Algebra and Discrete Mathematics
Volume 11 **(2011)**. Number 1. pp. 17 – 22

RESEARCH ARTICLE

Some fixed point theorems for pseudo ordered sets

S. Parameshwara Bhatta, Shiju George

Communicated by A. I. Kashu

Abstract. In this paper, it is shown that for an isotone map f on a pseudo ordered set A, the set of all fixed points of f inherits the properties of A, namely, completeness, chain-completeness and weakly chain-completeness, as in the case of posets.

1. Introduction

In 1955 Tarski [6] proved that the set of all fixed points of an order preserving map in a complete lattice constitutes a complete lattice. Later in 1976, Markowsky [2] generalized this result by proving that the set of all fixed points of an order preserving map in a chain-complete poset forms a chain-complete poset. We extend these results to generalized structures like trellises and pseudo ordered sets. Further, a counterexample is given to show that the least fixed point property does not imply weakly chain-completness even for an acyclic pseudo ordered set. This, in particular gives a negative solution to Problem 2 in [1].

2. Notations and definitions

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a non- empty set A is called a *pseudo order*. The set A together with this pseudo order $\trianglelefteq$ is called a *pseudo ordered set* or a *psoset*. For a subset of A, the notions of a lower bound, an upper bound, the greatest lower bound (or meet), the least upper

2000 Mathematics Subject Classification: *06B05.*
Key words and phrases: *pseudo ordered set, trellis, completeness, isotone map.*

bound (or join), the minimum (or the least) element and the maximum (or the greatest) element are defined analogous to the corresponding notions in a poset. Let B be a subset of A. Then for a subset X of B, the join of X in B is denoted by $\bigvee_B X$. For any two elements $a, b \in A$, if $a \trianglelefteq b$ and $a \neq b$, then we denote it as $a \triangleleft b$. If $a \trianglelefteq b$ does not hold, then we denote it by $a \ntrianglelefteq b$.

A *trellis* is a psoset, any two of whose elements have a join and a meet. A trellis is said to be a *complete trellis* if every subset of A has a meet and a join. An extensive investigation of the notions of psosets and related concepts can be found in H.L. Skala [3] and H. Skala [4].

A subset C of A, including $C = \phi$, is called a *chain* in A if the restriction of $\trianglelefteq$ to C is a complete order (i.e. it is a partial order on C such that every pair of elements of C are comparable). A chain C in A is said to be *well ordered* if every non-empty subset of C has the least element. A psoset A is said to be *chain-complete* if every chain in A has a join. A is said to be *weakly chain-complete* if every well ordered chain in A has a join. Eventhough the notions of chain-completeness and weakly chain-completeness coincide in posets, it is not known to date whether they are equivalent in case of psosets. The above definitions are due to Bhatta [1].

A map $f : A \to A$ is said to be *isotone* if $a \trianglelefteq b$ implies $f(a) \trianglelefteq f(b)$. An element $a \in A$ is said to be a *fixed point* for f if $f(a) = a$. If every isotone map of A into itself has a fixed point (the least fixed point), then A is said to have the *fixed point property* (*the least fixed point property*). The composition maps $f \circ f, f \circ f \circ f, \ldots$ are denoted by $f^2, f^3, \ldots$ respectively.

3. Results

The notion of an f-chain starting at a point p, comparable to its image, is well known for posets [5]. The following generalization of this definition helps us in our further discussion

Definition 3.1. *Let $< A, \trianglelefteq >$ be a psoset, $f : A \to A$ be an isotone map and B be a subset of $F_A = \{x \in A : f(x) = x\}$. For an ordinal ξ, a subset $S_\xi = \{x_\eta : \eta < \xi\}$ of A is called an f-chain on B if for any $\alpha < \xi$ we have*

$$x_\alpha = \begin{cases} \bigvee_A [B \cup \{x_\eta : \eta < \alpha\}] & \text{if } \alpha \text{ is a limit ordinal;} \\ f(x_\beta) & \text{otherwise, where } \alpha = \beta + 1. \end{cases}$$

To have more versatility later on, we shall not assume any sort of completeness on the psoset for the following lemmas.

