2-Galois groups and the Kaplansky radical

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ABSTRACT. An accurate description of the Galois group $G_F(2)$ of the maximal Galois 2-extension of a field F may be given for fields F admitting a 2-henselian valuation ring. In this note we generalize this result by characterizing the fields for which $G_F(2)$ decomposes as a free pro-2 product $\mathcal{F} * \mathcal{H}$ where $\mathcal{F}$ is a free closed subgroup of $G_F(2)$ and $\mathcal{H}$ is the Galois group of a 2-henselian extension of F. The free product decomposition of $G_F(2)$ is equivalent to the existence of a valuation ring compatible with the Kaplansky radical of F. Fields with Kaplansky radical fulfilling prescribed conditions are constructed, as an application.

Introduction

A field F is said to be 2-*henselian* in case F admits a valuation ring A which extends uniquely to the quadratic closure $F(2)$ of F ($F(2)$ is the inductive limit of the direct system of all Galois extensions of F of finite degree a power of 2 within a fixed algebraically closed field). Henselianity depends on A and it is more appropriate to say that the pair (F, A) is a 2-*henselian valued field*. For a 2-henselian valued field (F, A) the Galois group $G_F(2) = \mathrm{Gal}(F(2); F)$ is well-known, in case the residue class field of A has characteristic $\neq 2$, [6, Proposition 1.1]. Additionally, if the value group of A is not 2-divisible, then 2-henselian valued fields can be characterized as those fields for which $G_F(2)$ has a sufficiently large

abelian normal subgroup, [6, Theorem 4.3]. A similar statement holds for absolute henselianity, as it is shown in [12].

This note concerns to the description of $G_F(2)$ for valued fields (F, A) which one may consider "almost" 2-henselian as we shall describe now. For a valued field (F, A), where A has residue class field of characteristic $\neq 2$, it is well-known that A is 2-henselian if and only if $1 + m_A \subset (F^\times)^2$, where m_A is the maximal ideal of A and $(F^\times)^2$ is the subgroup consisting of non-zero squares of the multiplicative group $F^\times = F \smallsetminus \{0\}$. To define a *pre-2-henselian* valuation ring we substitute $(F^\times)^2$ by the Kaplansky radical $R(F)$ of F (see Section 1). Thus we call a valuation ring A of F *pre-2-henselian* if the residue class field k_A of A has characteristic $\neq 2$ and $1 + m_A \subset R(F)$.

In the trivial case $R(F) = (F^\times)^2$, we get a 2-henselian valuation ring. In general, take an extension $A(2)$ of A to $F(2)$ and write F_h for the decomposition field of $A(2)$ over F [5, §15]. Set $A_h = A(2) \cap F_h$. It follows from [5, 15.7] that (F_h, A_h) is a 2-henselian valued field. Moreover, since we are assuming that k_A has characteristic $\neq 2$, Theorem 15.7 of [5] implies that (F_h, A_h) is the smallest 2-henselian valued extension of (F, A) for which $A(2)$ lies over A_h. The pair (F_h, A_h) is then called a 2-*henselization* of (F, A). A pre-2-henselian valued field (F, A) has many significant properties determined by the corresponding properties of (F_h, A_h). Notably, properties concerning quadratic forms, quaternion algebras, Brauer groups, and orderings. The strong connection between (F, A) and (F_h, A_h) can be characterized by the decomposition $G_F(2) = \mathcal{F} * G_{F_h}(2)$ as a free pro-2 product of $G_{F_h}(2)$ and a closed free subgroup of $G_F(2)$ (see theorems 4.2 and 4.7). Moreover, A can be chosen in a natural way which determines uniquely $G_{F_h}(2)$ (up to inner automorphisms of $G_F(2)$). The rank of $\mathcal{F}$ is equal to the rank of the quotient group $(1 + m_A)(F^\times)^2/(F^\times)^2$, which equals the rank of $R(F)/(F^\times)^2$ in most cases.

In the next section we summarize significant properties of $R(F)$ which are necessary for the paper. The pre-2-henselian properties connecting the objects $G_F(2)$, $R(F)$, and the valuation ring A, are the subject of Section 2. There we also show how to construct pre-2-henselian valuation rings. In Section 3 the connection between the Brauer groups of F and F_h, a 2-henselization of F, is established. We derive from this result a characterization of pre-2-henselian fields by means of its Brauer group which will be necessary to prove, in Section 4, the main results of the paper. In the last section, as an application of theorems 4.2 and 4.7, we show how to produce examples of pre-2-henselian fields whose radical satisfies prescribed conditions. For instance, for any given pair of positive integers m and n, $n > 1$, there exists a pre-2-henselian field F such that

$F^\times/R(F)$ and $R(F)/(F^\times)^2$ have respectively rank m and n. Examples constructed in this way cover previous examples constructed by Berman [2] and Kula [13].

We finish the introduction remarking that *all fields considered have characteristic $\neq 2$*, (char $\neq 2$) and all subgroups of profinite groups are closed. For quadratic forms we use the standard terminology as is found in [14]. For instance given a quadratic form $\langle a_1, \ldots, a_n \rangle$ over a field F let $D_F\langle a_1, \ldots, a_n \rangle$ be the set of non-zero elements of F represented by $\langle a_1, \ldots, a_n \rangle$.

1.   On Kaplansky radical of a field

In a paper from 1969, Kaplansky [10] introduced the radical $R(F) = \{a \in F^\times \mid 1 \in D_F\langle a, b \rangle$ for every $b \in F^\times\}$ of a field F. Cordes [3] called $R(F)$ Kaplansky radical of F and rewrite the definition as

$$R(F) = \bigcap_{a \in F^\times} D_F\langle 1, a \rangle.$$

Alternatively, $x \in R(F)$ if and only if $D_F\langle 1, -x \rangle = F^\times$. Clearly $R(F)$ is a subgroup of $F^\times$ containing $(F^\times)^2$. Some further significant properties of $R(F)$ are:

1.1. *$R(F)$ is additively closed if and only if $R(F) = D_F\langle 1, 1 \rangle = \sum(F^\times)^2$ is the set of all finite sums of squares of $F^\times$, [14, Corollary 6.5(2), p 452].*

Fields for which $R(F)$ is additively closed are classified in two types:

1.2. *If F is non-real, i.e., $-1 \in \sum(F^\times)^2$, then $R(F) = F^\times$.*

1.3. *In the formally real case, $R(F)$ is a preordering (preorderings can be found in [14, p 289]).*

1.4. *$(F^\times : R(F)) = 2$ if and only if F is formally real, uniquely ordered by $R(F)$, and there exists a unique, up to isomorphism, quaternion division algebra over F, [14, Theorem 6.10, p 454]. In this case Lam [14, Definition 6.7, p 453] called F a pre-Hilbertian field.*

2.   Pre-2-henselianity

Fields admitting a proper pre-2-henselian valuation ring will be obtained by applying to the subgroup $R(F)$ the methods developed in [1]. To this end, let us first define $R(F)$-*rigid* elements as being those $a \in F^\times \smallsetminus \pm R(F)$ for which $\{x + ay \neq 0 \mid x, y \in R(F) \cup \{0\}\} = R(F) \cup aR(F)$. If x and

$-x$ are both $R(F)$-rigid, then we say that x (and $-x$ too) is $R(F)$-*birigid*. Non-$R(F)$-birigid elements also play an important role in constructing pre-2-henselian valuation rings. Denote $B(R(F)) = \{x \in F^\times \mid x$ is non-R(F)-birigid $\}$. The elements of $B(R(F))$ are called *basics*. A surprisingly fact is that $B(R(F))$ is a subgroup of $F^\times$. In the next result we formalize properties of $R(F)$-rigid and basic elements which will be crucial to establish the existence of a pre-2-henselian valuation ring.

