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Symbolic Rees algebras, vertex covers and irreducible representations of Rees cones

Luis A. Dupont and Rafael H. Villarreal

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Abstract. Let G be a simple graph and let $I_c(G)$ be its ideal of vertex covers. We give a graph theoretical description of the irreducible b-vertex covers of G, i.e., we describe the minimal generators of the symbolic Rees algebra of $I_c(G)$. Then we study the irreducible b-vertex covers of the blocker of G, i.e., we study the minimal generators of the symbolic Rees algebra of the edge ideal of G. We give a graph theoretical description of the irreducible binary b-vertex covers of the blocker of G. It is shown that they correspond to irreducible induced subgraphs of G. As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G. In particular we obtain all odd holes and antiholes. We study irreducible graphs and give a method to construct irreducible b-vertex covers of the blocker of G with high degree relative to the number of vertices of G.

Introduction

A *clutter* $\mathcal{C}$ with vertex set $X = \{x_1, \ldots, x_n\}$ is a family of subsets of X, called edges, none of which is included in another. The set of vertices and edges of $\mathcal{C}$ are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. A basic example of a clutter is a graph. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The *edge ideal* of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of R generated

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by all monomials $\prod_{x_i\in e} x_i$ such that $e \in E(\mathcal{C})$. The assignment $\mathcal{C} \mapsto I(\mathcal{C})$ establishes a natural one to one correspondence between the family of clutters and the family of square-free monomial ideals. Let $\mathcal{C}$ be a clutter and let $F = \{x^{v_1}, \ldots, x^{v_q}\}$ be the minimal set of generators of its edge ideal $I = I(\mathcal{C})$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The $n \times q$ matrix with column vectors $v_1, \ldots, v_q$ will be denoted by A, it is called the *incidence matrix* of $\mathcal{C}$.

The *blowup algebra* studied here is the *symbolic Rees algebra*:

$$R_s(I) = R \oplus I^{(1)}t \oplus \cdots \oplus I^{(i)}t^i \oplus \cdots \subset R[t],$$

where t is a new variable and $I^{(i)}$ is the *ith* symbolic power of I. Closely related to $R_s(I)$ is the *Rees algebra* of I:

$$R[It] := R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t].$$

The study of symbolic powers of edge ideals was initiated in [21] and further elaborated on in [1, 9, 12, 13, 14, 22, 28]. By a result of Lyubeznik [17], $R_s(I)$ is a K-algebra of finite type. In general the minimal set of generators of $R_s(I)$ as a K-algebra is very hard to describe in terms of $\mathcal{C}$ (see [1]). There are two exceptional cases. If the clutter $\mathcal{C}$ has the max-flow min-cut property, then by a result of [13] we have $I^i = I^{(i)}$ for all $i \geq 1$, i.e., $R_s(I) = R[It]$. If G is a perfect graph, then the minimal generators of $R_s(I(G))$ are in one to one correspondence with the cliques (complete subgraphs) of G [28]. We shall be interested in studying the minimal set of generators of $R_s(I)$ using polyhedral geometry. Let G be a graph and let $I_c(G)$ be the Alexander dual of $I(G)$, see definition below. Some of the main results of this paper are graph theoretical descriptions of the minimal generators of $R_s(I(G))$ and $R_s(I_c(G))$. In Sections 1 and 2 we show that both algebras encode combinatorial information of the graph which can be decoded using integral Hilbert bases.

The *Rees cone* of I, denoted by $\mathbb{R}_+(I)$, is the polyhedral cone consisting of the non-negative linear combinations of the set

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the *ith* unit vector.

A subset $C \subset X$ is called a *vertex cover* of the clutter $\mathcal{C}$ if every edge of $\mathcal{C}$ contains at least one vertex of C. A subset $C \subset X$ is called a *minimal vertex cover* of the clutter $\mathcal{C}$ if C is a vertex cover of $\mathcal{C}$ and no proper subset of C is a vertex cover of $\mathcal{C}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of the edge ideal $I = I(\mathcal{C})$ and let

$$C_k = \{x_i |\, x_i \in \mathfrak{p}_k\} \quad (k = 1, \ldots, s)$$

be the corresponding minimal vertex covers of $\mathcal{C}$, see [27, Proposition 6.1.16]. Recall that the primary decomposition of the edge ideal of $\mathcal{C}$ is given by

$$I(\mathcal{C}) = (C_1) \cap (C_2) \cap \cdots \cap (C_s),$$

where (C_k) denotes the ideal of R generated by C_k. In particular observe that the height of $I(\mathcal{C})$ equals the number of vertices in a minimum vertex cover of $\mathcal{C}$. This number is called the *vertex covering number* of $\mathcal{C}$ and is denoted by $\alpha_0(\mathcal{C})$. The *ith symbolic power* of I is given by

$$I^{(i)} = S^{-1}I^i \cap R \ \text{ for } i \geq 1,$$

where $S = R \setminus \cup_{k=1}^{s} \mathfrak{p}_i$ and $S^{-1}I^i$ is the localization of I^i at S. In our situation the ith symbolic power of I has a simple expression:

$$I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i,$$

see [27]. The Rees cone of I is a finitely generated rational cone of dimension $n+1$. Hence by the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap H_{\ell_2}^+ \cap \cdots \cap H_{\ell_r}^+ \qquad (1)$$

such that each ℓ_k is in $\mathbb{Z}^{n+1}$, the non-zero entries of each ℓ_k are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \ldots, H_{e_{n+1}}^+, H_{\ell_1}^+, \ldots, H_{\ell_r}^+$ can be omitted from the intersection. Here H_a^+ denotes the closed halfspace $H_a^+ = \{x | \, \langle x, a\rangle \geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a, where $\langle\, ,\, \rangle$ denotes the standard inner product. The *facets* (i.e., the proper faces of maximum dimension or equivalently the faces of dimension n) of the Rees cone are exactly:

$$F_i = H_{e_i} \cap \mathbb{R}_+(I), i = 1, \ldots, n+1, H_{\ell_1} \cap \mathbb{R}_+(I), \ldots, H_{\ell_r} \cap \mathbb{R}_+(I).$$

According to [9, Lemma 3.1] we may always assume that $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for $1 \leq k \leq s$, i.e., each minimal vertex cover of $\mathcal{C}$ determines a facet of the Rees cone and every facet of the Rees cone satisfying $\langle \ell_k, e_{n+1}\rangle = -1$ must be of the form $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for some minimal vertex cover C_k of $\mathcal{C}$. This is quite interesting because this is saying that the Rees cone of $I(\mathcal{C})$ is a carrier of combinatorial information of the clutter $\mathcal{C}$. Thus we can extract the primary decomposition of $I(\mathcal{C})$ from the irreducible representation of $\mathbb{R}_+(I(\mathcal{C}))$.

Rees cones have been used to study algebraic and combinatorial properties of blowup algebras of square-free monomial ideals and clutters [9, 12, 29]. Blowup algebras are interesting objects of study in algebra and geometry [25].

The ideal of *vertex covers* of $\mathcal{C}$ is the square-free monomial ideal

$$I_c(\mathcal{C}) = (x^{u_1}, \ldots, x^{u_s}) \subset R,$$

where $x^{u_k} = \prod_{x_i \in C_k} x_i$. Often the ideal $I_c(\mathcal{C})$ is called the *Alexander dual* of $I(\mathcal{C})$. The clutter $\Upsilon(\mathcal{C})$ associated to $I_c(\mathcal{C})$ is called the *blocker* of $\mathcal{C}$, see [5]. Notice that the edges of $\Upsilon(\mathcal{C})$ are precisely the minimal vertex covers of $\mathcal{C}$. If G is a graph, then $R_s(I_c(G))$ is generated as a K-algebra by elements of degree in t at most two [14, Theorem 5.1]. One of the main result of Section 1 is a graph theoretical description of the minimal generators of $R_s(I_c(G))$ (see Theorem 1.7). As an application we recover an explicit description [24], in terms of closed halfspaces, of the edge cone of a graph (Corollary 1.10).

The symbolic Rees algebra of the ideal $I_c(\mathcal{C})$ can be interpreted in terms of "k-vertex covers" [14] as we now explain. Let $a = (a_1, \ldots, a_n) \neq 0$ be a vector in $\mathbb{N}^n$ and let $b \in \mathbb{N}$. We say that a is a *b-vertex cover* of I (or $\mathcal{C}$) if $\langle v_i, a\rangle \geq b$ for $i = 1, \ldots, q$. Often we will call a b-vertex cover simply a *b-cover*. This notion plays a role in combinatorial optimization [20, Chapter 77, p. 1378] and algebraic combinatorics [14, 15].

