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Steadiness of polynomial rings

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Abstract. A module M is said to be small if the functor $\mathrm{Hom}(M,-)$ commutes with direct sums and right steady rings are exactly those rings whose small modules are necessary finitely generated. We give several results on steadiness of polynomial rings, namely we prove that polynomials over a right perfect ring such that $End_R(S)$ is finitely generated over its center for every simple module S form a right steady ring iff the set of variables is countable. Moreover, every polynomial ring in uncountably many variables is non-steady.

The notion of a *small module* is one of the most natural variant of a concept of compactness in categories of modules. It is defined as a module M for which the covariant functor $\mathrm{Hom}(M,-)$ commutes with all direct sums of modules, which exactly means that every homomorphism of M into an arbitrary direct sum $\bigoplus_{i\in I} N_i$ of modules can be factorized through a suitable direct sum $\bigoplus_{i\in F} N_i$ where $F \subseteq I$ is finite. The first systematic research concerning this notion was published in [8], however a non-categorical characterization as well as an observation that there are examples of infinitely generated small modules had appeared in [2, p.54]. Small modules are studied under various terms (module of type Σ, dually slender, U-compact) and motivation of the research have come in particular from the theory of representable equivalences of module categories ([4], [5], [6] [9], [10] etc.) and the structure theory of graded rings [7] and almost free modules [11].

It is easy to see (and it will be obvious from Lemma 1.1) that every small module is finitely generated. As many examples of infinitely generated

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small modules are known it seems to be useful to define a *right steady* ring as a ring over which classes of all small modules and of all finitely generated modules coincide. However any general ring-theoretical criterion of right steadiness is not available, there are known large (and frequently studied) classes of right steady rings (right noetherian, right perfect, right semiartinian with countable socle length) as well as right non-steady rings (infinite products of rings, endomorphism rings of infinitely generated free module, simple rings of infinite right rank). Moreover, characterization of right steady rings is known in some special cases (for commutative semiartinian, serial, and self injective regular rings) and a module-theoretic criterion of steadiness via products of simple modules end their injective envelops is proved in [13].

The present paper focuses on the question, how polynomial rings with commuting variables reflect steadiness. Obviously, polynomials over a right non-steady ring form a right non-steady ring. As every right noetherian ring is right steady by [8, 7^o], right steadiness of polynomials over a noetherian ring in finitely many variables follows from Hilbert basis theorem. Moreover, Rentschler proved in the cited work using the classical commutative prime-ideal calculus that polynomial rings in countably many variables over commutative noetherian rings are steady. Nevertheless, the existence of infinitely generated small modules over $R[X]$ for a general right steady ring R remains to be an open problem not only for countable but even for finite X. Our main result extends the Rentschler's quoted theorem to a "less commutative" case; namely, we prove that polynomial ring $T[X]$ is right steady, if X is countable and T is a skew field finitely generated over its center (Theorem 2.1). It implies right steadiness of polynomials $R[X]$ in countably many variables whenever R is a right perfect such that $End_R(S)$ is finitely generated right module over its center for every simple S (Theorem 2.7). On the other hand, we show that polynomials in uncountably many variables forms non-steady ring in general (Proposition 3.2).

Throughout the paper a *ring* R means an associative ring with unit, and a *module* means a right R-module. We say that N is a subfactor of M if it is a submodule of a suitable factor of M. A submodule N of M is called superfluous if $N + X \neq M$ for every proper submodule X of M (note that we will use the term small exclusively in the sense defined above). We denote by $J(R)$ Jacobson radical and by $Z(R)$ the center of R. The symbols ω and ω_1 respectively means the first infinite and the first uncountable ordinal respectively. Note that we identify cardinals with the least ordinals of the corresponding cardinality and "countable" means finite or infinitely countable.

For non-explained terminology we refer to [1].

1. Preliminaries

We start with well-known characterization of small modules (see e.g. [9, Lemma 1.2], [5, Lemma 1.1] and [8, 1°]).