Lemma 2.1. *Let F be a field and $a \in F^\times \smallsetminus \pm R(F)$. Then:*

(1) *a is $R(F)$-rigid if and only if $D_F\langle 1, a \rangle = R(F) \cup aR(F)$.*

(2) *$B(R(F))$ is a subgroup of $F^\times$.*

Proof. For a quadratic form $\langle 1, a \rangle$ it follows from [3, Proposition 1] that $D_F\langle 1, a \rangle = \{x + ay \neq 0 \mid x, y \in R(F) \cup \{0\}\}$; the equality implies (1). Also it implies that $\{x + ay \neq 0 \mid x, y \in R(F) \cup \{0\}\}$ is a subgroup of $F^\times$. Therefore, by [19, Proposition 2.4], $B(R(F))$ is a subgroup of $F^\times$.   □

In the trivial case $R(F) = (F^\times)^2$ we get the usual rigid elements (or $(F^\times)^2$-rigid). We know that 2-henselian valuation rings arise from the existence of enough rigid elements (see [1] or [19]). Pre-2-henselian valuation rings are obtained by replacing rigid elements by $R(F)$-rigid elements in the usual procedures.

Theorem 2.2. *Let F be a field such that $B(R(F)) \neq F^\times$. Then at least one of the following conditions holds:*

(1) *F admits a proper pre-2-henselian valuation ring A with non-2-divisible value group Γ_A.*

(2) *F is formally real with at most two orderings and $R(F) = \sum(F^\times)^2$. Consequently $(F^\times : R(F)) \leq 4$.*

The proof of Theorem 2.2 is partially based on Theorem 3.9 from [1] which requires the characterization of the "exceptional case," according to the designation given by Arason et al. [1, Definition 2.15].

We say that $R(F)$ is *exceptional* if $B(R(F)) = R(F) \cup -R(F)$ and either $-1 \in R(F)$ or $R(F)$ is additively closed.

If $R(F) + R(F) \subset R(F)$ and F is formally real we know from (1.3) that $R(F)$ is a preordering. If additionally $R(F)$ is exceptional, then $R(F)$ will possess a new interesting property, namely $R(F)$ will be a fan. See [15, §5] for more about fans.

In the non-real case, if $R(F)$ is exceptional, then $B(R(F)) = R(F)$. Thus assertion (1.4) implies either $B(R(F)) = F^\times$ or $(F^\times : B(R(F))) > 2$. For further reference let us remark the above two cases:

Lemma 2.3. *Assume $R(F) \neq F^\times$ is exceptional.*

(1) *If F is non-real, then $(F^\times : B(R(F))) > 2$.*

(2) *If F is formally real, then $R(F)$ is a fan.*

The basic connection between $R(F)$ and valuation rings of F is given in the next proposition. For a valuation ring A of F, let $A^\times = A \smallsetminus m_A$, $\pi_A : A \to k_A$, Γ_A, and v_A be respectively the group of units, the canonical homomorphism, the value group and a valuation corresponding to A.

Proposition 2.4. *For a field F with radical $R(F) \neq F^\times$ let A be a valuation ring of F.*

(1) $\pi_A(R(F) \cap A^\times) \subset R(k_A)$.

(2) *If A is pre-2-henselian and $\Gamma_A = 2\Gamma_A$, then $\pi_A(R(F) \cap A^\times) = R(k_A)$. Additionally $R(k_A) \neq k_A^\times$.*

(3) *If $R(F) \not\subset (1 + m_A)(F^\times)^2$, then $(\Gamma_A : 2\Gamma_A) \leq 2$. Moreover, $(\Gamma_A : 2\Gamma_A) = 2$ implies $k_A^\times = (k_A^\times)^2$.*

(4) *If $(1 + m_A)(F^\times)^2 \subsetneqq R(F)$, then $\Gamma_A = 2\Gamma_A$.*

Proof. (1) For $u \in (1 + m_A)(F^\times)^2 \cap A^\times$, $\pi_A(u) \in (k_A^\times)^2 \subset R(k_A)$. Take then $u \in R(F) \cap A^\times$ and $u \notin (1 + m_A)(F^\times)^2$. For any $z \in A^\times$ there are $a, b \in (F^\times)^2$ such that $z = a - ub$ (since $u \in R(F)$). From $v_A(a - ub) = 0$ one gets $v_A(a), v_A(ub) \geq 0$. Otherwise, $v_A(a) < 0$ or $v_A(ub) = v_A(b) < 0$ would imply $v_A(a) = v_A(b)$ (< 0) [1]. Then $u = (b^{-1}a)(1 - za^{-1}) \in (1 + m_A)(F^\times)^2$ contradicting the choice of u. Hence $\pi_A(z) = \pi_A(a) - \pi_A(u)\pi_A(b) \in D_{k_A}\langle 1, -\pi_A(u)\rangle$. Thus $k_A^\times = D_{k_A}\langle 1, -\pi_A(u)\rangle$ which implies $\pi_A(u) \in R(k_A)$, as required.

(2) Due to the previous item (1) it remains to show the inclusion $R(k_A) \subset \pi_A(R(F) \cap A^\times)$ to prove the equality. For $u \in A^\times$ such that $\pi_A(u) \in R(k_A)$ we have that $D_{k_A}\langle 1, -\pi_A(u)\rangle = k_A^\times$. Therefore given $z \in A^\times$ there exist $a, b \in (A^\times)^2$ such that $\pi_A(z) = \pi_A(a) - \pi_A(u)\pi_A(b)$. Thus $z(a - ub)^{-1} \in 1 + m_A \subset R(F)$. Since $R(F) \subset D_F\langle 1, -u\rangle$ it follows that $z \in D_F\langle 1, -u\rangle$. Hence $A^\times \subset D_F\langle 1, -u\rangle$. The assumption $\Gamma_A = 2\Gamma_A$ will then imply $F^\times = A^\times(F^\times)^2 \subset D_F\langle 1, -u\rangle$. Consequently $u \in R(F)$ and $\pi_A(u) \in \pi_A(R(F) \cap A^\times)$, as required.

Finally, $R(k_A) = k_A^\times$ would imply $\pi_A(R(F) \cap A^\times) = \pi_A(A^\times)$ and so $A^\times \subset R(F)$. Thus $F^\times = A^\times(F^\times)^2 = R(F)$, a contradiction.

(3) Take $r \in R(F) \smallsetminus (1 + m_A)(F^\times)^2$ and $a, b \in (F^\times)^2$. If $v_A(a) = v_A(rb) < v_A(a - rb)$, then $v_A(1 - rba^{-1}) > 0$. Consequently $rba^{-1} \in 1 + m_A$,

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[1] $v_A(a) \neq v_A(rb)$ implies $v_A(a - rb) = \min\{v_A(a), v_A(rb)\}$

a contradiction. Thus $v_A(a-rb) = \min\{v_A(a), v_A(rb)\}$, for all $a, b \in (F^\times)^2$. As $F^\times = D_F\langle 1, -r\rangle$, $v_A(x) \in 2\Gamma_A \cup (v(r) + 2\Gamma_A)$, for every $x \in F^\times$. Thus

$$\Gamma_A \subset 2\Gamma_A \cup (v(r) + 2\Gamma_A) \tag{1}$$

and so $(\Gamma_A : 2\Gamma_A) \leq 2$ as desired.

We next prove that $(\Gamma_A : 2\Gamma_A) = 2$ implies $k_A^\times = (k_A^\times)^2$. In fact, for $r \in R(F) \smallsetminus (1 + m_A)(F^\times)^2$, as above, $v_A(r) \notin 2\Gamma_A$ by equation (1). Moreover, for $y \in A^\times$ and $a, b \in (F^\times)^2$ satisfying $y = a - rb$ we have seen that $0 = v_A(y) = \min\{v_A(a), v_A(rb)\}$. The case $v_A(rb) = 0$ cannot occur, otherwise $v_A(r) = -v_A(b) \in 2\Gamma_A$. Consequently $v_A(rb) > 0$ and $v_A(a) = 0$. Hence $a \in (A^\times)^2$ and $\pi_A(y) = \pi_A(a) \in (k_A^\times)^2$, as required.