The *algebra of covers* of I (or $\mathcal{C}$), denoted by $R_c(I)$, is the K-subalgebra of $K[t]$ generated by all monomials $x^a t^b$ such that a is a b-cover of I. We say that a b-cover a of I is *reducible* if there exists an i-cover c and a j-cover d of I such that $a = c + d$ and $b = i + j$. If a is not reducible, we call a *irreducible*. The irreducible 0 and 1 covers of $\mathcal{C}$ are the unit vector $e_1, \ldots, e_n$ and the incidence vectors $u_1, \ldots, u_s$ of the minimal vertex covers of $\mathcal{C}$, respectively. The minimal generators of $R_c(I)$ as a K-algebra correspond to the irreducible covers of I. Notice the following dual descriptions:

$$\begin{aligned} I^{(b)} &= (\{x^a \mid \langle a, u_i\rangle \geq b \text{ for } i = 1, \ldots, s\}), \\ J^{(b)} &= (\{x^a \mid \langle a, v_i\rangle \geq b \text{ for } i = 1, \ldots, q\}), \end{aligned}$$

where $J = I_c(\mathcal{C})$. Hence $R_c(I) = R_s(J)$ and $R_c(J) = R_s(I)$.

In general each ℓ_i occurring in Eq. (1) determines a minimal generator of $R_s(I_c(\mathcal{C}))$. Indeed if we write $\ell_i = (a_i, -d_i)$, where $a_i \in \mathbb{N}^n$, $d_i \in \mathbb{N}$, then a_i is an irreducible d_i-cover of I (Lemma 1.8). Let F_{n+1} be the facet of $\mathbb{R}_+(I)$ determined by the hyperplane $H_{e_{n+1}}$. Thus we have a map ψ:

$$\begin{aligned} \{\text{Facets of } \mathbb{R}_+(I(\mathcal{C}))\} \setminus \{F_{n+1}\} &\stackrel{\psi}{\longrightarrow} R_s(I_c(\mathcal{C})) \\ H_{\ell_k} \cap \mathbb{R}_+(I) &\stackrel{\psi}{\longrightarrow} x^{a_k} t^{d_k}, \text{ where } \ell_k = (a_k, -d_k) \\ H_{e_i} \cap \mathbb{R}_+(I) &\stackrel{\psi}{\longrightarrow} x_i \end{aligned}$$

whose image provides a good approximation for the minimal set of generators of $R_s(I_c(\mathcal{C}))$ as a K-algebra. Likewise the facets of $\mathbb{R}_+(I_c(\mathcal{C}))$ give an approximation for the minimal set of generators of $R_s(I(\mathcal{C}))$. In Example 1.9 we show a connected graph G for which the image of the map ψ does not generates $R_s(I_c(G))$. For balanced clutters, i.e., for clutters without odd cycles, the image of the map ψ generates $R_s(I_c(\mathcal{C}))$. This follows from [12, Propositions 4.10 and 4.11]. In particular the image of the map ψ generates $R_s(I_c(\mathcal{C}))$ when $\mathcal{C}$ is a bipartite graph. It would be interesting to characterize when the irreducible representation of the Rees cone determine the irreducible covers.

The *Simis cone* of I is the rational polyhedral cone:

$$\mathrm{Cn}(I) = H^+_{e_1} \cap \cdots \cap H^+_{e_{n+1}} \cap H^+_{(u_1,-1)} \cap \cdots \cap H^+_{(u_s,-1)},$$

Simis cones were introduced in [9] to study symbolic Rees algebras of square-free monomial ideals. If $\mathcal{H}$ is an integral Hilbert basis of $\mathrm{Cn}(I)$, then $R_s(I(\mathcal{C}))$ equals $K[\mathbb{N}\mathcal{H}]$, the semigroup ring of $\mathbb{N}\mathcal{H}$ (see [9, Theorem 3.5]). This result is interesting because it allows us to compute the minimal generators of $R_s(I(\mathcal{C}))$ using Hilbert bases. The program *Normaliz* [3] is suitable for computing Hilbert bases. There is a description of $\mathcal{H}$ valid for perfect graphs [28]. Perfect graphs are defined in Section 2.

If G is a perfect graph, the irreducible b-covers of $\Upsilon(G)$ correspond to cliques of G [28] (cf. Corollary 2.5). In this case, setting $\mathcal{C} = \Upsilon(G)$, it turns out that the image of ψ generates $R_s(I_c(\Upsilon(G)))$. Notice that $I_c(\Upsilon(G))$ is equal to $I(G)$.

In Section 2 we introduce and study the concept of an irreducible graph. A b-cover $a = (a_1, \dots, a_n)$ is called *binary* if $a_i \in \{0, 1\}$ for all i. We present a graph theoretical description of the irreducible binary b-vertex covers of the blocker of G (see Theorem 2.9). It is shown that they are in one to one correspondence with the irreducible induced subgraphs of G. As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G (see Corollary 2.12). In particular we obtain all induced odd cycles and all induced complements of odd cycles. These cycles are called the *odd holes* and *odd antiholes* of the graph. It was shown recently [10] that $\mathfrak{p}$ is an associated prime of $I_c(G)^2$ if and only if $\mathfrak{p}$ is generated by the vertices of an edge of G or $\mathfrak{p}$ is generated by the vertices of an odd hole of G. The proof of this remarkable result makes use of Theorem 1.7. Odd holes and antiholes play a major role in graph theory. In [4] it is shown that a graph G is perfect if and only if G has no odd holes or antiholes of length at least five. We give a procedure to build irreducible graphs (Proposition 2.18) and a method to construct irreducible b-vertex covers of the blocker of G with high degree relative to the number of vertices of G (see Corollaries 2.24 and 2.25).

Along the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology we refer to [6, 18, 25].

1. Blowup algebras of ideals of vertex covers

Let G be a simple graph with vertex set $X = \{x_1, \ldots, x_n\}$. In what follows we shall always assume that G has no isolated vertices. Here we will give a graph theoretical description of the irreducible b-covers of G, i.e., we will describe the symbolic Rees algebra of $I_c(G)$.

Let S be a set of vertices of G. The *neighbor set* of S, denoted by $N_G(S)$, is the set of vertices of G that are adjacent with at least one vertex of S. The set S is called *independent* if no two vertices of S are adjacent. The empty set is regarded as an independent set whose incidence vector is the zero vector. Notice the following duality: S is a maximal independent set of G (with respect to inclusion) if and only if $X \setminus S$ is a minimal vertex cover of G.

Lemma 1.1. *If $a = (a_i) \in \mathbb{N}^n$ is an irreducible k-cover of G, then $0 \leq k \leq 2$ and $0 \leq a_i \leq 2$ for $i = 1, \ldots, n$.*

Proof. Recall that a is a k-cover of G if and only if $a_i + a_j \geq k$ for each edge $\{x_i, x_j\}$ of G. If $k = 0$ or $k = 1$, then by the irreducibility of a it is seen that either $a = e_i$ for some i or $a = e_{i_1} + \cdots + e_{i_r}$ for some minimal vertex cover $\{x_{i_1}, \ldots, x_{i_r}\}$ of G. Thus we may assume that $k \geq 2$.

Case (I): $a_i \geq 1$ for all i. Clearly $\mathbf{1} = (1, \ldots, 1)$ is a 2-cover. If $a - \mathbf{1} \neq 0$, then $a - \mathbf{1}$ is a $k - 2$ cover and $a = \mathbf{1} + (a - \mathbf{1})$, a contradiction to a being an irreducible k-cover. Hence $a = \mathbf{1}$. Pick any edge $\{x_i, x_j\}$ of G. Since a is a k-cover, we get $2 = a_i + a_j \geq k$ and k must be equal to 2.