Lemma 1.1. *The following conditions are equivalent for an arbitrary module M:*

(1) *M is small,*

(2) *if $M = \bigcup_{i<\omega} M_i$ for an increasing chain of submodules $M_i \subseteq M_{i+1} \subseteq M$, then there exists n such that $M = M_n$,*

(3) *if $M = \sum_{i<\omega} M_i$ for a system of submodules $M_i \subseteq M$, then there exists n such that $M = \sum_{i<n} M_i$.*

We will use freely several easy consequences of the previous lemma:

Corollary 1.2. *Let M be a small module.*

(1) *Any factor of M is small,*

(2) *if M is countable generated, M is finitely generated,*

(3) *if $M = \bigoplus_{i\in I} M_i$, then I is finite,*

(4) *if M is a submodule of $\sum_{i<\omega} N_i$, there exists n such that $M \subseteq \sum_{i<n} N_i$,*

(5) *if M_i are submodules of M such that $M/\sum_{i<n} M_i$ is infinitely generated for each n, $M/\sum_{i<\omega} M_i$ is infinitely generated.*

Before we apply the fact that no infinitely generated small module is countable generated we make several technical observations about countably generated modules and ideals. First one is a "countable" analogue of Hilbert basis theorem.

Lemma 1.3. *Let R be a ring whose all right ideals are countably generated.*

(1) *Every submodule of every countably generated module is countably generated.*

(2) *If X is a countable set of variables, every right ideal of the polynomial ring $R[X]$ is countably generated.*

Proof. (1) Fix a countable set $\{m_0, m_1, \dots\}$ of generators of M. Put $M_0 = 0$, $M_n = \sum_{i<n} m_i R$ for $n > 0$ and assume that N is an uncountably generated submodule of M. Since $N = \bigcup_{n<\omega}(N \cap M_n)$, there exists n such that $N \cap M_n$ is uncountably generated; take such a minimal n. As $N \cap M_{n-1}$

is countably generated, $(N \cap M_n)/(N \cap M_{n-1})$ is uncountable generated. Finally, M_n/M_{n-1} is a cyclic module and $((N \cap M_n)+M_{n-1})/M_{n-1} \cong (N \cap M_n)/(N \cap M_{n-1})$ is its uncountable generated submodule, a contradiction with the hypothesis.

(2) Since $R[X]$ is countably generated as a right R-module, every its R-submodule is countably generated by (1), hence every right $R[X]$-submodule of $R[X]$ is countable generated as well. □

The following technical lemma, which generalizes [12, Lemma 6], uses the similar argument as Lemma 1.3.

Lemma 1.4. *Let R be a ring, M a module and $M = \sum_{i<\omega} M_i$ where M_i are submodules of M such that no subfactor of M_i is an infinitely generated small module. Then no subfactor of M is an infinitely generated small module.*

Proof. Since every small submodule of any factor module M/X is a submodule of $\sum_{i\leq n}(M_i + X/X)$ for a suitable n by Corollary 1.2(4), it is enough to show that there exists no infinitely generated small subfactor of $\sum_{i\leq n} M_i$.

Assume that there exists an infinitely generated small submodule N of some factor $\sum_{i\leq n} M_i/Y$, fix a minimal such n. We may suppose that $Y = 0$. Since $N + \sum_{i<n} M_i/\sum_{i<n} M_i$ is a submodule of the module $\sum_{i\leq n} M_i/\sum_{i<n} M_i \cong M_n/(M_n \cap \sum_{i<n} M_i)$, it is finitely generated because no small subfactor of M_n is infinitely generated. Hence there exists a finitely generated module such that $F + \sum_{i<n} M_i = N + \sum_{i<n} M_i$. Now, N/F is an infinitely generated small submodule of $\sum_{i<n} M_i + F/F$, which is a contradiction with the minimality of n. □

Lemma 1.5. *Let R be a ring whose all right ideals are countably generated, M an infinitely generated small module, and κ an infinite cardinal. Suppose that $\bigoplus_{\alpha<\kappa} N_\alpha$ is a submodule of M such that $M/(\bigoplus_{\alpha\in K} N_\alpha)$ is infinitely generated for each finite subset $K \subset \kappa$. Then $M/(\bigoplus_{\alpha<\kappa} N_\alpha)$ is an infinitely generated module.*

Proof. The assertion for countable cardinals follows immediately from Corollary 1.2(5).