(4) Assuming $\Gamma_A \neq 2\Gamma_A$, it follows from the previous item (3) that $(\Gamma_A : 2\Gamma_A) = 2$ and $k_A^\times = (k_A^\times)^2$. The last equality implies $A^\times = (1 + m_A)(A^\times)^2$. Thus $A^\times \subset R(F)$ (†).

From $(\Gamma_A : 2\Gamma_A) = 2$ and equation (1) we can write $F^\times = A^\times(F^\times)^2 \cup rA^\times(F^\times)^2$ (‡), for any $r \in R(F) \smallsetminus (1 + m_A)(F^\times)^2$.

Combining (†) and (‡) we get a contradiction. □

Corollary 2.5. *Keep the notation of Proposition 2.4 and let A be a pre-2-henselian valuation ring of F. If $\Gamma_A \neq 2\Gamma_A$, then $(1 + m_A)(F^\times)^2 = R(F)$.*

We now present the proof of Theorem 2.2.

Proof of Theorem 2.2. We shall consider two cases. The first case corresponds to $R(F)$ non-exceptional or $R(F)$ exceptional and F is non-real. The other case corresponds to $R(F)$ exceptional and F is formally real.

In the first case, [1, Theorem 3.9] guarantees the existence of a valuation ring A, pre-2-henselian, such that $A^\times R(F) = B(R(F))$ if $R(F)$ is not exceptional and $(A^\times R(F) : B(R(F))) \leq 2$, otherwise. Particularly, if $R(F)$ is exceptional and F is non-real, then $(F^\times : A^\times R(F)) \geq 2$, by Lemma 2.3 (1). Therefore, in both cases, $F^\times \neq A^\times R(F)$ and A is a proper subring of F. Moreover, since $A^\times(F^\times)^2 \subset A^\times R(F)$ we also get $F^\times \neq A^\times(F^\times)^2$ and so $\Gamma_A \neq 2\Gamma_A$.

To complete the proof in this case it remains to be seen that char $k_A \neq 2$, i.e., the first case implies (1) of Theorem 2.2. By [1, Lemma 4.4], char $k_A \neq 2$ follows from $(F^\times : R(F)) > 2$. In the non-exceptional case, as $R(F) \subsetneqq B(R(F))$ and $B(R(F)) \neq F^\times$ (by assumption), $(F^\times : R(F)) > 2$ holds. In the non-real exceptional case, Lemma 2.3 (1) implies $(F^\times : R(F)) > 2$ because $R(F) \subset B(R(F))$.

Assume finally that $R(F)$ is exceptional and F is formally real (i.e., $B(R(F)) = \pm R(F)$, $-1 \notin R(F)$, and $R(F) + R(F) \subset R(F)$). Then Lemma 2.3 (2) shows that $R(F)$ is a fan. By Bröcker's Trivialization

Theorem [15, Theorem 5.13], F admits a valuation ring A which is pre-2-henselian and such that $\pi_A(R(F))$ is a trivial fan, i.e., a preordering such that $(k_A^\times : \pi_A(R(F))) \leq 4$. Since k_A is a formally real field we get char $k_A \neq 2$.

If $\Gamma_A \neq 2\Gamma_A$ we get again the conclusion (1) of Theorem 2.2.

Next, we assume that Γ_A is 2-divisible and prove that the conclusion (2) of Theorem 2.2 holds. For $\Gamma_A = 2\Gamma_A$, Proposition 2.4 (2) implies $R(k_A) = \pi_A(R(F))$, which we know to be additively closed. Applying now assertion (1.1) to both radicals we get $R(F) = \sum(F^\times)^2$ and $R(k_A) = \sum(k_A^\times)^2$. The last equality, $R(k_A) = \sum(k_A^\times)^2$, combined with $(k_A : R(k_A)) \leq 4$ implies that k_A has at most two orderings. On the other hand, the equality $R(F) = \sum(F^\times)^2$ implies that A is compatible with every ordering of F [15, Definition 2.4]. Consequently, by [15, Corollary 3.11], the orderings of F correspond bijectively with the orderings of k_A if we keep in mind that $\Gamma_A = 2\Gamma_A$. Hence F has at most two orderings and $(F^\times : R(F)) \leq 4$. □

In case (2) of Theorem 2.2, $R(F) = \sum(F^\times)^2$ is a trivial fan and fields for which $\sum(F^\times)^2$ is a trivial fan are well-know. Moreover, such a field F may or may not admit a pre-2-henselian valuation ring with non-2-divisible value group. For instance, if F admits an Archimedean ordering, then no valuation ring could be pre-2-henselian. On the other hand, the existence of a pre-2-henselian valuation ring with non-2-divisible value group can only occur if $(\Gamma_A : 2\Gamma_A) = 2$ and k_A is uniquely ordered, by [15, Corollary 3.11]. The ordering of k_A is given by $R(k_A)$ and so k_A is a pre-Hilbertian field (see (1.4)).

We shall now investigate the relations between various pre-2-henselian valuation rings of a field. Like for 2-henselian valuation rings we shall see for a field $F \neq F(2)$ that any two pre-2-henselian valuation rings C and D are comparable provided that k_C or k_D is non-quadratically closed[2]. Moreover if exactly one of them, say k_D, is quadratically closed, then $D \subset C$. In order to describe the set of all pre-2-henselian valuation rings of F we set up some notations.

We denote by Ω_1 the set of all pre-2-henselian valuation rings of F with non-quadratically closed residue class field and by Ω_2 the set of all valuation rings having residue class field quadratically closed. Set $\Omega = \Omega_1 \cup \Omega_2$. Before we describe the properties of Ω_1 and Ω_2 let us recall, for any two valuation rings C and D, that C is said to be *coarser* than D (or D is said to be *finer* than C) if $D \subset C$.

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[2]Since we are assuming char $\neq 2$ for every field there is no distinction between quadratically closed and quadratic separably closed.

Proposition 2.6. *Keep notations as above and assume that $R(F) \neq F^\times$.*

(1) *If a valuation ring C of F is coarser than $A \in \Omega$, then $C \in \Omega$.*

(2) *Ω_1 is totally ordered (by inclusion) and closed by the coarsening relation.*

(3) *If $A \in \Omega_2$ is coarser than a valuation ring C of F, then $C \in \Omega_2$. Moreover, for $C, D \in \Omega_2$, their compositum $C \cdot D$ lies in Ω_2 as well.*

(4) *For any pair $C \in \Omega_2$ and $A \in \Omega_1$, A is coarser than C.*

(5) *If $\Omega_2 \neq \emptyset$, then for every $A \in \Omega_2$, $\Gamma_A \neq 2\Gamma_A$ and $(1 + m_A)(F^\times)^2 = R(F)$.*

We need the following two lemmas to prove the proposition.

Lemma 2.7. *For a field F let T be a subgroup of $F^\times$ and let A and C be independent valuation rings of F, i.e., $A \cdot C = F$. If $1 + m_A \subset T$ and $1 + m_C \subset T$, then $T = F^\times$.*

Proof. By the Approximation Theorem [7, Theorem 2.4.1] for every $x \in F^\times$ there exists $y \in F^\times$ such that $v_C(y - x) > v_C(x)$ and $v_D(y - 1) > 0$. Thus $yx^{-1} \in 1 + m_C$ and $y \in 1 + m_D$. Hence $x = y(yx^{-1})^{-1} \in T$. □

Lemma 2.8. *Let A and C be incomparable (by inclusion) pre-2-henselian valuation rings of a field F. Write $D = A \cdot C$, for their compositum and assume that $R(F) \neq F^\times$. Then A, C, $D \in \Omega_2$.*

Proof. From $A \subset D$ it follows that $m_D \subset m_A$. Thus D is also pre-2-henselian. Going to residue class field of D we set $\overline{R(F)} = \pi_D(R(F) \cap D^\times)$, $\overline{A} = \pi_D(A)$, and $\overline{C} = \pi_D(C)$.