Case (II): $a_i = 0$ for some i. We may assume $a_i = 0$ for $1 \leq i \leq r$ and $a_i \geq 1$ for $i > r$. Notice that the set $S = \{x_1, \ldots, x_r\}$ is independent because if $\{x_i, x_j\}$ is an edge and $1 \leq i < j \leq r$, then $0 = a_i + a_j \geq k$, a contradiction. Consider the neighbor set $N_G(S)$ of S. We may assume that $N_G(S) = \{x_{r+1}, \ldots, x_s\}$. Observe that $a_i \geq k \geq 2$ for $i = r+1, \ldots, s$, because a is a k-cover. Write

$$a = (0, \ldots, 0, a_{r+1} - 2, \ldots, a_s - 2, a_{s+1} - 1, \ldots, a_n - 1) + (\underbrace{0, \ldots, 0}_{r}, \underbrace{2, \ldots, 2}_{s-r}, \underbrace{1, \ldots, 1}_{n-s}) = c + d.$$

Clearly d is a 2-cover. If $c \neq 0$, using that $a_i \geq k \geq 2$ for $r + 1 \leq i \leq s$ and $a_i \geq 1$ for $i > s$ it is not hard to see that c is a $(k - 2)$-cover. This gives a contradiction, because $a = c + d$. Hence $c = 0$. Therefore $a_i = 2$ for $r < i \leq s$, $a_i = 1$ for $i > s$, and $k = 2$. □

The next result complements the fact that the symbolic Rees algebra of $I_c(G)$ is generated by monomials of degree in t at most two [14, Theorem 5.1].

Corollary 1.2. *$R_s(I_c(G))$ is generated as a K-algebra by monomials of degree in t at most two and total degree at most $2n$.*

Proof. Let x^at^k be a minimal generator of $R_s(I_c(G))$ as a K-algebra. Then $a=(a_1,\ldots,a_n)$ is an irreducible k-cover of G. By Lemma 1.1 we obtain that $0\le k\le 2$ and $0\le a_i\le 2$ for all i. If $k=0$ or $k=1$, we get that the degree of x^at^k is at most n. Indeed when $k=0$ or $k=1$, one has $a=e_i$ or $a=\sum_{x_i\in C}e_i$ for some minimal vertex cover C of G, respectively. If $k=2$, by the proof of Lemma 1.1 either $a=\mathbf{1}$ or $a_i=0$ for some i. Thus $\deg(x^a)\le 2(n-1)$. □

Let $I=I(G)$ be the edge ideal of G. For use below consider the vectors $\ell_1,\ldots,\ell_r$ that occur in the irreducible representation of $\mathbb{R}_+(I)$ given in Eq. (1).

Corollary 1.3. *If $\ell_i=(\ell_{i1},\ldots,\ell_{in},-\ell_{i(n+1)})$, then $0\le\ell_{ij}\le 2$ for $j=1,\ldots,n$ and $1\le\ell_{i(n+1)}\le 2$.*

Proof. It suffices to observe that $(\ell_{i1},\ldots,\ell_{in})$ is an irreducible $\ell_{i(n+1)}$-cover of G and to apply Lemma 1.1. □

Lemma 1.4. *$a=(1,\ldots,1)$ is an irreducible 2-cover of G if and only if G is non bipartite.*

Proof. $\Rightarrow$) We proceed by contradiction assuming that G is a bipartite graph. Then G has a bipartition (V_1,V_2). Set $a'=\sum_{x_i\in V_1}e_i$ and $a''=\sum_{x_i\in V_2}e_i$. Since V_1 and V_2 are minimal vertex covers of G, we can decompose a as $a=a'+a''$, where a' and a'' are 1-covers, which is impossible.

$\Leftarrow$) Notice that a cannot be the sum of a 0-cover and a 2-cover. Indeed if $a=a'+a''$, where a' is a 0-cover and a'' is a 2-cover, then a'' has an entry a_i equal to zero. Pick an edge $\{x_i,x_j\}$ incident with x_i, then $\langle a'',e_i+e_j\rangle\le 1$, a contradiction. Thus we may assume that $a=c+d$, where c,d are 1-covers. Let C_r be an odd cycle of G of length r. Notice that any vertex cover of C_r must contain a pair of adjacent vertices because r is odd. Clearly a vertex cover of G is also a vertex cover of the subgraph C_r. Hence the vertex covers of G corresponding to c and d must contain a pair of adjacent vertices, a contradiction because c and d are complementary vectors and the complement of a vertex cover is an independent set. □

Definition 1.5. Let A be the incidence matrix of a clutter $\mathcal{C}$. A clutter $\mathcal{C}$ has a *cycle* of length r if there is a square sub-matrix of A of order $r\ge 3$ with exactly two 1's in each row and column. A clutter without odd cycles is called *balanced*.

Proposition 1.6 ([12, Proposition 4.11])**.** *If $\mathcal{C}$ is a balanced clutter, then*

$$R_s(I_c(\mathcal{C})) = R[I_c(\mathcal{C})t].$$

This result was first shown for bipartite graphs in [11, Corollary 2.6] and later generalized to balanced clutters [12] using an algebro combinatorial description of clutters with the max-flow min-cut property [13].

Let S be a set of vertices of a graph G, the *induced subgraph* on S, denoted by $\langle S\rangle$, is the maximal subgraph of G with vertex set S. The next result has been used in [10] to show that any associated prime of $I_c(G)^2$ is generated by the vertices of an edge of G or it is generated by the vertices of an odd hole of G.

We come to the main result of this section.

Theorem 1.7. *Let $0 \neq a = (a_i) \in \mathbb{N}^n$ and let $\Upsilon(G)$ be the family of minimal vertex covers of a graph G.*

(i) *If G is bipartite, then a is an irreducible b-cover of G if and only if $b = 0$ and $a = e_i$ for some $1 \leq i \leq n$ or $b = 1$ and $a = \sum_{x_i\in C} e_i$ for some $C \in \Upsilon(G)$.*

(ii) *If G is non-bipartite, then a is an irreducible b-cover if and only if a has one of the following forms*:

(a) (0-*covers*) *$b = 0$ and $a = e_i$ for some $1 \leq i \leq n$,*

(b) (1-*covers*) *$b = 1$ and $a = \sum_{x_i\in C} e_i$ for some $C \in \Upsilon(G)$,*

(c) (2-*covers*) *$b = 2$ and $a = (1, \ldots, 1)$,*

(d) (2-*covers*) *$b = 2$ and up to permutation of vertices*

$$a = (\underbrace{0, \ldots, 0}_{|A|}, \underbrace{2, \ldots, 2}_{|N_G(A)|}, 1, \ldots, 1)$$

for some independent set of vertices $A \neq \emptyset$ of G such that

(d$_1$) *$N_G(A)$ is not a vertex cover of G and $V \neq A \cup N_G(A)$,*

(d$_2$) *the induced subgraph $\langle V \setminus (A \cup N_G(A))\rangle$ has no isolated vertices and is not bipartite.*

Proof. (i) $\Rightarrow$) Since G is bipartite, by Proposition 1.6, we have the equality $R_s(J) = R[Jt]$, where $J = I_c(G)$ is the ideal of vertex covers of G. Thus the minimal set of generator of $R_s(J)$ as a K-algebra is the set

$$\{x_1, \ldots, x_n, x^{u_1}t, \ldots, x^{u_s}t\},$$

where $u_1, \ldots, u_s$ are the incidence vectors of the minimal vertex covers of G. By hypothesis a is an irreducible b-cover of G, i.e., $x^a t^b$ is a minimal generator of $R_s(I_c(\mathcal{C}))$. Therefore either $a = e_i$ for some i and $b = 0$ or $a = u_i$ for some i and $b = 1$. The converse follows readily and is valid for any graph or clutter.

(ii) $\Rightarrow$) By Lemma 1.1 we have $0 \leq b \leq 2$ and $0 \leq a_i \leq 2$ for all i. If $b = 0$ or $b = 1$, then clearly a has the form indicated in (a) or (b) respectively.