Let κ be uncountable and assume that there exists a finitely generated module F such that $M = F + \bigoplus_{\alpha<\kappa} N_\alpha$. Put $N_U = \bigoplus_{\alpha\in U} N_\alpha$ for a arbitrary subset $U \subset \kappa$ and $N = \bigoplus_{\alpha<\kappa} N_\alpha$. By the hypothesis and by Lemma 1.3(1) all submodules of F are countably generated, hence there exists a countable set $C \subset \kappa$ such that $F \cap N \subseteq N_C$, which implies that $(F + N_C) \cap N_{\kappa\setminus C} = 0$. Hence $M/(F + N_C) = (F + N)/(F + N_C) \cong$

$N_{\kappa\setminus C}/((F + N_C) \cap N_{\kappa\setminus C}) \cong N_{\kappa\setminus C}$. But M is small, a contradiction with Corollary 1.2(3). □

Recall several well-known properties of the class of all right steady rings.

Proposition 1.6. *Let R be a right steady ring and R_i a ring for each $i \in I$.*

(1) *Every factor of R is right steady,*

(2) *every ring Morita equivalent to R is right steady,*

(3) *$\prod_{i\in I} R_i$ is right steady iff I is finite and each R_i is right steady.*

Proof. (1) [5, Lemma 1.9], (2) [10, Theorem 2.5], (3) [6, Lemma 1.7]. □

2. Polynomials in countably many variables

Let R be a ring, M a module and $r \in R$. We say that M is *r-torsion-free* provided $mr \neq 0$ for each nonzero $m \in M$. If $S \subset R$ we say that M is *S-torsion-free* if M is r-torsion-free for each $r \in S$. Denote by $\mathcal{M}(X)$ a set of all monic monomials of the polynomial ring $R[X]$.

Theorem 2.1. *Let T be a skew field finitely generated over its center $Z(T)$ and X be a countable set of variables. Then the polynomial ring $T[X]$ is right steady.*

Proof. Assume that $T[X]$ is not right steady, i.e. there exists an infinitely generated small right $T[X]$-module. As all ideals of $T[X]$ are countably generated by Lemma 1.3(2) we may apply [12, Lemma 11] which says that there exist a two-sided prime ideal I and a module M such that

(+) M is infinitely generated and small, $MI = 0$ and $M/MpT[X]$ is finitely generated for every $p \in T[X] \setminus I$.

It is easy to see that every infinitely generated factor of M satisfies the condition (+) as well.

Before we finish the proof of Theorem 2.1, we prove three technical lemmas, in which we will deal with one fixed ideal I for some modules M satisfying the condition (+).

For convenience we introduce some new notation. Put $C = Z(T)$ and $S = Z(T)[X] \setminus I$. Obviously, C is a field and S is a multiplicative set of the ring $T[X]$. Finally, put $L_p = \{m \in L;\ mp = 0\}$ for every $p \in S$ and every module L. Clearly, L_p is a module because p is a central polynomial.

Lemma 2.2. *Assume that a module M satisfies the condition* $(+)$ *and let* $Y = \{y_0, \ldots, y_n\} \subset \mathcal{M}(X)$. *Then there exists a submodule* $M_Y \subseteq M$ *such that* M/M_Y *is an infinitely generated small module and* M/M_Y *is* s*-torsion-free for each polynomial of the form* $s = \sum_{i<n} a_i y_i \in S$.

Proof. We will prove the assertion by induction on the cardinality of the set Y. If $Y = \emptyset$, the claim is true for $M_Y = 0$. Suppose that the assertion holds true for all modules M satisfying the condition $(+)$ and for every Y such that $card(Y) \leq n$. Let $card(Y) = n + 1$ where $Y = \{y_0, \ldots, y_n\}$. Define two sets $U = \{\sum_{i\leq n} a_i y_i \in S\}$ and $V = \{y_0, \ldots, y_{n-1}\}$.

