lambda)\}\big)$ where $R_m = \{r_1, r_2, \ldots, r_m\}$ is a right zero semigroup. We define

$$f : Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big) \to Cay\big(\langle p_{\lambda i}g^{'}\rangle \times R_m, \{(p_{\lambda i}g^{'}, r_\lambda)\}\big)$$

$$\text{by } f\big((p_{\beta j}g)^r, r_\mu\big) = \begin{cases} \big((p_{\lambda i}g^{'})^r, r_\lambda\big) & if \mu = \beta \\ \big((p_{\lambda i}g^{'})^r, r_\beta\big) & if \mu = \lambda \\ \big((p_{\lambda i}g^{'})^r, r_\alpha\big) & \text{otherwise} \end{cases}$$

Since $|p_{\beta j}g| = |p_{\lambda i}g^{'}|$, f is an isomorphism. □

Now we give the conditions for a Cayley graph of a finite simple semigroup with a one-element connection set to be connected.

Theorem 5. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $a = (g, j, \beta) \in S$. Then $Cay(S, \{a\})$ is connected if and only if $G = \langle p_{\beta j}g\rangle$ and $|I| = 1$, in particular this means that S is a right group.*

Proof. ($\Rightarrow$) Let $Cay(S, \{a\})$ is connected. By Lemma 2, we get $|I| = 1$ and $G = \langle p_{\beta j}g\rangle$.

($\Leftarrow$) Assume that $G = \langle p_{\beta j}g\rangle$ and $|I| = 1$. We will prove that $Cay(S, \{a\})$ is connected. Let $(g_1, i, \lambda_1), (g_2, i, \lambda_2) \in S$. Hence $g_1, g_2 \in \langle p_{\beta j}g\rangle$. Therefore $g_1 = (p_{\beta j}g)^r$ and $g_2 = (p_{\beta j}g)^q$ for some $r, q \in \{1, 2, ..., |p_{\beta j}g|\}$. Then $r \le q$ or $r > q$.

(case 1) For $\lambda_1 = \lambda_2 = \beta$. If $r \le q$, then $q = r + t$ for some $t \in \mathbb{N} \cup \{0\}$. Then we get $(g_1, i, \lambda_1) = \big((p_{\beta j}g)^r, i, \beta\big), \big((p_{\beta j}g)^{r+1}, i, \beta\big), \ldots, \big((p_{\beta j}g)^{r+t}, i, \beta\big) = (g_2, i, \lambda_2)$ is a path from (g_1, i, λ_1) to (g_2, i, λ_2) in $Cay(S, \{a\})$. Similarly, if $r > q$, there is a path from (g_2, i, λ_2) to (g_1, i, λ_1) in $Cay(S, \{a\})$.

(case 2) For $\lambda_1 = \beta, \lambda_2 \ne \beta$. Then $(g_2, i, \lambda_2)(g, j, \beta) = (g_2 p_{\lambda_2 j} g, i, \beta)$, and thus $\big((g_2, i, \lambda_2), (g_2 p_{\lambda_2 j} g, i, \beta)\big)$ is an arc in $Cay(S, \{a\})$. Since $G = \langle p_{\beta j}g\rangle$ and $g_2 p_{\lambda_2 j} g \in G$, $g_2 p_{\lambda_2 j} g = (p_{\beta j}g)^u$ for some $u \in \{1, 2, ..., |p_{\beta j}g|\}$. By case 1, there is a path from (g_1, i, λ_1) to $(g_2 p_{\lambda_2 j} g, i, \beta)$ or from $(g_2 p_{\lambda_2 j} g, i, \beta)$ to (g_1, i, λ_1). Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

(case 3) For $\lambda_1 \ne \beta, \lambda_2 = \beta$. Then $(g_1, i, \lambda_1)(g, j, \beta) = (g_1 p_{\lambda_1 j} g, i, \beta)$, and so $\big((g_1, i, \lambda_1), (g_1 p_{\lambda_1 j} g, i, \beta)\big)$ is an arc in $Cay(S, \{a\})$. Since

$G = \langle p_{\beta j}g\rangle$ and $g_1p_{\lambda_1 j}g \in G$, $g_1p_{\lambda_1 j}g = (p_{\beta j}g)^v$ for some $v \in \{1, 2, ..., |p_{\beta j}g|\}$. By case 1, there is a path from (g_2, i, λ_2) to $(g_1p_{\lambda_1 j}g, i, \beta)$ or from $(g_1p_{\lambda_1 j}g, i, \beta)$ to (g_2, i, λ_2). Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

(case 4) For $\lambda_1 \neq \beta, \lambda_2 \neq \beta$. Then $(g_1, i, \lambda_1)(g, j, \beta) = (g_1p_{\lambda_1 j}g, i, \beta)$ and $(g_2, i, \lambda_2)(g, j, \beta) = (g_2p_{\lambda_2 j}g, i, \beta)$.
Thus $\big((g_1, i, \lambda_1), (g_1p_{\lambda_1 j}g, i, \beta)\big) and \big((g_2, i, \lambda_2), (g_2p_{\lambda_2 j}g, i, \beta)\big)$ are arcs in $Cay(S, \{a\})$. Since $g_1p_{\lambda_1 j}g, g_2p_{\lambda_2 j}g \in G = \langle p_{\beta j}g\rangle$, $g_1p_{\lambda_1 j}g = (p_{\beta j}g)^w$ and $g_2p_{\lambda_2 j}g = (p_{\beta j}g)^z$ for some $w, z \in \{1, 2, ..., |p_{\beta j}g|\}$. By case 1, there is a path from $(g_1p_{\lambda_1 j}g, i, \beta)$ to $(g_2p_{\lambda_2 j}g, i, \beta)$ or from $(g_2p_{\lambda_2 j}g, i, \beta)$ to $(g_1p_{\lambda_1 j}g, i, \beta)$.
Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

By the above four cases we conclude that $Cay(S, \{a\})$ is connected. □

Acknowledgment

The first author would like to thank the Thai Government Scholaships in the Area of Science and Technology (Ministry of Science and Technology) and the Graduate School, Chiang Mai University. The second author the Centre of excellence in mathematics, the Commission on Higher Education, Thailand for their financial support during the preparation of this paper.

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Received by the editors: 02.06.2010
and in final form 30.09.2011.