Assume $b = 2$. If $a_i \geq 1$ for all i, the $a_i = 1$ for all i, otherwise if $a_i = 2$ for some i, then $a - e_i$ is a 2-cover and $a = e_i + (a - e_i)$, a contradiction. Hence $a = \mathbf{1}$. Thus we may assume that a has the form

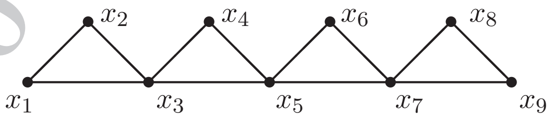
$$a = (0, \ldots, 0, 2, \ldots, 2, 1, \ldots, 1).$$

We set $A = \{x_i |\, a_i = 0\} \neq \emptyset$, $B = \{x_i |\, a_i = 2\}$, and $C = V \setminus (A \cup B)$. Observe that A is an independent set because a is a 2-cover and $B = N_G(A)$ because a is irreducible. Hence it is seen that conditions (d_1) and (d_2) are satisfied. Using Lemma 1.4, the proof of the converse is direct. $\square$

Lemma 1.8. *Let $\mathcal{C}$ be a clutter and let $I = I(\mathcal{C})$ be its edge ideal. If $\ell_k = (a_k, -d_k)$ is any of the vectors that occur in Eq. (1), where $a_k \in \mathbb{N}^n$, $d_k \in \mathbb{N}$, then a_k is an irreducible d_k-cover of $\mathcal{C}$.*

Proof. We proceed by contradiction assume there is a d_k'-cover a_k' and a d_k''-cover a_k'' such that $a_k = a_k' + a_k''$ and $d_k = d_k' + d_k''$. Set $F' = H_{(a_k', -d_k')} \cap \mathbb{R}_+(I)$ and $F'' = H_{(a_k'', -d_k'')} \cap \mathbb{R}_+(I)$. Clearly F', F'' are proper faces of $\mathbb{R}_+(I)$ and $F = \mathbb{R}_+(I) \cap H_{\ell_k} = F' \cap F''$. Recall that any proper face of $\mathbb{R}_+(I)$ is the intersection of those facets that contain it (see [30, Theorem 3.2.1(vii)]). Applying this fact to F' and F'' it is seen that $F' \subset F$ or $F'' \subset F$, i.e., $F = F'$ or $F = F''$. We may assume $F = F'$. Hence $H_{(a_k', -d_k')} = H_{\ell_k}$. Taking orthogonal complements we get that $(a_k', -d_k') = \lambda(a_k, -d_k)$ for some $\lambda \in \mathbb{Q}_+$, because the orthogonal complement of H_{ℓ_k} is one dimensional and it is generated by ℓ_k. Since the non-zero entries of ℓ_k are relatively prime, we may assume that $\lambda \in \mathbb{N}$. Thus $d_k' = \lambda d_k \geq d_k \geq d_k'$ and λ must be equal to 1. Hence $a_k = a_k'$ and a_k'' must be zero, a contradiction. $\square$

Example 1.9. Consider the following graph G:



Using *Normaliz* [3] it is seen that the vector $a = (1, 1, 2, 0, 2, 1, 1, 1, 1)$ is an irreducible 2-cover of G such that the supporting hyperplane $H_{(a,-2)}$ does not define a facet of the Rees cone of $I(G)$. Thus, in general, the image of ψ described in the introduction does not determine $R_s(I_c(G))$. We may use Lemma 1.4 to construct non-connected graphs with this property.

Edge cones of graphs. Let G be a connected simple graph and let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of all vectors $e_i + e_j$ such that $\{x_i, x_j\}$ is an edge of G. The *edge cone* of G, denoted by $\mathbb{R}_+\mathcal{A}$, is defined as the cone generated by $\mathcal{A}$. Below we give an explicit combinatorial description of the edge cone.

Let A be an *independent set* of vertices of G. The supporting hyperplane of the edge cone of G defined by

$$\sum_{x_i \in N_G(A)} x_i - \sum_{x_i \in A} x_i = 0$$

will be denoted by H_A.

Edge cones and their representations by closed halfspaces are a useful tool to study a-invariants of edge subrings [23, 26]. The following result is a prototype of these representations. As an application we give a direct proof of the next result using Rees cones.

Corollary 1.10 ([24, Corollary 2.8]). *A vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is in $\mathbb{R}_+\mathcal{A}$ if and only if a satisfies the following system of linear inequalities*

$$\begin{array}{rcll} a_i & \geq & 0, & i = 1, \ldots, n; \\ \sum_{x_i \in N_G(A)} a_i - \sum_{x_i \in A} a_i & \geq & 0, & \textit{for all independent sets } A \subset V(G). \end{array}$$

Proof. Set $\mathcal{B} = \{(v_1, 1), \ldots, (v_q, 1)\}$ and $I = I(G)$. Notice the equality

$$\mathbb{R}_+(I) \cap \mathbb{R}\mathcal{B} = \mathbb{R}_+\mathcal{B}, \qquad (2)$$

where $\mathbb{R}\mathcal{B}$ is $\mathbb{R}$-vector space spanned by $\mathcal{B}$. Consider the irreducible representation of $\mathbb{R}_+(I)$ given in Eq. (1) and write $\ell_i = (a_i, -d_i)$, where $0 \neq a_i \in \mathbb{N}^n$, $0 \neq d_i \in \mathbb{N}$. Next we show the equality:

$$\mathbb{R}_+\mathcal{A} = \mathbb{R}\mathcal{A} \cap \mathbb{R}^n_+ \cap H^+_{(2a_1/d_1 - \mathbf{1})} \cap \cdots \cap H^+_{(2a_r/d_r - \mathbf{1})}, \qquad (3)$$

where $\mathbf{1} = (1, \ldots, 1)$. Take $\alpha \in \mathbb{R}_+\mathcal{A}$. Clearly $\alpha \in \mathbb{R}\mathcal{A} \cap \mathbb{R}^n_+$. We can write

$$\alpha = \lambda_1 v_1 + \cdots + \lambda_q v_q \;\Rightarrow\; |\alpha| = 2(\lambda_1 + \cdots + \lambda_q) = 2b.$$

Thus $(\alpha, b) = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1)$, i.e., $(\alpha, b) \in \mathbb{R}_+\mathcal{B}$. Hence from Eq. (2) we get $(\alpha, b) \in \mathbb{R}_+(I)$ and

$$\langle(\alpha, b), (a_i, -d_i)\rangle \geq 0 \ \Rightarrow\ \langle\alpha, a_i\rangle \geq bd_i = (|\alpha|/2)d_i = |\alpha|(d_i/2).$$

Writing $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $a_i = (a_{i1}, \ldots, a_{in})$, the last inequality gives:

$$\alpha_1 a_{i_1} + \cdots + \alpha_n a_{in} \geq (\alpha_1 + \cdots + \alpha_n)(d_i/2) \ \Rightarrow\ \langle\alpha, a_i - (d_i/2)\mathbf{1}\rangle \geq 0.$$

Then $\langle\alpha, 2a_i/d_i - \mathbf{1}\rangle \geq 0$ and $\alpha \in H^+_{(2a_i/d_i-\mathbf{1})}$ for all i, as required. This proves that $\mathbb{R}_+\mathcal{A}$ is contained in the right hand side of Eq. (3). The other inclusion follows similarly. Now by Lemma 1.8 we obtain that a_i is an irreducible d_i-cover of G. Therefore, using the explicit description of the irreducible b-covers of G given in Theorem 1.7, we get the equality

$$\mathbb{R}_+\mathcal{A} = \left(\bigcap_{A\in\mathcal{F}} H^+_A\right) \bigcap \left(\bigcap_{i=1}^{n} H^+_{e_i}\right),$$

where $\mathcal{F}$ is the collection of all the independent sets of vertices of G. From this equality the assertion follows at once. □

The edge cone of G encodes information about both the Hilbert function of the edge subring $K[G]$ (see [23]) and the graph G itself. As a simple illustration, we recover the following version of the marriage theorem for bipartite graphs, see [2]. Recall that a pairing by an independent set of edges of all the vertices of a graph G is called a *perfect matching* or a 1-*factor*.

Corollary 1.11. *If G is a bipartite connected graph, then G has a perfect matching if and only if $|A| \leq |N_G(A)|$ for every independent set of vertices A of G.*

Proof. Notice that the graph G has a perfect matching if and only if the vector $\beta = (1, 1, \ldots, 1)$ is in $\mathbb{N}\mathcal{A}$. By [23, Lemma 2.9] we have the equality $\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}$. Hence β is in $\mathbb{N}\mathcal{A}$ if and only if $\beta \in \mathbb{R}_+\mathcal{A}$. Thus the result follows from Corollary 1.10. □

2. Symbolic Rees algebras of edge ideals

Let G be a graph with vertex set $X = \{x_1, \ldots, x_n\}$ and let $I = I(G)$ be its edge ideal. As before we denote the clutter of minimal vertex covers of G by $\Upsilon(G)$. The clutter $\Upsilon(G)$ is called the *blocker* of G. Recall that the symbolic Rees algebra of $I(G)$ is given by

$$R_s(I(G)) = K[x^a t^b \,|\, a \text{ is an irreducible } b\text{-cover of } \Upsilon(G)], \qquad (4)$$

where the set $\{x^at^b|\, a \text{ is an irreducible } b\text{-cover of } \Upsilon(G)\}$ is the minimal set of generators of $R_s(I(G))$ as a K-algebra. The main purpose of this section is to study the symbolic Rees algebra of $I(G)$ via graph theory. We are interested in finding combinatorial representations for the minimal set of generators of this algebra.