We define inductively two chains $\{P_i\}_{i<\omega}$ and $\{N_i\}_{i<\omega}$ of submodules of M such that $P_{i-1} \subseteq N_{i-1} \subseteq P_i$, $P_i/N_{i-1} = (M/N_{i-1})_V$ and M/N_i is infinitely generated. Put $P_0 = N_0 = 0$. Suppose that P_{i-1} and N_{i-1} are defined and note that there exists module $(M/N_{i-1})_V$ by the induction hypothesis since M/N_{i-1} is infinitely generated and it satisfies $(+)$. Hence we may define P_i as the submodule of M containing N_{i-1} for which $P_i/N_{i-1} = (M/N_{i-1})_V$ and N_i the submodule of M containing P_i such that $N_i/P_i = \sum_{p\in U}(M/P_i)_p$. Note that $M/P_i \cong (M/N_{i-1})/(M/N_{i-1})_V$ is infinitely generated by the induction premise. It remains to prove that M/N_i is infinitely generated as well.

Put $\overline{M} = M/P_i$ and fix an arbitrary $p \in S$. Since p is a central element, it acts on $\overline{M}$ as an endomorphism; denote by ϱ_p the endomorphism induced by the multiplication by p. Then $\overline{M}_p = ker\,\varrho_p$ and $\overline{M}/\overline{M}p$ is a finitely generated module by $(+)$. Hence $\overline{M}/\overline{M}_p \cong \varrho_p(\overline{M}) = \overline{M}p$ is infinitely generated.

Now, fix $p,\ q \in S$ such that $p = \sum_{i\leq n} a_i y_i$, $q = \sum_{i\leq n} b_i y_i$ are C-linear combinations of monomials from Y. Suppose that $m \in \overline{M}_p \cap \overline{M}_q$ is a nonzero element. As $\overline{M} = M/P_i \cong (M/N_{i-1})/(M/N_{i-1})_V$, polynomials p and q are not C-linear combinations of monomials from V, i.e. both a_n and b_n are nonzero. Moreover, $m(p - qb_n^{-1}a_n) = 0$ because $mp = mq = 0$ and $p - qb_n^{-1}a_n$ is a C-linear combination of elements from the set V. It implies that $(p - qb_n^{-1}a_n) \in I$ by the induction hypothesis and because $\overline{M}$ is not $(p - qb_n^{-1}a_n)$-torsion-free. We have proved that $(p - qb_n^{-1}a_n)$ annihilates $\overline{M}$. Hence $mp = 0$ iff $mqb_n^{-1}a_n = 0$ and, obviously, it holds true iff $mq = 0$, or equivalently formulated $\overline{M}_p = \overline{M}_{qb_n^{-1}a_n} = \overline{M}_q$.

We define an equivalence $\sim$ on $U = \{\sum_{i\leq n} a_i y_i \in S\}$. For $p, q \in U$ we have $p \sim q$ provided there exists a non-zero $a \in C$ such that $p - qa \in I$. We have proved that $p \not\sim q$ iff $\overline{M}_p \cap \overline{M}_q = 0$, and $p \sim q$ iff $\overline{M}_p = \overline{M}_q$. Moreover, $\sum_{p\in U} \overline{M}_p = \bigoplus_{[p]\in U/\sim} \overline{M}_p$. Indeed, suppose that $\sum_{i=0}^{k} m_i = 0$ where $0 \neq m_i \in \overline{M}_{p_i}$, $p_i \in S$, $p_i \not\sim p_j$ for every $i \neq j$ and fix the minimal positive k satisfying this condition. Then $0 = m_0 p_0 = -\sum_{i=1}^{k} m_i p_0$, hence by minimality of k it holds true that $m_i p_0 = 0$ for each $i > 0$. Since it implies that $m_i \in M_{p_0}$, we obtain a contradiction.