Clearly $1 + m_{\overline{A}} \subset \overline{R(F)}$ and $1 + m_{\overline{C}} \subset \overline{R(F)}$ hold. On the other hand $\overline{A}$ and $\overline{C}$ are independent valuation rings of k_D, by construction. Applying Lemma 2.7 to them one gets $\overline{R(F)} = k_D^\times$. By Proposition 2.4 (1) $\overline{R(F)} \subset R(k_D)$. Therefore $R(k_D) = k_D^\times$.

On the other hand, under the assumption $(1 + m_D)(F^\times)^2 \subsetneq R(F)$ we would get from Proposition 2.4 (4) that $\Gamma_D = 2\Gamma_D$. But this would contradict item (2) of Proposition 2.4. Therefore $(1 + m_D)(F^\times)^2 = R(F)$. Applying now the conclusions of the previous paragraph we get $k_D^\times = \overline{R(F)} = (k_D^\times)^2$. Thus $D \in \Omega_2$.

Finally since A and $\overline{A}$ have the same residue class field, also k_A is quadratically closed. Hence $A \in \Omega_2$ follows from $A \subset D$. Similarly $C \in \Omega_2$. □

We are now ready to prove Proposition 2.6.

Proof of Proposition 2.6. (1) and the first part of (3) are consequences of general valuation theory; (2), the last part of (3), and (4) follow directly from the above Lemma 2.8. To prove (5), observe first that if $A \in \Omega_2$ then $A^\times = (1 + m_A)(A^\times)^2$. Under the condition $\Gamma_A = 2\Gamma_A$, it follows that $F^\times = A^\times (F^\times)^2 = (1 + m_A)(F^\times)^2 \subset R(F) \subset F^\times$. Thus $F^\times = R(F)$, contrary to the assumption. Hence $\Gamma_A \neq 2\Gamma_A$, for every $A \in \Omega_2$. Finally, by Corollary 2.5, $(1 + m_A)(F^\times)^2 = R(F)$.   □

Observe that $F \in \Omega$ is always true and so $\Omega \neq \emptyset$. For $F \neq F(2)$ we cannot say $\Omega_2 \neq \emptyset$, however. In Section 4 we shall see that the existence of $A \in \Omega$, $A \neq F$, is the main requirement to describe $G_F(2)$.

Remark 2.9. For further use, we end this section remarking that $(F_h)^\times = F^\times (F_h^\times)^2$ and $(1 + m_A)(F^\times)^2 = (F_h^\times)^2 \cap F^\times$ hold for a 2-henselization (F_h, A_h) of a valued field (F, A). They are consequence of two facts: (F_h, A_h) is an immediate extension of (F, A) [5, Theorem 15.8], and $1 + m_{A_h} \subset (F_h^\times)^2$.

3. Brauer group of pre-2-henselian valued fields

This section is dedicated to study the relative Brauer group $_2\mathrm{Br}(F)$ consisting of all classes of central simple F-algebras split by $F(2)$. Setting $\mathrm{Br}(F)$ for the Brauer group of F we may describe $_2\mathrm{Br}(F)$ as $_2\mathrm{Br}(F) = \{ z \in \mathrm{Br}(F) \mid 2z = 0 \}$. In 1981 Merkurjev [16] proved the long standing conjecture that $_2\mathrm{Br}(F)$ is generated by the set of quaternion algebras $(F; a, b)$ defined over F. Recall that for $a, b \in F^\times$ the quaternion algebra $(F; a, b)$ is the F-algebra having generators i, j and relations $i^2 = a$, $j^2 = b$ and $ij = -ji$ (see for example [14, Chapter III, p 51]). Quaternion algebras over F are connected with the Kaplansky radical of F as follows: the correspondence from $F^\times /(F^\times)^2 \times F^\times /(F^\times)^2$ to the Brauer group $\mathrm{Br}(F)$ of F which assigns to each pair $(a(F^\times)^2, b(F^\times)^2)$ the class of the quaternion algebra $(F; a, b)$ in $\mathrm{Br}(F)$ is a symmetric bimultiplicative pairing. The radical of this pairing consisting of $\{x \in F^\times \mid (F; x, b)$ splits for every $b \in F^\times\}$ is the Kaplansky radical $R(F)$ of F, [14, Proposition 6.1].

For a 2-henselization (F_h, A_h) of (F, A) let $\mathrm{Res} : {_2\mathrm{Br}(F)} \to {_2\mathrm{Br}(F_h)}$ be the *restriction* map which assigns to the class $[S]$ of a central simple F-algebra S the class $[S \otimes_F F_h] \in {_2\mathrm{Br}(F_h)}$. Observe that $\ker(\mathrm{Res}) = \mathrm{Br}(F_h; F)$ is the relative Brauer group associated with the extension $F_h|F$. Our first aim is to prove Theorem 3.2 which gives a description of $\ker(\mathrm{Res})$ for an arbitrary field F (char $F \neq 2$). The main goal in this section follows from this result; a characterization of a pre-2-henselian valued field F by means of $_2\mathrm{Br}(F)$ (Corollary 3.4). Corollary 3.4, which has its own interest,

provides one of the tools necessary to prove the main theorems of the paper (results 4.2 and 4.7).

We start with a technical lemma. Recall first that $\langle a_1, \ldots, a_n \rangle$ represents the diagonal quadratic form with coefficients $a_1, \ldots, a_n$ in a field F. Also the n-fold Pfister form $\bigotimes_{i=1}^{n} \langle 1, -a_i \rangle$ will be denoted by $\langle\langle -a_1, \ldots, -a_n \rangle\rangle$. As usual $W(F)$ stands for the Witt ring of all isometric classes of anisotropic quadratic forms over F. Following Lam [14], IF denotes the ideal consisting of classes of even-dimensional quadratic forms and $I^n F$ is the n-power of IF. All unexplained notations can be found in [14].

Lemma 3.1. *Let F be a field and $a_1, \ldots, a_n \in F^\times$, for some $n \geq 2$. Then there exists $q_n \in I^3 F$ such that*

$$\langle\langle -a_1 \cdots a_n \rangle\rangle = \sum_{i=1}^{n} \langle\langle -a_i \rangle\rangle \;-\; \sum_{1 \leq i < j \leq n} \langle\langle -a_i, -a_j \rangle\rangle \;+\; q_n.$$

Proof. We argue by induction on n. For $n = 2$,

$$\langle\langle -a_1 \rangle\rangle + \langle\langle -a_2 \rangle\rangle = \langle\langle -a_1, -a_2 \rangle\rangle + \langle\langle -a_1 a_2 \rangle\rangle.$$

Next, for $n > 2$,

$$\langle\langle -a_1 \cdots a_n \rangle\rangle = \langle\langle -a_1 \cdots a_{n-1} \rangle\rangle + \langle\langle -a_n \rangle\rangle - \langle\langle -a_n \rangle\rangle \langle\langle -a_1 \cdots a_{n-1} \rangle\rangle.$$

Applying the induction hypothesis to $\langle\langle -a_1 \cdots a_{n-1} \rangle\rangle$ one gets $q_{n-1} \in I^3 F$ such that $\langle\langle -a_1 \cdots a_n \rangle\rangle$ can be written as

$$\sum_{i=1}^{n-1} \langle\langle -a_i \rangle\rangle - \sum_{1 \leq i < j \leq n-1} \langle\langle -a_i, -a_j \rangle\rangle - \langle\langle -a_n \rangle\rangle \Big(\sum_{i=1}^{n-1} \langle\langle -a_i \rangle\rangle - \sum_{1 \leq i < j \leq n-1} \langle\langle -a_i, -a_j \rangle\rangle \Big) +$$

$$\langle\langle -a_n \rangle\rangle + q_{n-1} - \langle\langle -a_n \rangle\rangle q_{n-1} =$$

$$= \sum_{i=1}^{n} \langle\langle -a_i \rangle\rangle - \sum_{1 \leq i < j \leq n} \langle\langle -a_i, -a_j \rangle\rangle + \sum_{1 \leq i < j \leq n-1} \langle\langle -a_n, -a_i, -a_j \rangle\rangle + q_{n-1} - \langle\langle -a_n \rangle\rangle q_{n-1}$$