Lemma 2.1. *Let $0 \neq a = (a_1, \ldots, a_m, 0, \ldots, 0) \in \mathbb{N}^n$ and let $a' = (a_1, \ldots, a_m)$. If $0 \neq b \in \mathbb{N}$, then a is an irreducible b-cover of $\Upsilon(G)$ if and only if a' is an irreducible b-cover of $\Upsilon(\langle S\rangle)$, where $S = \{x_1, \ldots, x_m\}$.*

Proof. It suffices to prove that a is a b-cover of the blocker of G if and only if a' is a b-cover of the blocker of $\langle S\rangle$.

$\Rightarrow$) The induced subgraph $\langle S\rangle$ is not a discrete graph. Take a minimal vertex cover C' of $\langle S\rangle$. Set $C = C' \cup (V(G) \setminus S)$. Since C is a vertex cover of G such that $C \setminus \{x_i\}$ is not a vertex cover of G for every $x_i \in C'$, there is a minimal vertex cover C_ℓ of G such that $C' \subset C_\ell \subset C$ and $C' = C_\ell \cap S$. Notice that

$$\sum_{x_i \in C'} a_i = \sum_{x_i \in C_\ell \cap S} a_i = \langle a, u_\ell\rangle \geq b,$$

where u_ℓ is the incidence vector of C_ℓ. Hence $\sum_{x_i \in C'} a_i \geq b$, as required.

$\Leftarrow$) Take a minimal vertex cover C_ℓ of G. Then $C_\ell \cap S$ contains a minimal vertex cover C'_ℓ of $\langle S\rangle$. Let u_ℓ (resp. u'_ℓ) be the incidence vector of C_ℓ (resp. C'_ℓ). Notice that

$$\langle a, u_\ell\rangle = \sum_{x_i \in C_\ell \cap S} a_i \geq \sum_{x_i \in C'_\ell} a_i = \langle a', u'_\ell\rangle \geq b.$$

Hence $\langle a, u_\ell\rangle \geq b$, as required. □

We denote a complete subgraph of G with r vertices by $\mathcal{K}_r$. If v is a vertex of G, we denote its neighbor set by $N_G(v)$.

Lemma 2.2. *Let G be a graph and let $a = (a_1, \ldots, a_n)$ be an irreducible b-cover of $\Upsilon(G)$ such that $a_i \geq 1$ for all i. If $\langle N_G(x_n)\rangle = \mathcal{K}_r$, then $a_i = 1$ for all i, $b = r$, $n = r + 1$, and $G = \mathcal{K}_n$.*

Proof. We may assume that $N_G(x_n) = \{x_1, \ldots, x_r\}$. We set

$$c = e_1 + \cdots + e_r + e_n; \quad d = (a_1 - 1, \ldots, a_r - 1, a_{r+1}, \ldots, a_{n-1}, a_n - 1).$$

Notice that $\langle x_1, \ldots, x_r, x_n\rangle = \mathcal{K}_{r+1}$. Thus c is an r-cover of $\Upsilon(G)$ because any minimal vertex cover of G must intersect all edges of $\mathcal{K}_{r+1}$. By the irreducibility of a, there exists a minimal vertex cover C_ℓ of G such that

$\sum_{x_i\in C_\ell} a_i = b$. Clearly we have $b \geq g \geq r$, where g is the height of $I(G)$. Let C_k be an arbitrary minimal vertex cover of G. Since C_k contains exactly r vertices of $\mathcal{K}_{r+1}$, from the inequality $\sum_{x_i\in C_k} a_i \geq b$ we get $\sum_{x_i\in C_k} d_i \geq b - r$, where $d_1, \ldots, d_n$ are the entries of d. Therefore $d = 0$; otherwise if $d \neq 0$, then d is a $(b-r)$-cover of $\Upsilon(G)$ and $a = c + d$, a contradiction to the irreducibility of a. It follows that $g = r$, $n = r + 1$, $a_i = 1$ for $1 \leq i \leq r$, $a_n = 1$, and $G = \mathcal{K}_n$. □

Notation We regard $\mathcal{K}_0$ as the empty set with zero elements. A sum over an empty set is defined to be 0.

Proposition 2.3. *Let G be a graph and let $J = I_c(G)$ be its ideal of vertex covers. Then the set*

$$F = \{(a_i) \in \mathbb{R}^{n+1} | \textstyle\sum_{x_i\in\mathcal{K}_r} a_i = (r-1)a_{n+1}\} \cap \mathbb{R}_+(J)$$

is a facet of the Rees cone $\mathbb{R}_+(J)$.

Proof. If $\mathcal{K}_r = \emptyset$, then $r = 0$ and $F = H_{e_{n+1}} \cap \mathbb{R}_+(J)$, which is clearly a facet because $e_1, \ldots, e_n \in F$. If $r = 1$, then $F = H_{e_i} \cap \mathbb{R}_+(J)$ for some $1 \leq i \leq n$, which is a facet because $e_j \in F$ for $j \notin \{i, n+1\}$ and there is at least one minimal vertex cover of G not containing x_i. We may assume that $X' = \{x_1, \ldots, x_r\}$ is the vertex set of $\mathcal{K}_r$ and $r \geq 2$. For each $1 \leq i \leq r$ there is a minimal vertex cover C_i of G not containing x_i. Notice that C_i contains $X' \setminus \{x_i\}$. Let u_i be the incidence vector of C_i. Since the rank of $u_1, \ldots, u_r$ is r, it follows that the set

$$\{(u_1, 1), \ldots, (u_r, 1), e_{r+1}, \ldots, e_n\}$$

is a linearly independent set contained in F, i.e., $\dim(F) = n$. Hence F is a facet of $\mathbb{R}_+(J)$ because the hyperplane that defines F is a supporting hyperplane. □

Proposition 2.4. *Let G be a graph and let $0 \neq a = (a_i) \in \mathbb{N}^n$. If*

(a) $a_i \in \{0, 1\}$ *for all i, and*

(b) $\langle\{x_i | a_i > 0\}\rangle = \mathcal{K}_{r+1}$,

then a is an irreducible r-cover of $\Upsilon(G)$.

Proof. By Proposition 2.3, the closed halfspace $H^+_{(a,-r)}$ occurs in the irreducible representation of the Rees cone $\mathbb{R}_+(J)$, where $J = I_c(G)$. Hence a is an irreducible r-cover by Lemma 1.8. □

A *clique* of a graph G is a set of vertices that induces a complete subgraph. We will also call a complete subgraph of G a clique. Symbolic Rees algebras are related to perfect graphs as is seen below. Let us recall the notion of perfect graph. A *colouring* of the vertices of G is an assignment of colours to the vertices of G in such a way that adjacent vertices have distinct colours. The *chromatic number* of G is the minimal number of colours in a colouring of G. A graph is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest complete subgraph of H. We refer to [5, 6, 20] and the references there for the theory of perfect graphs.

Notation The *support* of $x^a = x_1^{a_1}\cdots x_n^{a_n}$ is $\operatorname{supp}(x^a) = \{x_i \,|\, a_i > 0\}$.

Corollary 2.5 ([28]). *If G is a graph, then*

$$K[x^a t^r \,|\, x^a \ \textit{square-free}\ ;\ \langle \operatorname{supp}(x^a)\rangle = \mathcal{K}_{r+1};\ 0 \le r < n] \subset R_s(I(G))$$

with equality if and only if G is a perfect graph.

Proof. The inclusion follows from Proposition 2.4. If G is a perfect graph, then by [28, Corollary 3.3] the equality holds. Conversely if the equality holds, then by Lemma 1.8 and Proposition 2.3 we have

$$\mathbb{R}_+(I_c(G)) = \left\{(a_i) \in \mathbb{R}^{n+1} \middle|\ \textstyle\sum_{x_i \in \mathcal{K}_r} a_i \ge (r-1)a_{n+1};\ \forall \mathcal{K}_r \subset G\right\}. \quad (5)$$

Hence a direct application of [28, Proposition 2.2] gives that G is a perfect graph. □

The *vertex covering number* of G, denoted by $\alpha_0(G)$, is the number of vertices in a minimum vertex cover of G (the cardinality of any smallest vertex cover in G). Notice that $\alpha_0(G)$ equals the height of $I(G)$. If H is a discrete graph, i.e., all the vertices of H are isolated, we set $I(H) = 0$ and $\alpha_0(H) = 0$.