Finally, we are ready to show that the hypothesis of Lemma 1.5 is satisfied for the submodule $\bigoplus_{[p]\in U/\sim} \overline{M}_p$ of the module M. Fix an arbitrary finite set of polynomials $p_1, \dots, p_r \in S$ and put $p = p_1 \cdot \ldots \cdot p_r \in S$. Since $\sum_{i=1}^r \overline{M}_{p_i} \subseteq \overline{M}_p$ and since $Mp \cong \overline{M}/\overline{M}_p$ is an infinitely generated homomorphic image of $\overline{M}/\sum_{i=1}^r \overline{M}_{p_i}$, we see that $\overline{M}/\sum_{i=1}^r \overline{M}_{p_i}$ is infinitely generated as well. We may apply Lemma 1.5 which says that $(M/N_i) \cong \overline{M}/\sum_{p\in U} \overline{M}_p = \overline{M}/\bigoplus_{[p]_\sim\in U/\sim} \overline{M}_p$ is an infinitely generated module.

Remind that we have constructed chains $\{P_i\}_{i<\omega}$ and $\{N_i\}_{i<\omega}$ of submodules of M such that $P_{i-1} \subseteq N_{i-1} \subseteq P_i$, $P_i/N_{i-1} = (M/N_{i-1})_V$ and M/N_i is infinitely generated. Applying Corollary 1.2(5) we see that $M/\bigcup_{i<\omega} P_i = M/\bigcup_{i<\omega} N_i$ is an infinitely generated small module. Finally, fix $p \in U$ and $mp \in \bigcup_{i<\omega} N_i$. Then there exists k such that $mp \in N_k \subseteq P_{k+1}$ thus $m \in N_{k+1}$, which implies that $M/\bigcup_{i<\omega} N_i$ is U-torsion-free over. Thus we may put $M_Y = \bigcup_{i<\omega} N_i$. □

Lemma 2.3. *If a module M satisfies the condition (+), there exists an infinitely generated factor of M which is S-torsion-free.*

Proof. Fix an increasing chain $(Y_i;\ i < \omega)$ of finite subsets $Y_i \subset Y_{i+1} \subset \mathcal{M}(X)$ such that $\bigcup_{i<\omega} Y_i = \mathcal{M}(X)$. Then applying Lemma 2.2 we construct a countable chain of submodules of M such that $M_0 = 0$ and $M_i/M_{i-1} = (M/M_{i-1})_{Y_i}$. Obviously, $M/M_i \cong (M/M_{i-1})/(M/M_{i-1})_{Y_i}$ is infinitely generated for each $i > 0$, hence $\overline{M} = M/(\bigcup_{i<\omega} M_i)$ is an infinitely generated small module by Corollary 1.2(5). It is easy to see that $\overline{M}$ is S-torsion-free. □

Recall that a ring is *semisimple* if it is (direct) sum of its simple submodules, or, equivalently, if it is isomorphic to a finite direct product of matrix rings over skew fields.

Lemma 2.4. *If a small module M is S-torsion-free and $MI = 0$, then M is finitely generated.*

Proof. By [3, Proposition 0.5.3] there exists a right ring of fractions R of the ring $T[X]/I$ with respect to the multiplicative set $((C[X] + I/I) \setminus \{I\}$. Note that there exists the natural embedding of $T[X]/I$ into R since the multiplicative set is contained in the center of $T[X]/I$.

First, we prove that R is a semisimple ring. Denote by Q the classical ring of fractions of $C[X]/I$. Obviously, Q is a field which is a subring of R and R is a finitely generated vector space over Q by the hypothesis of Theorem 2.1. Since each right ideal in R is a Q-subspace of R, every strictly decreasing chain of right ideals is finite, hence R is right artinian. Since I is a prime ideal, no nonzero ideal of $T[X]/I$ and so of R is nilpotent, hence $J(R) = 0$, which implies that $R = R/J(R)$ is semisimple by Hopkins theorem.

Now, we show that no factor of R contains as a right $T[X]/I$-module an infinitely generated small submodule. Assume to contrary that some factor of R contains such a submodule. Since $T[X]/I \subseteq R$, there exists an infinitely generated small submodule M of a suitable factor of $R/(T[X]/I)$, otherwise some factor of cyclic module $T[X]/I$ contains an infinitely generated (so uncountably generated) small module which contradicts to Lemma 1.3. Obviously any subfactor of $R/(T[X]/I)$ is torsion (i.e. for every $m \in M$ there exists nonzero $r \in T[X]/I$ such that $mr = 0$). As M has the natural structure of (infinitely generated small) $T[X]$-module such that $MI = 0$, we may use [12, Lemma 11], which claims that there exists a factor $\overline{M}$ of M and a nontrivial ideal J containing I such that the condition $(+)$ holds true for $\overline{M}$. Applying Lemma 2.3 we obtain a S'-torsion-free factor of $\overline{M}$ where $S' = C[X] \setminus J$, which is a contradiction because $\overline{M}$ should be torsion.