Setting $q_n = \displaystyle\sum_{1 \leq i < j \leq n-1} \langle\langle -a_n, -a_i, -a_j \rangle\rangle + q_{n-1} - \langle\langle -a_n \rangle\rangle q_{n-1}$, one gets the desired decomposition for $\langle\langle -a_1 \cdots a_n \rangle\rangle$. $\qquad\square$

For the next theorem let us recall that the map $\langle\langle -a, -b \rangle\rangle \mapsto (F; a, b)$ extends to a group homomorphism $\gamma : I^2 F / I^3 F \to {}_2\mathrm{Br}(F)$ which, in light of Merkurjev's Theorem [16], is an isomorphism. Moreover, this is

a functorial isomorphism. In particular, the extension $F_h|F$ gives rise to the following commutative diagram

$$
\begin{array}{ccc}
I^2F/I^3F & \xrightarrow{\ \bar{\mathrm{r}}\ } & I^2F_h/I^3F_h \\
\gamma \downarrow \wr & & \wr \downarrow \gamma_h \\
{}_2\mathrm{Br}(F) & \xrightarrow{\ \mathrm{Res}\ } & {}_2\mathrm{Br}(F_h)
\end{array}
\tag{2}
$$

where $\bar{\mathrm{r}}$ is induced by $F \subset F_h$ as follows: the inclusion induces naturally a ring homomorphism $\mathrm{r} : W(F) \to W(F_h)$. Thus $\mathrm{r}(I^nF) \subset I^nF_h$, for every $n \geq 0$. Actually, $\mathrm{r}(I^nF) = I^nF_h$; just observe that r is a surjective map by Remark 2.9 and IF is a maximal ideal. Therefore, $\bar{\mathrm{r}}$ is the quotient homomorphism constructed from r. Note that $\bar{\mathrm{r}}$ is also a surjective map and consequently so is Res.

Theorem 3.2. *For a valued field (F, A) let (F_h, A_h) be a 2-henselization. Then* $\mathrm{Br}(F_h; F)$, *the kernel of* Res, *is generated by the set*

$$
\{ (F, a, t) \,|\, a \in F^\times,\ t \in 1 + m_A \}.
$$

Proof. Let us first check that $\ker(\bar{r})$ is generated by the set

$$
\{ \langle b \rangle \langle\langle -a, -t \rangle\rangle + I^3F \,|\, a,\, b \in F^\times,\ \text{and}\ t \in 1 + m_A \}.
\tag{3}
$$

We know from [11, Proposition 2.4] that $\ker(\mathrm{r})$ is generated by the set $\{ \langle\langle -t \rangle\rangle \,|\, t \in 1 + m_A \}$. Observe next that for every $x + I^3F \in \ker(\bar{\mathrm{r}})$ we may choose $x \in \ker(\mathrm{r}) \cap I^2F$.[3] Hence we may write $x = \sum_{i=1}^{n} \langle a_i \rangle \langle\langle -t_i \rangle\rangle$, for some $a_i \in F^\times$ and $t_i \in 1 + m_A$. The quadratic form has determinant $d(x) = (-1)^n t_1 \cdots t_n (F^\times)^2$ and dimension $2n$. However, according to [14, Corollary 2.2, p 32] $d(x) = (-1)^{n(2n-1)}(F^\times)^2$, for $x \in I^2F$. Combining both facts we obtain $t_1 \cdots t_n \in (F^\times)^2$. By replacing t_n with $t_1 \cdots t_{n-1}$ in the description of x and then applying Lemma 3.1 we deduce the following

---

[3]For $x \in I^2F$ and $y \in I^3F$ such that $\mathrm{r}(x) = \mathrm{r}(y)$ it follows that $z = x - y \in \ker(\mathrm{r})$. Then $z \in I^2F$ and $z - x = -y \in I^3F$. Hence $x + I^3F = z + I^3F$.

equations:

$$x = \sum_{i=1}^{n-1} \langle a_i \rangle \langle\langle -t_i \rangle\rangle + \langle\langle -t_1 t_2 \dots t_{n-1} \rangle\rangle \langle a_n \rangle$$

$$= \sum_{i=1}^{n-1} \langle a_i \rangle \langle\langle -t_i \rangle\rangle + \Big(\sum_{i=1}^{n-1} \langle\langle -t_i \rangle\rangle - \sum_{1 \le i < j \le n-1} \langle\langle -t_i, -t_j \rangle\rangle + q_{n-1} \Big) \langle a_n \rangle,$$

for some $q_{n-1} \in I^3 F$

$$= \sum_{i=1}^{n-1} \langle a_i, a_n \rangle \langle\langle -t_i \rangle\rangle - \sum_{1 \le i < j \le n-1} \langle\langle -t_i, -t_j \rangle\rangle \langle a_n \rangle + \langle a_n \rangle q_{n-1},$$

for some $q_{n-1} \in I^3 F$.

Since each of the above summands belongs to $I^2 F$ the equation below makes sense in $I^2 F / I^3 F$

$$x + I^3 F = \sum_{i=1}^{n-1} \langle a_i, a_n \rangle \langle\langle -t_i \rangle\rangle - \sum_{1 \le i < j \le n-1} \langle\langle -t_i, -t_j \rangle\rangle \langle a_n \rangle + I^3 F.$$

Finally, it follows from $\langle a_i, a_n \rangle \langle\langle -t_i \rangle\rangle + I^3 F = -\langle\langle a_i a_n, -t_i \rangle\rangle + I^3 F$ that the set displayed in (3) generates $\ker(\bar{r})$, as announced.

To finish the proof observe that the commutative diagram (2) implies that $\ker(\mathrm{Res}) = \gamma(\ker(\bar{r}))$ and [14, Lemma 3.2, p 114] yields $\gamma(\langle b \rangle \langle\langle -a, -t \rangle\rangle + I^3 F) = (F, a, t)$, for every $a \in F^\times$, and $t \in 1 + m_A$. $\qquad \square$

Corollary 3.3. *Let (F, A) be a valued field and fix a 2-henselization (F_h, A_h). Then*

$$_2\mathrm{Br}(F) \cong \Big\langle \big\{ (F, a, t) \mid a \in F^\times, t \in 1 + m_A \big\} \Big\rangle \oplus {}_2\mathrm{Br}(F_h),$$

where $\Big\langle \{ \cdots \} \Big\rangle$ stands for the subgroup (of $_2\mathrm{Br}(F)$) generated by the set $\{ \cdots \}$.

Proof. The surjectivity of Res implies the exactness of the sequence

$$0 \to \mathrm{Br}(F_h; F) \longrightarrow {}_2\mathrm{Br}(F) \xrightarrow{\mathrm{Res}} {}_2\mathrm{Br}(F_h) \to 0$$

which splits when considered as a sequence of $\mathbb{F}_2$-vector spaces. $\qquad \square$

A cohomological characterization of pre-2-henselian fields follows immediately from Corollary 3.3.

Corollary 3.4. *For a valued field (F, A) let (F_h, A_h) be a 2-henselization. Then, (F, A) is pre-2-henselian if and only if $\mathrm{Res} : {}_2\mathrm{Br}(F) \to {}_2\mathrm{Br}(F_h)$ is an isomorphism.*

4. Galois characterization of pre-2-henselian fields

In this section we give a characterization of a pre-2-henselian field (F, A) by means of the Galois group $G_F(2)$. The instrument is the cohomological characterization of free products of pro-p groups established by Neukirch [17, Sätze 4.2, 4.3]. A convenient form of Neukirch's criterium, translated to the case under consideration in the paper, is as follows:

Theorem 4.1. *Let F be a field and K_1, K_2 be two extensions of F in $F(2)$. Then $G_F(2) \cong G_{K_1}(2) * G_{K_2}(2)$ if and only if*

(1) *the canonical map*

$$\mathrm{res}^1 : F^\times/(F^\times)^2 \to K_1^\times/(K_1^\times)^2 \times K_2^\times/(K_2^\times)^2$$

induced by the inclusions is an isomorphism;

(2) *the canonical map*

$$\mathrm{res}^2 : {}_2\mathrm{Br}(F) \to {}_2\mathrm{Br}(K_1) \times {}_2\mathrm{Br}(K_2)$$

induced by $\mathrm{Res} : {}_2\mathrm{Br}(F) \to {}_2\mathrm{Br}(K_1)$ and $\mathrm{Res} : {}_2\mathrm{Br}(F) \to {}_2\mathrm{Br}(K_2)$ is a monomorphism.