Lemma 2.6. *Let G be a graph. If $a = e_1 + \cdots + e_r$ is an irreducible b-cover of $\Upsilon(G)$, then $b = \alpha_0(H)$, where $H = \langle x_1, \ldots, x_r\rangle$.*

Proof. The case $b = 0$ is clear. Assume $b \ge 1$. Let $C_1, \ldots, C_s$ be the minimal vertex covers of G and let $u_1, \ldots, u_s$ be their incidence vectors. Notice that $\langle a, u_i\rangle = b$ for some i. Indeed if $\langle a, u_i\rangle > b$ for all i, then $a - e_1$ is a b-cover of $\Upsilon(G)$ and $a = (a - e_1) + e_1$, a contradiction. Hence

$$b = \langle a, u_i\rangle = |\{x_1, \ldots, x_r\} \cap C_i| \ge \alpha_0(H).$$

This proves that $b \geq \alpha_0(H)$. Notice that H is not a discrete graph. Then we can pick a minimal vertex cover A of H such that $|A| = \alpha_0(H)$. The set

$$C = A \cup (V(G) \setminus \{x_1, \ldots, x_r\})$$

is a vertex cover of G. Hence there is a minimal vertex cover C_ℓ of G such that $A \subset C_\ell \subset C$. Observe that $C_\ell \cap \{x_1, \ldots, x_r\} = A$. Thus we get $\langle a, u_\ell \rangle = |A| \geq b$, i.e., $\alpha_0(H) \geq b$. Altogether we have $b = \alpha_0(H)$. □

This result has been recently extended to clutters using the notion of parallelization [7]. Let $\mathcal{C}$ be a clutter on the vertex set $X = \{x_1, \ldots, x_n\}$ and let $x_i \in X$. Then *duplicating* x_i means extending X by a new vertex x_i' and replacing $E(\mathcal{C})$ by

$$E(\mathcal{C}) \cup \{(e \setminus \{x_i\}) \cup \{x_i'\} |\, x_i \in e \in E(\mathcal{C})\}.$$

The *deletion* of x_i, denoted by $\mathcal{C} \setminus \{x_i\}$, is the clutter formed from $\mathcal{C}$ by deleting the vertex x_i and all edges containing x_i. A clutter obtained from $\mathcal{C}$ by a sequence of deletions and duplications of vertices is called a *parallelization*. If $w = (w_i)$ is a vector in $\mathbb{N}^n$, we denote by $\mathcal{C}^w$ the clutter obtained from $\mathcal{C}$ by deleting any vertex x_i with $w_i = 0$ and duplicating $w_i - 1$ times any vertex x_i if $w_i \geq 1$. The map $w \mapsto \mathcal{C}^w$ gives a one to one correspondence between $\mathbb{N}^n$ and the parallelizations of $\mathcal{C}$.

Example 2.7. Let G be the graph whose only edge is $\{x_1, x_2\}$ and let $w = (3, 3)$. Then $G^w = \mathcal{K}_{3,3}$ is the complete bipartite graph with bipartition $V_1 = \{x_1, x_1^2, x_1^3\}$ and $V_2 = \{x_2, x_2^2, x_2^3\}$. Notice that x_i^k is a vertex, i.e., k is an index not an exponent.

Proposition 2.8 ([7]). *Let $\mathcal{C}$ be a clutter and let $\Upsilon(\mathcal{C})$ be the blocker of $\mathcal{C}$. If $w = (w_i)$ is an irreducible b-cover of $\Upsilon(\mathcal{C})$, then*

$$b = \min\left\{ \sum_{x_i \in C} w_i \,\middle|\, C \in \Upsilon(\mathcal{C}) \right\} = \alpha_0(\mathcal{C}^w).$$

The next result gives a nice graph theoretical description for the irreducible binary b-vertex covers of the blocker of G.

Theorem 2.9. *Let G be a graph and let $a = (1, \ldots, 1)$. Then a is a reducible $\alpha_0(G)$-cover of $\Upsilon(G)$ if and only if there are H_1 and H_2 induced subgraphs of G such that*

(i) *$V(G)$ is the disjoint union of $V(H_1)$ and $V(H_2)$, and*

(ii) $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$.

Proof. $\Rightarrow$) We may assume that $a_1 = e_1 + \cdots + e_r$, $a_2 = a - a_1$, a_i is a b_i-cover of $\Upsilon(G)$, $b_i \geq 1$ for $i = 1, 2$, and $\alpha_0(G) = b_1 + b_2$. Consider the subgraphs $H_1 = \langle x_1, \ldots, x_r \rangle$ and $H_2 = \langle x_{r+1}, \ldots, x_n \rangle$. Let A be a minimal vertex cover of H_1 with $\alpha_0(H_1)$ vertices. Since

$$C = A \cup (V(G) \setminus \{x_1, \ldots, x_r\})$$

is a vertex cover G, there is a minimal vertex cover C_k of G such that $A \subset C_k \subset C$. Hence

$$|A| = |C_k \cap \{x_1, \ldots, x_r\}| = \langle a_1, u_k \rangle \geq b_1,$$

and $\alpha_0(H_1) \geq b_1$. Using a similar argument we get that $\alpha_0(H_2) \geq b_2$. If C_ℓ is a minimal vertex cover of G with $\alpha_0(G)$ vertices, then $C_\ell \cap V(H_i)$ is a vertex cover of H_i. Therefore

$$b_1 + b_2 = \alpha_0(G) = |C_\ell| = \sum_{i=1}^{2} |C_\ell \cap V(H_i)| \geq \sum_{i=1}^{2} \alpha_0(H_i) \geq b_1 + b_2,$$

and consequently $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$.

$\Leftarrow$) We may assume that $V(H_1) = \{x_1, \ldots, x_r\}$ and $V(H_2) = V(G) \setminus V(H_1)$. Set $a_1 = e_1 + \cdots + e_r$ and $a_2 = a - a_1$. For any minimal vertex cover C_k of G, we have that $C_k \cap V(H_i)$ is a vertex cover of H_i. Hence

$$\langle a_1, u_k \rangle = |C_k \cap \{x_1, \ldots, x_r\}| \geq \alpha_0(H_1),$$

where u_k is the incidence vector of C_k. Consequently a_1 is an $\alpha_0(H_1)$-cover of $\Upsilon(G)$. Similarly we obtain that a_2 is an $\alpha_0(H_2)$-cover of $\Upsilon(G)$. Therefore a is a reducible $\alpha_0(G)$-cover of $\Upsilon(G)$. □

Definition 2.10. A graph satisfying conditions (i) and (ii) is called a *reducible* graph. If G is not reducible, it is called *irreducible.*

These notions appear in [8]. As far as we know there is no structure theorem for irreducible graphs. Examples of irreducible graphs include complete graphs, odd cycles, and complements of odd cycles. Below we give a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G.

By [16, Lemma 5.4] there exists a finite set $\mathcal{H} \subset \mathbb{N}^{n+1}$ such that

(a) $\mathrm{Cn}(I(G)) = \mathbb{R}_+\mathcal{H}$, and

(b) $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{H} = \mathbb{N}\mathcal{H}$,

where $\mathbb{N}\mathcal{H}$ is the additive subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{H}$.

Definition 2.11. The set $\mathcal{H}$ is called a *Hilbert basis* of $\mathrm{Cn}(I(G))$.

If we require $\mathcal{H}$ to be minimal (with respect inclusion), then $\mathcal{H}$ is unique [19].

Corollary 2.12. *Let G be a graph and let $\alpha = (a_1, \ldots, a_n, b)$ be a vector in $\{0,1\}^n \times \mathbb{N}$. Then α is an element of the minimal integral Hilbert basis of $\mathrm{Cn}(I(G))$ if and only if the induced subgraph $H = \langle \{x_i | \, a_i = 1\} \rangle$ is irreducible with $b = \alpha_0(H)$.*

Proof. The map $(a_1, \ldots, a_n, b) \mapsto x_1^{a_1} \cdots x_n^{a_n} t^b$ establishes a one to one correspondence between the minimal integral Hilbert basis of $\mathrm{Cn}(I(G))$ and the minimal generators of $R_s(I(G))$ as a K-algebra. Thus the result follows from Lemma 2.1 and Theorem 2.9. □

The next result shows that irreducible graphs occur naturally in the theory of perfect graphs.

Proposition 2.13. *A graph G is perfect if and only if the only irreducible induced subgraphs of G are the complete subgraphs.*

Proof. $\Rightarrow$) Let H be an irreducible induced subgraph of G. We may assume that $V(H) = \{x_1, \ldots, x_r\}$. Set $a' = (1, \ldots, 1) \in \mathbb{N}^r$ and $a = (a', 0 \ldots, 0) \in \mathbb{N}^n$. By Theorem 2.9, a' is an irreducible $\alpha_0(H)$ cover of $\Upsilon(H)$. Then by Lemma 2.1, a' is an irreducible $\alpha_0(H)$ cover of $\Upsilon(G)$. Since $x_1 \cdots x_r t^{\alpha_0(H)}$ is a minimal generator of $R_s(I(G))$, using Corollary 2.5 we obtain that $\alpha_0(H) = r - 1$ and that H is a complete subgraph of G on r vertices.