Lemma 3.2. *Let $< A, \trianglelefteq >$ be a psoset, $f : A \to A$ be an isotone map and B be a subset of $F_A = \{x \in A : f(x) = x\}$. Then any f-chain on B is well ordered and is contained in the set of upper bounds of B in A.*

Proof. Assume the contrary. Let B^Δ denote the set of all upper bounds of B in A. Choose α to be the least ordinal for which either $x_\eta \ntrianglelefteq x_\alpha$ for some $\eta < \alpha$ or $x_\alpha \notin B^\Delta$. Clearly α is not a limit ordinal. Hence $\alpha = \beta + 1$ for some ordinal number β. Since $x_\beta \in B^\Delta$, it follows that $x_\alpha = f(x_\beta) \in B^\Delta$. Choose γ to be the least ordinal for which $x_\gamma \ntrianglelefteq x_\alpha$. Since x_α is is an upper bound for $\{x_\eta : \eta < \gamma\}$, γ cannot be a limit ordinal. Hence $\gamma = \delta + 1$ for an ordinal δ. As $x_\delta \trianglelefteq x_\beta$ and f is order preserving, we get $x_\gamma = f(x_\delta) \trianglelefteq f(x_\beta) = x_\alpha$, a contradiction. □

If $S_\alpha = \{x_\eta : \eta < \alpha\}$ and $S_\beta = \{x_\eta : \eta < \beta\}$ are two f-chains on B, then by transfinite induction it can be shown that $x_\gamma = y_\gamma$ for every $\gamma < \alpha, \beta$. Hence either both of them are equal or one should be an initial segment of the other. Thus there is a unique maximal f-chain on B.

Lemma 3.3. *Let $< A, \trianglelefteq >$ be a psoset and $f : A \to A$ be an isotone map. For a subset B of $F_A = \{x \in A : f(x) = x\}$, let S be the unique maximal f-chain on B. If $u = \bigvee_A (B \cup S)$ exists, then $u = \bigvee_{F_A} B$.*

Proof. We have $x \trianglelefteq f(x)$ for every $x \in S$. Further, $u = \bigvee_A (B \cup S) \in S$ so that $f(u) \in S$. Thus $f(u) = u$. Hence u is an upper bound for B in F_A. If y is any upper bound for B in F_A, then by transfinite induction, it follows that y is an upper bound for $B \cup S$ in A. Hence $u \trianglelefteq y$. □

The following theorems are direct consequences of Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *Let $< A, \trianglelefteq >$ be a chain-complete psoset and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a chain-complete psoset in the induced order.*

Corollary 3.5. (Theorem 9, [2]) *Let P be a chain-complete poset, $f : P \to P$ isotone and $F_P = \{x \in P : f(x) = x\}$ be the set of all fixed points of f. Then*

(i) there is a least element $0^* \in F_P$.

(ii) for all $y \in P$, if $f(y) \leq y$, then $0^* \leq y$.

(iii) F_P is a chain-complete poset in the induced order.

Theorem 3.6. *Let $< A, \trianglelefteq >$ be a weakly chain-complete psoset and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a weakly chain-complete psoset in the induced order.*

Corollary 3.7. (Theorem, [1]) *Every weakly chain-complete psoset has the least fixed point property.*

Eventhough the following theorem follows directly from Lemma 3.3, a much shorter proof is given below.

Theorem 3.8. *Let $< A, \trianglelefteq >$ be a complete trellis and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a complete trellis in the induced order.*

Proof. Let B be a subset of F_A. Let B^Δ denote the set of all upper bounds of B in A. Then B^Δ is a complete trellis in the induced order and f is a self map on B^Δ. By Corollary 3.7 f has the least fixed point say u in B^Δ. Clearly, $u = \bigvee_{F_A} B$. Hence F_A is a complete trellis. □

Corollary 3.9.(Theorem 37, [4]) *If f is an isotone map of a complete trellis A onto itself such that $a \trianglelefteq f(a)$ for each a in A, then with respect to the same pseudo order on A, the set of all fixed points of f constitutes a complete trellis.*