With the help of Corollary 3.4 and Theorem 4.1 we obtain a Galois characterization of those fields F for which there exists $A \in \Omega$ such that $\Gamma_A \neq 2\Gamma_A$.

Theorem 4.2. *Let (F, A) be a pre-2-henselian valued field with $\Gamma_A \neq 2\Gamma_A$. Then $G_F(2)$ admits a decomposition as free pro-2 product $G_F(2) = \mathcal{F} * \mathcal{H}$ of closed subgroups $\mathcal{F}$ and $\mathcal{H}$ such that*

(i) *$\mathcal{F}$ is a free pro-2 group;*

(ii) *$\mathcal{H}$ has a non-trivial abelian normal closed subgroup.*

Moreover the subgroups $\mathcal{F}$, $\mathcal{H}$ are naturally associated with $R(F)$ and A:

(I) *$rank(\mathcal{F}) = rank\left(R(F)/(F^\times)^2\right)$.*

(II) *Set $\mu = rank\left(\Gamma_A/2\Gamma_A\right)$. Then $\mathcal{H} = \mathbb{Z}_2^\mu \rtimes G_{k_A}(2)^4$.*

Conversely, let F be a field such that $G_F(2) = \mathcal{F} * \mathcal{H}$ where

(a) *$\mathcal{F}$ is a free pro-2 group;*

---

[4]The action of $G_{k_A}(2)$ on the abelian component is described in [6, Proposition 1.1].

(b) $rank(\mathcal{H}) \geq 2$, $\mathcal{H} \not\cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{H}$ has a non-trivial abelian normal closed subgroup.

Then F admits a pre-2-henselian valuation ring A with $\Gamma_A \neq 2\Gamma_A$.

Finally, the decomposition $G_F(2) = \mathcal{F} * \mathcal{H}$ does not depend on the valuation ring A and $\mathcal{H}$ is uniquely determined (up to inner automorphisms of $G_F(2)$).

The proof of Theorem 4.2 depends essentially on the next three results which also have their own interest. We start by characterizing those fields F such that $G_F(2)$ is a free pro-2 group by means of $R(F)$.

Proposition 4.3. For a field F the following conditions are equivalent:

(1) $R(F) = F^\times$.

(2) $_2\mathrm{Br}(F) = 0$.

(3) $G_F(2)$ is a free pro-2 group.

Proof. The equivalence $(1) \Leftrightarrow (2)$ is clear if we observe that $R(F) = F^\times$ occurs if and only if $(F; a, b)$ split for all $a, b \in F^\times$ which turns to be equivalent to $_2\mathrm{Br}(F) = 0$, by Merkurjev's Theorem [16]. The proof of (2) $\Leftrightarrow$ (3) follows from [18, Corollary 3.8, p 262, and Corollary 3.2, p 255]. $\square$

Proposition 4.4. For a pre-2-henselian valued field (F, A), $A \neq F$, fix a 2-henselization (F_h, A_h). Then there exists an extension $K \subset F(2)$ satisfying the following conditions:

(1) $R(K) = K^\times$ and the inclusion $F \subset K$ induces an isomorphism

$$K^\times/(K^\times)^2 \cong (1 + m_A)(F^\times)^2/(F^\times)^2.$$

(2) $G_F(2) = G_K(2) * G_{F_h}(2)$.

Proof. Take an extension $K \subset F(2)$ of F such that the natural map $F^\times/(F^\times)^2 \to K^\times/(K^\times)^2$ restricts to an injective map on group $(1 + m_A)(F^\times)^2/(F^\times)^2$ and K is maximal with this property. We claim this map induces an isomorphism $(1+m_A)(F^\times)^2/(F^\times)^2 \cong K^\times/(K^\times)^2$. In fact, the map is injective by construction and what remains to be proved is the surjectivity. Take any $x \in K^\times$ and assume, for the sake of contradiction, that $x(K^\times)^2$ is not in the image of $(1 + m_A)(F^\times)^2/(F^\times)^2$. Form the quadratic extension $L = K(\sqrt{x})$. Since $(L^\times)^2 \cap K = (K^\times)^2 \cup x(K^\times)^2$ it follows that $(1 + m_A)(F^\times)^2/(F^\times)^2$ is sent injectively into $L^\times/(L^\times)^2$,

contradicting the maximality of K. The last argument can also be displayed as

$$K^\times = (1 + m_A)(K^\times)^2. \tag{4}$$

Thus if $1 + m_A \subset R(K)$, then $K^\times = R(K)$, as desired. However, if $x \in 1 + m_A \subset R(F)$, then $F^\times \subset D_F\langle 1, -x\rangle \subset D_K\langle 1, -x\rangle$. Hence $1 + m_A \subset D_K\langle 1, -x\rangle$ which combined with the above equation (4) implies $K \subset D_K\langle 1, -x\rangle$. So $1 + m_A \subset R(K)$ already occurs.

To finish the proof it remains to show that item (2) of Proposition 4.4 also holds. To prove the decomposition $G_F(2) = G_K(2) * G_{F_h}(2)$ we have to show that the conditions (1) and (2) from Theorem 4.1 are fulfilled.

• res^1 is injective. Assume $x \in (K^\times)^2$ and $x \in (F_h^\times)^2$ for some $x \in F^\times$. According to Remark 2.9, $(F_h^\times)^2 \cap F = (1 + m_A)(F^\times)^2$. On the other hand, it follows from the very construction of K that $(K^\times)^2 \cap (1 + m_A)(F^\times)^2 = (F^\times)^2$. Consequently $x \in (F^\times)^2$, proving that res^1 is injective.

• res^1 is surjective. Let $x \in K^\times$ and $y \in F_h^\times$, be arbitrarily chosen. From Remark 2.9 there exists $w \in F^\times$ such that $w^{-1}y \in (F_h^\times)^2$. Next, by applying the above condition (4) to $w^{-1}x \in K^\times$ we get $u \in (1 + m_A)$ such that $u^{-1}w^{-1}x \in (K^\times)^2$. Since $1 + m_A \subset (F_h^\times)^2$ we also have $u^{-1}w^{-1}y \in (F_h^\times)^2$. Therefore, putting the things together we get res$^1(wu(F^\times)^2) = \big(x(K^\times)^2, y(F_h^\times)^2\big)$, proving the required surjectivity.

• res^2 is injective. $R(K) = K^\times$ was obtained at the beginning of the proof. Thus, according to Proposition 4.3, $_2\mathrm{Br}(K) = 0$. Therefore res$^2 = \mathrm{Res}$, which is an isomorphism by Corollary 3.4. □

Proposition 4.5. *Let F be a field with $R(F) \neq F^\times$. Let K, $H \subset F(2)$ be two extensions of F such that $G_K(2)$ is free and H admits a 2-henselian valuation ring $C \neq H$. If $G_F(2) = G_K(2) * G_H(2)$, then $A = C \cap F$ is a pre-2-henselian valuation ring of F. If Γ_A is not 2-divisible, then this decomposition implies that (H, C) is a 2-henselization of (F, A). In the case $\Gamma_A = 2\Gamma_A$, (H, C) contain a 2-henselization (F_h, A_h) of (F, A) such that $G_F(2) = G_{K_1}(2) * G_{F_h}(2)$, for a 2-extension $F \subset K_1 \subset F(2)$ with $G_{K_1}(2)$ free.*

Proof. Let us call $R = (H^\times)^2 \cap F^\times$. We first prove that $R \subset R(F)$. For $r \in R$ we shall show that $(F; r, x)$ splits for every $x \in F^\times$. In fact, given $x \in F^\times$, we have that $(H; r, x)$ splits, because $r \in (H^\times)^2$. On the other hand, by Proposition 4.3, $_2\mathrm{Br}(K) = 0$. So $(K; r, x)$ also splits. Consequently, in the notation of Theorem 4.1, the injectivity of res^2 implies that $(F; r, x)$ splits, as required. Thus $R \subset R(F)$, as claimed.