$\Leftarrow$) In [4] it is shown that G is a perfect graph if and only if no induced subgraph of G is an odd cycle of length at least five or the complement of one. Since odd cycles and their complements are irreducible subgraphs. It follows that G is perfect. □

Definition 2.14. A graph G is called *vertex critical* if $\alpha_0(G \setminus \{x_i\}) < \alpha_0(G)$ for all $x_i \in V(G)$.

Remark 2.15. If x_i is any vertex of a graph G and $\alpha_0(G \setminus \{x_i\}) < \alpha_0(G)$, then $\alpha_0(G \setminus \{x_i\}) = \alpha_0(G) - 1$

Lemma 2.16. *If the graph G is irreducible, then it is connected and vertex critical*

Proof. Let $G_1,\ldots,G_r$ be the connected components of G. Since $\alpha_0(G)$ is equal to $\sum_i \alpha_0(G_i)$, we get $r=1$. Thus G is connected. To complete the proof it suffices to prove that $\alpha_0(G\setminus\{x_i\})<\alpha_0(G)$ for all i (see Remark 2.15). If $\alpha_0(G\setminus\{x_i\})=\alpha_0(G)$, then $G=H_1\cup H_2$, where $H_1=G\setminus\{x_i\}$ and $V(H_2)=\{x_i\}$, a contradiction. □

Definition 2.17. The *cone* $C(G)$, over a graph G, is obtained by adding a new vertex v to G and joining every vertex of G to v.

The next result can be used to build irreducible graphs. In particular it follows that cones over irreducible graphs are irreducible.

Proposition 2.18. *Let G be a graph with n vertices and let H be a graph obtained from G by adding a new vertex v and some new edges joining v with $V(G)$. If $a=(1,\ldots,1)\in\mathbb{N}^n$ is an irreducible $\alpha_0(G)$-cover of $\Upsilon(G)$ such that $\alpha_0(H)=\alpha_0(G)+1$, then $a'=(a,1)$ is an irreducible $\alpha_0(H)$-cover of $\Upsilon(H)$.*

Proof. Clearly a' is an $\alpha_0(H)$-cover of $\Upsilon(H)$. Assume that $a'=a_1'+a_2'$, where $0\neq a_i'$ is a b_i'-cover of $\Upsilon(H)$ and $b_1'+b_2'=\alpha_0(H)$. We may assume that $a_1'=(1,\ldots,1,0,\ldots,0)$ and $a_2'=(0,\ldots,0,1,\ldots,1)$. Let a_i be the vector in $\mathbb{N}^n$ obtained from a_i' by removing its last entry. Set $v=x_{n+1}$. Take a minimal vertex cover C_k of G and consider $C_k'=C_k\cup\{x_{n+1}\}$. Let u_k' (resp. u_k) be the incidence vector of C_k' (resp. C_k). Then

$$\langle a_1,u_k\rangle=\langle a_1',u_k'\rangle\geq b_1' \text{ and } \langle a_2,u_k\rangle+1=\langle a_2',u_k'\rangle\geq b_2',$$

consequently a_1 is a b_1'-cover of $\Upsilon(G)$. If $b_2'=0$, then a_1 is an $\alpha_0(H)$-cover of $\Upsilon(G)$, a contradiction; because if u is the incidence vector of a minimal vertex cover of G with $\alpha_0(G)$ elements, then we would obtain $\alpha_0(G)\geq\langle u,a_1\rangle\geq\alpha_0(H)$, which is impossible. Thus $b_2'\geq 1$, and a_2 is a $(b_2'-1)$-cover of $\Upsilon(G)$ if $a_2\neq 0$. Hence $a_2=0$, because $a=a_1+a_2$ and a is irreducible. This means that $a_2'=e_{n+1}$ is a b_2'-cover of $\Upsilon(H)$, a contradiction. Therefore a' is an irreducible $\alpha_0(H)$-cover of $\Upsilon(H)$, as required. □

Definition 2.19. A graph G is called *edge critical* if $\alpha_0(G\setminus e)<\alpha_0(G)$ for all $e\in E(G)$.

Proposition 2.20. *If G is a connected edge critical graph, then G is irreducible.*

Proof. Assume that G is reducible. Then there are induced subgraphs H_1, H_2 of G such that $V(H_1)$, $V(H_2)$ is a partition of $V(G)$ and $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$. Since G is connected there is an edge $e = \{x_i, x_j\}$ with x_i a vertex of H_1 and x_j a vertex of H_2. Pick a minimal vertex cover C of $G\setminus e$ with $\alpha_0(G)-1$ vertices. As $E(H_i)$ is a subset of $E(G\setminus e) = E(G)\setminus\{e\}$ for $i = 1, 2$, we get that C covers all edges of H_i for $i = 1, 2$. Hence C must have at least $\alpha_0(G)$ elements, a contradiction. □

Corollary 2.21. *The following hold for any connected graph*:

$$\textit{edge critical} \implies \textit{irreducible} \implies \textit{vertex critical.}$$

Finding generators of symbolic Rees algebras using cones The *cone* $C(G)$, over the graph G, is obtained by adding a new vertex t to G and joining every vertex of G to t.

Example 2.22. A pentagon and its cone:

In [1] Bahiano showed that if $H = C(G)$ is the graph obtained by taking a cone over a pentagon G with vertices $x_1, \ldots, x_5$, then

$$R_s(I(H)) = R[I(H)t][x_1 \cdots x_5 t^3, x_1 \cdots x_6 t^4, x_1 \cdots x_5 x_6^2 t^5].$$

This simple example shows that taking a cone over an irreducible graph tends to increase the degree in t of the generators of the symbolic Rees algebra. Other examples using this "cone process" have been shown in [14, Example 5.5].

Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. The aim here is to give a general procedure—based on the irreducible representation of the Rees cone of $I_c(G)$—to construct generators of $R_s(I(H))$ of high degree in t, where H is a graph constructed from G by recursively taking cones over graphs already constructed.

By the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I_c(G)) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\alpha_1}^+ \cap H_{\alpha_2}^+ \cap \cdots \cap H_{\alpha_p}^+ \qquad (6)$$

such that each α_k is in $\mathbb{Z}^{n+1}$, the non-zero entries of each α_k are relatively prime, and none of the closed halfspaces $H^+_{e_1}, \ldots, H^+_{e_{n+1}}, H^+_{\alpha_1}, \ldots, H^+_{\alpha_p}$ can be omitted from the intersection. For use below we assume that α is any of the vectors $\alpha_1, \ldots, \alpha_p$ that occur in the irreducible representation. Thus we can write $\alpha = (a_1, \ldots, a_n, -b)$ for some $a_i \in \mathbb{N}$ and for some $b \in \mathbb{N}$.

Lemma 2.23. *Let H be the cone over G. If*

$$\beta = (a_1, \ldots, a_n, (\textstyle\sum_{i=1}^n a_i) - b, -\sum_{i=1}^n a_i) = (\beta_1, \ldots, \beta_{n+1}, -\beta_{n+2})$$

and $a_i \geq 1$ for all i, then $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$.