Finally, there exists an embedding $M \hookrightarrow M \otimes_{T[X]/I} R \cong_R \bigoplus_{\alpha<\kappa} S_\alpha$ by [3, Proposition 0.6.1], where κ is a cardinal and every $S_\alpha,\ \alpha < \kappa$, is a simple right R-module, since R is semisimple. Note that we have proved that no $T[X]/I$-factor of any S_α contains an infinitely generated small submodule. Since M is small , there exists $n < \omega$ and $\alpha_i < \kappa$, for each $i < n$, such that $M \hookrightarrow \bigoplus_{i<k} S_{\alpha_i}$. As no factor of S_α, for each $\alpha < \kappa$, contains an infinitely generated small submodule, M is finitely generated by Lemma 1.4. □

We can to finish the proof of Theorem 2.1. Lemma 2.3 claims that there exists an infinitely generated small S-torsion-free module which is a factor of M. As it contradicts to Lemma 2.4, $T[X]$ is right steady. □

Note that the premise of Theorem 2.1 that T is finitely generated over its center is necessary only in Lemma 2.4, in another words, Lemmas 2.2 and 2.3 holds true for a general skew field T.

Recall that an ideal I is *right T-nilpotent* if for every sequence $\{a_i\}_{i<\omega}$ of elements of I there exists n such that $a_n \cdot \ldots \cdot a_1 = 0$. As every right T-nilpotent ideal is right steady in the sense of paper [12], we may prove the following three assertions.

Lemma 2.5. *Let R be a ring, X a set of variables and I a right T-nilpotent ideal of R. Then $R[X]$ is right steady iff $(R/J(R))[X]$ is right steady.*

Proof. Since $(R/J(R))[X]$ is isomorphic to a factor of $R[X]$, it is enough to prove the reverse implication. Assume that there exists an infinitely generated small $R[X]$-module M. Note that MJ is an $R[X]$-submodule of M because all variables commutes with elements from J. Moreover, MJ is superfluous in M as R-module by [1, Lemma 28.3], hence it superfluous in M as $R[X]$-module. Thus M/MJ is an infinitely generated small $R/J[X]$-module, which finishes the proof. □

Proposition 2.6. *Let R a be right perfect ring and X a finite set of variables. Then $R[X]$ is right steady.*

Proof. First note that $J(R)$ is right T-nilpotent by [1, Theorem 28.4] and $R/J(R)$ is semisimple. Applying Lemma 2.5 we see it is enough to check steadiness of the ring $R/J(R)[X]$. As $R/J(R)[X]$ is right noetherian by Hilbert basis theorem, the conclusion follows from [8, 7^o] or [4, Proposition 1.9]. □

Theorem 2.7. *Let X be a countable set of variables and R a right perfect ring such that $End_R(S)$ is finitely generated as a right module over its center for every simple module S. Then $R[X]$ is right steady.*

Proof. Using Lemma 2.5 as in the proof of Proposition 2.6, it suffices to show that $R/J(R)[X]$ is right steady. By the Wedderburn-Artin theorem the ring $R/J(R)$ is isomorphic to a finite product of matrix rings $M_{k_i}(T_i)$ over finitely generated skew-fields T_i where T_i is isomorphic to $End_R(S_i)$ for a suitable simple R-module S_i. Since $T_i[X]$ is right steady by Theorem 2.1 and right steadiness is a Morita invariant property by Proposition 1.6(2), $M_{k_i}(T_i[X]) \cong M_{k_i}(T_i)[X]$ is right steady. Finally, any finite product of right steady rings is right steady as well by Proposition 1.6(3). □

3. Non-steady polynomial rings

Recall that a module M is said to be ω_1*-reducing* if for every countably generated submodule N of M there exists a finitely generated submodule F of M such that $N \subseteq F$. Applying Lemma 1.1 it is easy to see that every ω_1-reducing module is an example of a small module. This notion gives us a natural construction of an infinitely generated small module as a union of an uncountable chain of finitely generated modules, which we use in the following example.