In the next step, observe that $1 + m_A \subset 1 + m_C \subset (H^\times)^2$ implies that $1 + m_A \subset R$ and so (F, A) is pre-2-henselian, as required.

Now we prove the final part of the proposition. Recall that the valued field (H, C) must contain a 2-henselization (F_h, A_h) of (F, A), because (H, C) is a 2-henselian extension of (F, A). Since $F_h \subset H$ are two subextensions of $F(2)$, we have either $F_h = H$ or there exists an intermediate extension $F_h \subset L \subset H$ such that $[L : F_h] = 2$. If the last case occurs, there exists $x \in F_h \smallsetminus (F_h^\times)^2$ such that $x \in (H^\times)^2$. According to Remark 2.9, we may take $x \in F$. By the first part of the proof we have $x \in R(F)$. Since $x \notin (F_h^\times)^2$ also $x \notin (1 + m_A)(F^\times)^2$. Then Corollary 2.5 implies $\Gamma_A = 2\Gamma_A$. Thus the inequality $F \neq H$ only occurs if $\Gamma_A = 2\Gamma_A$. In this case, we apply the Proposition 4.4 to get the desired decomposition. $\qquad\square$

Now that the connection between free products of suitable subgroups of $G_F(2)$ and pre-2-henselian valuation rings was established the proof of our first main result is just a matter of a proper combination of all ingredients.

Proof of Theorem 4.2. Proposition 4.4 gives us the free product decomposition $G_F(2) = G_K(2) * G_{F_h}(2)$ for suitable extensions K, $F_h \subset F(2)$ of F. Let us check that $\mathcal{F} = G_K(2)$ and $\mathcal{H} = G_{F_h}(2)$ have the properties announced in (i), (ii), (I) and (II).

That $\mathcal{F} = G_K(2)$ is a free pro-2 group follows from Proposition 4.3. From $\Gamma_A \neq 2\Gamma_A$ we deduce by means of Corollary 2.5 that $R(F) = (1 + m_A)(F^\times)^2 \neq F^\times$. Hence $K^\times/(K^\times)^2 \cong R(F)/(F^\times)^2$, by Proposition 4.4. From [18, Proposition 6.2, Theorem 6.8] and Galois Theory we have that rank $(\mathcal{F})$ is equal to rank $(K^\times/(K^\times)^2)$. Thus (I) holds.

The existence of a non-trivial abelian normal subgroup of $G_{F_h}(2)$ follows from valuation theory. In fact, a proof of it, as well as a proof of (II), can be found in § 1 of [6] during the demonstration of Proposition 1.1.

Conversely, assume $G_F(2) \cong \mathcal{F} * \mathcal{H}$ as stated in the theorem. Let K and H be respectively the fixed fields of $\mathcal{F}$ and $\mathcal{H}$. We now use Proposition 4.5. We already know that $G_K(2)$ is free. By applying [6, Theorem 4.3] to the Galois extension $F(2)|H$ in the case $p = 2$ we obtain that H carries a 2-henselian valuation ring C with value group $\Gamma \neq 2\Gamma$ and residue class field k such that char $k \neq 2$. Thus $C \neq H$, as desired. Then $A = C \cap F$ is pre-2-henselian and $\Gamma_A \neq 2\Gamma_A$, because the same is true for Γ. Hence (H, C) is a 2-henselization of (F, A) (Proposition 4.5).

The last statement of the theorem says that the decomposition does not depend on A and $\mathcal{H}$ is unique. To prove it we write a preparatory lemma.

For any valuation ring C of $F(2)$ let $G^h(C)$ stand for the decomposition group of C over F [5, §15].

Lemma 4.6. *For a field F assume there are A, $C \in \Omega$ with non-2-divisible value group. Then we can choose extensions $A(2)$ and $C(2)$ of A and C to $F(2)$, respectively, such that $G^h(A(2)) = G^h(C(2))$.*

The lemma above enables us to finish the proof of Theorem 4.2.

We have already proved that any decomposition $G_F(2) = \mathcal{F}_1 * \mathcal{H}_1$ which satisfies the assumptions (i) and (ii) of Theorem 4.2 yields $C \in \Omega$ such that $\Gamma_C \neq 2\Gamma_C$ and $\mathcal{H}_1 = G^h(C(2))$, for some extension $C(2)$ of C to $F(2)$. Consequently, Lemma 4.6 above says that $\mathcal{H}_1$ does not depend on a particular choice of $C \in \Omega$ with non 2-divisible value group.

Set next $G_F(2) = \mathcal{F} * \mathcal{H}$ and $G_F(2) = \mathcal{F}_1 * \mathcal{H}_1$ where $\mathcal{H} = G^h(A(2))$ and $\mathcal{H}_1 = G^h(C(2))$ for valuation rings $A(2)$ and $C(2)$ of $F(2)$ lying, respectively, over two valuation rings A, $C \in \Omega$ of F, with non 2-divisible value groups.

By Lemma 4.6 we can choose $A(2)'$ and $C(2)'$, respectively the extensions of A and C to $F(2)$ such that $G^h(A(2)') = G^h(C(2)')$. By the Conjugation Theorem [7, 3.2.15] there exist $g, h \in G_F(2)$ such that $A(2)' = g(A(2))$ and $C(2)' = h(C(2))$. We derive from these equalities $G^h(A(2)') = gG^h(A(2))g^{-1}$ and $G^h(C(2)') = hG^h(C(2))h^{-1}$. Thus $\mathcal{H} = g^{-1}h\mathcal{H}_1h^{-1}g$, are conjugated.   □

Proof of Lemma 4.6. Assume first that A and C are comparable by inclusion; say C is finer than A. Take an extension $C(2)$ of C to $F(2)$ finer than $A(2)$, an extension of A to $F(2)$. We shall show that $G^h(A(2)) = G^h(C(2))$, and additionally, this equality implies that $C(2)$ is the **unique extension** of C to $F(2)$ finer than $A(2)$. Let $F^h(C(2))$ and $F^h(A(2))$ be the fixed fields of $G^h(C(2))$ and $G^h(A(2))$, respectively. From $C(2) \subset A(2)$ one gets $G^h(C(2)) \subset G^h(A(2))$ and so $F^h(A(2)) \subset F^h(C(2))$. To conclude the equality $F^h(A(2)) = F^h(C(2))$ we have to prove for $x \in F^h(A(2))$ and $x \in (F^h(C(2)))^2$ that $x \in (F^h(A(2)))^2$. In other words there is no quadratic extension between $F^h(A(2))$ and $F^h(C(2))$. Take x which satisfies these conditions. From Remark 2.9 we can assume, without loss of generality, that $x \in F^\times$. Then $x \in (F^h(C(2)))^2 \cap F^\times = (1 + m_C)(F^\times)^2$. Since by Corollary 2.5, $(1+m_A)(F^\times)^2 = R(F) = (1+m_C)(F^\times)^2$ it follows that $x \in (1 + m_A)(F^\times)^2 \subset (F^h(A(2)))^2$, proving the desired equality.

We now prove the uniqueness of such an extension $C(2)$. According to the Conjugation Theorem [7, 3.2.15] for two extensions $C(2)$ and $C(2)'$ of C to $F(2)$ there exists $g \in G_F(2)$ such that $C(2)' = g(C(2))$. If $C(2)$ and $C(2)'$ are finer than $A(2)$, then $g(A(2))$ and $A(2)$ are both coarser than $C(2)'$, which implies that $g(A(2))$ and $A(2)$ have to be comparable by inclusion. Thus $g(A(2)) = A(2)$, by [7, Lemma 3.2.8]. Hence $g \in G^h(A(2))$. Since we know $G^h(A(2)) = G^h(C(2))$, $g \in G^h(C(2))$ and $C(2)' = C(2)$, as claimed.