Proof. First we prove that $\mathbb{R}_+(I_c(H)) \subset H^+_\beta$, i.e., H_β is a supporting hyperplane of the Rees cone. By Lemma 1.8, $(a_1, \ldots, a_n)$ is an irreducible b-cover of $\Upsilon(G)$. Hence there is $C \in \Upsilon(G)$ such that $\sum_{x_i \in C} a_i = b$. Therefore β_{n+1} is greater or equal than 1. This proves that $e_1, \ldots, e_{n+1}$ are in H^+_β. Let C be any minimal vertex cover of H and let $u = \sum_{x_i \in C} e_i$ be its characteristic vector. Case (i): If $x_{n+1} \notin C$, then $C = \{x_1, \ldots, x_n\}$ and

$$\sum_{x_i \in C} \beta_i = \sum_{i=1}^n a_i = \beta_{n+2},$$

that is, $(u, 1) \in H^+_\beta$. Case (ii): If $x_{n+1} \in C$, then $C_1 = C \setminus \{x_{n+1}\}$ is a minimal vertex cover of G. Hence

$$\sum_{x_i \in C} \beta_i = \sum_{x_i \in C_1} \beta_i + \beta_{n+1} \geq b + \beta_{n+1} = \beta_{n+2},$$

that is, $(u, 1) \in H^+_\beta$. Therefore $\mathbb{R}_+(I_c(H)) \subset H^+_\beta$. To prove that F is a facet we must show it has dimension $n+1$ because the dimension of $\mathbb{R}_+(I_c(H))$ is $n+2$. We denote the characteristic vector of a minimal vertex cover C_k of G by u_k. By hypothesis there are minimal vertex covers $C_1, \ldots, C_n$ of G such that the vectors $(u_1, 1), \ldots, (u_n, 1)$ are linearly independent and

$$\langle (a, -b), (u_k, 1) \rangle = 0 \iff \langle a, u_k \rangle = b, \tag{7}$$

for $k = 1, \ldots, n$. Therefore

$$\langle (\beta_1, \ldots, \beta_{n+1}), (u_k, 1) \rangle = \beta_{n+2} \quad \text{and} \quad \langle (\beta_1, \ldots, \beta_{n+1}), (1, \ldots, 1, 0) \rangle = \beta_{n+2},$$

i.e., the set $\mathcal{B} = \{(u_1, 1), \ldots, (u_n, 1), (1, \ldots, 1, 0\}$ is contained in H_β. Since

$$C_1 \cup \{x_{n+1}\}, \ldots, C_n \cup \{x_{n+1}\}, \{x_1, \ldots, x_n\}$$

are minimal vertex covers of H, the set $\mathcal{B}$ is also contained in $\mathbb{R}_+(I_c(H))$ and consequently in F. Thus its suffices to prove that $\mathcal{B}$ is linearly independent. If $(1,\ldots,1,0)$ is a linear combination of $(u_1,1),\ldots,(u_n,1)$, then we can write

$$(1,\ldots,1) = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

for some scalars $\lambda_1,\ldots,\lambda_n$ such that $\sum_{i=1}^n \lambda_i = 0$. Hence from Eq. (7) we get

$$|a| = \langle (1,\ldots,1), a\rangle = \lambda_1\langle u_1, a\rangle + \cdots + \lambda_n\langle u_n, a\rangle = (\lambda_1 + \cdots + \lambda_n)b = 0,$$

a contradiction. □

Corollary 2.24. *If $a_i \geq 1$ for all i, then $x_1^{\beta_1}\cdots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$.*

Proof. By Lemma 2.23, $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$. Therefore using Lemma 1.8, the vector $(\beta_1,\ldots,\beta_{n+1})$ is an irreducible β_{n+2}-cover of $\Upsilon(H)$, i.e., $x_1^{\beta_1}\cdots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$. □

Corollary 2.25. *Let $G_0 = G$ and let G_r be the cone over G_{r-1} for $r \geq 1$. If $\alpha = (1,\ldots,1,-g)$, then*

$$(\underbrace{1,\ldots,1}_{n},\underbrace{n-g,\ldots,n-g}_{r})$$

is an irreducible $n + (r-1)(n-g)$ cover of G_r. In particular $R_s(I(G_r))$ has a generator of degree in t equal to $n + (r-1)(n-g)$.

As a very particular example of our construction consider:

Example 2.26. Let $G = C_s$ be an odd cycle of length $s = 2k+1$. Note that $\alpha_0(C_s) = (s+1)/2 = k+1$. Then by Corollary 2.25

$$x_1\cdots x_s x_{s+1}^k \cdots x_{s+r}^k t^{rk+k+1}$$

is a minimal generator of $R_s(I(G_r))$. This illustrates that the degree in t of the minimal generators of $R_s(I(G_r))$ is much larger than the number of vertices of the graph G_r [14].

References

[1] C. Bahiano, Symbolic powers of edge ideals, J. Algebra **273 (2)** (2004), 517-537.

[2] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics **184** Springer-Verlag, New York, 1998.

[3] W. Bruns and B. Ichim, Normaliz 2.0, Computing normalizations of affine semigroups 2008. Available from `http://www.math.uos.de/normaliz`.

[4] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, Ann. of Math. **(2) 164** (2006), no. 1, 51–229.

[5] G. Cornuéjols, *Combinatorial optimization: Packing and covering*, CBMS-NSF Regional Conference Series in Applied Mathematics **74**, SIAM (2001).

[6] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics **173**, Springer-Verlag, New York, 2nd ed., 2000.

[7] L. A. Dupont, E. Reyes and R. H. Villarreal, Cohen-Macaulay clutters with combinatorial optimization properties and parallelizations of normal edge ideals, São Paulo J. Math. Sci. **3** (2009), no. 1, 61–75.

[8] P. Erdös and T. Gallai, On the minimal number of vertices representing the edges of a graph, Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 181–203.

[9] C. Escobar, R. H. Villarreal and Y. Yoshino, Torsion freeness and normality of blowup rings of monomial ideals, *Commutative Algebra*, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 69-84.

[10] C. A. Francisco, H. T. Hà and A. Van Tuyl, Associated primes of monomial ideals and odd holes in graphs, J. Algebraic Combin. **32** (2010), no. 2, 287-301.

[11] I. Gitler, E. Reyes and R. H. Villarreal, Blowup algebras of ideals of vertex covers of bipartite graphs, Contemp. Math. **376** (2005), 273–279.

[12] I. Gitler, E. Reyes and R. H. Villarreal, Blowup algebras of square–free monomial ideals and some links to combinatorial optimization problems, Rocky Mountain J. Math. **39** (2009), no. 1, 71–102.

[13] I. Gitler, C. Valencia and R. H. Villarreal, A note on Rees algebras and the MFMC property, Beiträge Algebra Geom. **48** (2007), No. 1, 141-150.

[14] J. Herzog, T. Hibi and N. V. Trung, Symbolic powers of monomial ideals and vertex cover algebras, Adv. Math. **210** (2007), 304–322.

[15] J. Herzog, T. Hibi and N. V. Trung, Vertex cover algebras of unimodular hypergraphs, Proc. Amer. Math. Soc. **137** (2009), 409-414.

[16] B. Korte and J. Vygen, *Combinatorial Optimization Theory and Algorithms*, Springer-Verlag, 2000.

[17] G. Lyubeznik, On the arithmetical rank of monomial ideals, J. Algebra **112** (1988), 86–89.

[18] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986.

[19] A. Schrijver, On total dual integrality, Linear Algebra Appl. **38** (1981), 27–32.

[20] A. Schrijver,*Combinatorial Optimization*, Algorithms and Combinatorics **24**, Springer-Verlag, Berlin, 2003.

[21] A. Simis, W. V. Vasconcelos and R. H. Villarreal, On the ideal theory of graphs, J. Algebra, **167** (1994), 389–416.

[22] S. Sullivant, Combinatorial symbolic powers, J. Algebra **319(1)** (2008), 115-142.

[23] C. Valencia and R. H. Villarreal, Canonical modules of certain edge subrings, European J. Combin. **24(5)** (2003), 471–487.

[24] C. Valencia and R. H. Villarreal, Explicit representations of the edge cone of a graph, Int. Journal of Contemp. Math. Sciences **1** (2006), no. 1–4, 53–66.

[25] W. V. Vasconcelos, *Arithmetic of Blowup Algebras*, London Math. Soc., Lecture Note Series **195**, Cambridge University Press, Cambridge, 1994.

[26] R. H. Villarreal, On the equations of the edge cone of a graph and some applications, Manuscripta Math. **97** (1998), 309–317.

[27] R. H. Villarreal, *Monomial Algebras*, Dekker, New York, N.Y., 2001.

[28] R. H. Villarreal, Rees algebras and polyhedral cones of ideals of vertex covers of perfect graphs, J. Algebraic Combin. **27(3)** (2008), 293-305.

[29] R. H. Villarreal, Rees cones and monomial rings of matroids, Linear Algebra Appl. **428** (2008), 2933-2940.

[30] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.

Contact information

L. A. Dupont
Departamento de Matemáticas
Centro de Investigación y de Estudios
Avanzados del IPN
Apartado Postal 14–740
07000 Mexico City, D.F.
E-Mail: lduponte@math.cinvestav.mx
URL: www.math.cinvestav.mx

R. H. Villarreal
Departamento de Matemáticas
Centro de Investigación y de Estudios
Avanzados del IPN
Apartado Postal 14–740
07000 Mexico City, D.F.
E-Mail: vila@math.cinvestav.mx
URL: www.math.cinvestav.mx

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