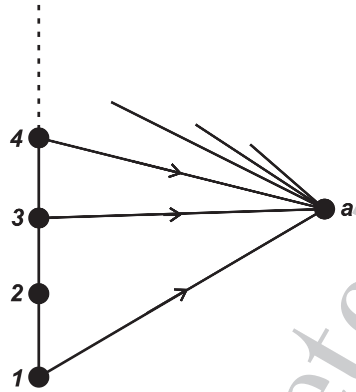
Corollary 3.10.(Theorem 1, [6]) *Let P be a complete lattice, f an isotone map of A to itself and F_P be the set of all fixed points of f. Then the set F_P is a complete lattice.*

Counterexample 3.11 The converse of Corollary 3.7 was posed as an open problem (Problem 2, [1]). A negative solution is given below to show that the converse doesn't hold even for an acyclic psoset.

Let $A = N \cup \{a\} = \{1, 2, 3, \ldots\} \cup \{a\}$. We define a pseudo-order $\trianglelefteq$ on A as follows. The elements of N are ordered by the usual natural order of the reals. Further, for any $k \in N, k \neq 2$ we have $k \triangleleft a$. This psoset is represented by the digraph in Figure 1.

Clearly, A is not weakly chain-complete as the well ordered chain $C = \{1, 2, 3, \ldots\}$ is not bounded above.

Let $f : A \to A$ be isotone. Suppose f does not have any fixed points. Since $1 \triangleleft f(1) \triangleleft f^2(1) \triangleleft \ldots$ is a chain in A, there is no $n \in N$ such that $f^n(1) = a$. For, if $f^n(1) = a$ for some $n \in N$, then we get $f(a) = a$, a contradiction. Thus $1 \triangleleft f(1) \triangleleft f^2(1) \triangleleft \ldots$ is a chain contained in N. Hence $f^2(1) \geq 3$ so that $\{f^2(1), f^3(1), \ldots\}$ is a chain contained in $\{3, 4, \ldots\}$. Since a is an upper bound for $\{f^2(1), f^3(1), \ldots\}$, $f(a)$ should be an upper bound for $\{f^3(1), f^4(1), \ldots\}$ in A. As a being the only upper bound of



$\{f^3(1), f^4(1), \ldots\}$ in A, we should have $f(a) = a$, a contradiction to the assumption that f has no fixed points. Hence f has a fixed point. Further, if $f(2) = 2$ and $f(a) = a$, then either $f(1) = 1$ or $f(1) = a$. Since $2 = f(2) \trianglelefteq f(3)$ and $f(1) \trianglelefteq f(3)$, it follows that $f(1) \neq a$. Thus A has the least fixed point property.

Acknowledgment

The second author gratefully acknowledges the financial assistance by CSIR, New Delhi, INDIA, under the Research Fellowship Scheme.

References

[1] S.P. Bhatta, Weak chain-completeness and fixed point property for pseudo-ordered sets, Czechoslovak Mathematical Journal 55 (130) (2005), 365-369.

[2] G. Markowsky, Chain-complete posets and directed sets with applications, Algebra Universalis 6 (1976), 53-68.

[3] H. L. Skala, Trellis Theory, Algebra Universalis 1 (1971), 218-233.

[4] H. Skala, Trellis Theory, Mem. Amer. Math. Soc.,1, Providence, 1972.

[5] B. Schröder, Ordered Sets - An Introduction, Birkhäuser, Boston, Massachusetts, 2002.

[6] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific J. Math. 5 (1955), 285-309.

Contact information

S. Parameshwara Bhatta

Department of Mathematics, Mangalore University, Mangalagangothri, 574 199, Karnataka State, INDIA
E-Mail: s_p_bhatta@yahoo.co.in

Shiju George Department of Mathematics, Mangalore University, Mangalagangothri, 574 199, Karnataka State, INDIA
E-Mail: shijugeorgem@rediffmail.com

Received by the editors: 09.09.2009
and in final form 04.05.2011.