Example 3.1. Let R be an arbitrary ring and consider the monoid $\mathbb{N}^{\omega_1}$ as the product of ω_1 copies of the monoid $(\mathbb{N}, +, 0)$ of natural numbers. For every $\alpha < \beta \leq \omega_1$ define $e_{\alpha\beta} \in \mathbb{N}^{\omega_1}$ by the rule $e_{\alpha\beta}(\gamma) = 1$ for $\gamma \in \langle \alpha, \beta)$

and $e_{\alpha\beta}(\gamma) = 0$ elsewhere. Denote by E the submonoid of $\mathbb{N}^{\omega_1}$ generated by the set $\{e_{\alpha\beta} | \alpha < \beta \leq \omega_1\}$.

Now, consider a monoid ring $S = R[E]$. Put $s_\alpha = 1e_{\alpha\kappa} \in S$ for every $\alpha < \omega_1$ where 1 is the unit of the ring R. Since $s_\alpha = s_\beta \cdot 1e_{\alpha\beta}$ whenever $\alpha < \beta$, $(s_\alpha S | \alpha < \omega_1)$ forms a strictly increasing chain of right principal ideals of S. Hence the ideal $\bigcup_{\alpha<\omega} e_\alpha S$ is ω_1-generated and ω_1-reducing as a right S-module, which proves that S is not right steady. The symmetric argument shows that S is not left steady.

As the cardinality of sets of generators of any infinitely generated monoid is the same as cardinality of the monoid, we may formulate the following consequence of the previous construction.

Proposition 3.2. *If R be a ring and X an uncountable set of variables, $R[X]$ is neither right nor left steady.*

Proof. Since there exists a surjective map of X onto the monoid E from Example 3.1, it can be extended to a surjective homomorphism from the free commutative monoid in free generators X to the monoid E. Moreover, this homomorphism of monoids can be naturally extended to a surjective homomorphism of the polynomial ring $R[X]$ to the monoid ring $R[E]$. Since the homomorphic image $R[E]$ of the ring $R[X]$ is neither right nor left steady by Example 3.1, the conclusion follows by Proposition 1.6(1). $\square$

Obviously, any direct sum of an infinitely generated small module and non-small module is not small, however it has an infinitely generated small factor. If we look at the reason of non-steadiness of polynomial rings in uncountably many variables more carefully, i.e. if we explicitly construct an infinitely generated small module over such a ring, we obtain an example of non-small module with infinitely generated small factor, which is far from containing small module as a direct summand.

Example 3.3. Let X be a set of variables of the cardinality ω_1 and $Y \subset X$ such that $|Y| = |X \setminus Y|$ $(= \omega_1)$. Fix a field F, consider the monoid E as it is defined in 3.1 and fix a surjective map $\varphi : X \to E$ such that $\varphi(Y) = \{e_{\alpha\kappa} | \alpha < \kappa\}$. Then φ can be extended to a ring homomorphism $F[X] \to F[E]$ as it is described in the proof of Proposition 3.2. Now, the module $U = \sum_{y\in Y} yF[X]$ is uniform (i.e. it contains no non-trivial direct summand) because $F[X]$ is an integral domain. As there exists an strictly increasing chain of subset $Y_i \subset Y_{i+1} \subset Y$ such that $Y = \bigcup_{i<\omega} Y_i$, the module $U = \bigcup_{i<\omega} \sum_{y\in Y_i} yF[X]$ is not small by Lemma 1.1. Since $\varphi(U)$ has the natural structure of an $F[X]$-module it is an infinitely generated small $F[X]$-module.

Combining Proposition 3.2 and Theorem 2.7 we obtain the final criterion.

Corollary 3.4. *Let X be a set of variables and R a right perfect ring such that $End_R(S)$ is finitely generated as a right module over its center for every simple module S. Then $R[X]$ is right steady iff X is countable.*

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