Consider next the case where A and C are not comparable. Then $A, C \in \Omega_2$, and we know by Proposition 2.6, (3) and (5), that $D = A \cdot C \in \Omega_2$ and $\Gamma_D \neq 2\Gamma_D$. Since A and D, respectively C and D, are comparable by inclusion, by the first part of the proof there is an extension $D(2)$ of D to $F(2)$ which contains uniquely determined extensions $A(2)$ and $C(2)$, respectively, of A and C to $F(2)$ for which $G^h(A(2)) = G^h(D(2))$ and $G^h(C(2)) = G^h(D(2))$. Thus $G^h(A(2)) = G^h(C(2))$. $\square$

In the next result we complete the study of pre-2-henselian valued fields (F, A) by considering the missing case $\Gamma_A = 2\Gamma_A$ in Theorem 4.2. Recall that Ω_1 is totally ordered by inclusion, Proposition 2.6 (2). Hence $A_{(1)} = \bigcap_{A \in \Omega_1} A$ is a valuation ring having maximal ideal $m_{(1)} = \bigcup_{A \in \Omega_1} m_A$. Thus $A_{(1)}$ is a pre-2-henselian valuation ring of F. Observe also that $A_{(1)}$ is comparable by inclusion to any other element of Ω, Proposition 2.6 (4).

Theorem 4.7. *Let F be a field with $R(F) \neq F^\times$ and $\Omega \neq \{F\}$, but each $C \in \Omega$ has 2-divisible value group. Let k be the residue class field of the valuation ring $A_{(1)}$ introduced above. Then $G_F(2) = \mathcal{F} * G_k(2)$, and the following conditions hold:*

(1) *$\mathcal{F}$ is a free pro-2 group and $rank(\mathcal{F}) = rank\big((1 + m_{(1)})(F^\times)^2/(F^\times)^2\big)$*

(2) *k does not admit any proper pre-2-henselian valuation ring. Moreover, $k^\times/(k^\times)^2 \cong F^\times/(1 + m_{(1)})(F^\times)^2$.*

Conversely, if there exists a decomposition

$$G_F(2) = \mathcal{F} * G_H(2)$$

where $\mathcal{F}$ is a free pro-2 group and H has a proper 2-henselian valuation ring C with $\Gamma_C = 2\Gamma_C$ and residue class field k which does not admit any proper pre-2-henselian valuation ring, then $R(F) \neq F^\times$, $\Omega \neq \{F\}$ and for every $A \in \Omega$, $\Gamma_A = 2\Gamma_A$.

Proof. Since $\Omega \neq \{F\}$, $A_{(1)}$ is a proper valuation ring of F. By applying Proposition 4.4 to $A_{(1)}$ we get $G_F(2) = G_K(2) * G_{F_h}(2)$, where $\mathcal{F} = G_K(2)$ is free and F_h is a 2-henselization of $(F, A_{(1)})$. Thus $(F, A_{(1)})$ and (F_h, A) have the same value group and the same residue class field. By assumption $A_{(1)}$, and consequently also A, have 2-divisible value group. Let $A(2)$ be the extension of A to $F(2)$. Since $G_{F_h}(2)$ is a pro-2 group and A has 2-divisible value group, we can conclude that $A(2)$ has a trivial inertia group over F_h. Thus $G_{F_h}(2) \cong G_k(2)$, where k is the residue class field of A and also of $A_{(1)}$ (see [5, Theorem 19.6]). To complete the first part of the

proof it remains to show that k does not admit any proper pre-2-henselian valuation ring.

According to Proposition 2.6 (5), $\Omega_2 = \emptyset$. Hence $\Omega = \Omega_1$ and therefore $A_{(1)} \in \Omega_1$ does not contain any proper pre-2-henselian valuation ring of F. Denote by π the canonical map corresponding to $A_{(1)}$ and k and recall from Proposition 2.4 (2) that $\pi(R(F)) = R(k)$. Since π induces a bijection between the set of all valuation rings of k and the set of all valuation rings of F finer than $A_{(1)}$ ([5, Theorem 8.7]), we can conclude that k does not admit any proper pre-2-henselian valuation ring, as claimed in (2). The isomorphism $k^\times/(k^\times)^2 \cong F^\times/(1 + m_{(1)})(F^\times)^2$ follows from $F^\times = A_{(1)}^\times (F^\times)^2$, because Γ_C is 2-divisible.

Conversely, given a decomposition $G_F(2) = \mathcal{F} * G_H(2)$ we can apply Proposition 4.5 to H and to the fixed field K of $\mathcal{F}$. Then $A = C \cap F$ is a proper pre-2-henselian valuation ring of F and we may assume that (H, C) is a 2-henselization of (F, A). Since Γ_C is a 2-divisible group and (H, C) is an immediate extension of (F, A), also $\Gamma_A = 2\Gamma_A$. Moreover $k_A = k_C$ does not admit any pre-2-henselian valuation ring. In particular k_A is not quadratically closed and so $A \in \Omega_1$. Moreover, the bijective correspondence between the set of valuation rings of k_A and the set of valuation rings of F finer than A implies that $A = A_{(1)}$ and $\Omega_2 = \emptyset$. Hence any element of Ω, being coarser than $A_{(1)}$ which has a 2-divisible value group, also has a 2-divisible value group. Finally, $R(F) \neq F^\times$ follows from the properties of A combined with (1) of Proposition 2.4. $\qquad\square$

5.   Examples and comments

The simplest possible description of $G_F(2)$ for a pre-2-henselian valued field (F, A) occurs for $A \in \Omega_2$. Then $G_F(2) = \mathcal{F} * \mathbb{Z}_2^\mu$ where $\mu = \mathrm{rank}(F^\times/R(F))$. Indeed, observe that $R(F) = (1 + m_A)(F^\times)^2$ (Proposition 2.6 (5)) and $A^\times \subset (1 + m_A)(F^\times)^2$ ($A \in \Omega_2$). Thus $A^\times(F^\times)^2 = R(F)$. Finally $\Gamma_A/2\Gamma_A \cong F^\times/A^\times(F^\times)^2$.

Theorems 4.2 and 4.7 can be seen as tools to produce fields with Kaplansky radical having some prescribed conditions. By [9, Theorem 3.6] given two pro-2 groups $\mathcal{F}$ and $\mathcal{H}$ which can be realized as Galois groups $G_K(2)$ and $G_H(2)$, for some fields K and H, there exists a field F such that $G_F(2) \cong \mathcal{F} * \mathcal{H}$. Thus, all we need to construct fields with a pre-2-henselian valuation ring is to choose fields K and H having convenient properties. Examples of fields K for which $G_K(2)$ is a free pro-2 group are well-known. Let us fix that K has characteristic $\neq 2$ and $G_K(2)$ is a free pro-2 group. To get suitable examples of 2-henselian valued fields it is enough to take $H = k((X))^\Gamma$, the field of generalized formal power series

over a field k of characteristic $\neq 2$ and with respect to a totally ordered abelian group Γ. H has a henselian valuation ring $A_H = k[[X]]^\Gamma$ which is a fortiori 2-henselian. Moreover k and Γ are, respectively, residue class field and value group of A_H.

Therefore if we choose k a pre-Hilbertian (as in 1.4) field with an Archimedean ordering and Γ a 2-divisible group, then we get a field F for which $G_F(2) = G_K(2) * G_H(2)$ exemplifies Theorem 4.7. Here $A_{(1)} = A_H \cap F$.

Similarly, the choice of k quadratically closed and $\Gamma \neq 2\Gamma$ will give us examples of pre-2-henselian valued fields F with $\Omega_2 \neq \emptyset$. In this case $G_F(2) = \mathcal{F} * \mathbb{Z}_2^\mu$, as described in the first paragraph.

Adding to K the property $-1 \notin (K^\times)^2$ while $-1 \in (k^\times)^2$ we get a pre-2-henselian valued field F with $-1 \in R(F) \smallsetminus (F^\times)^2$. Hence $F^\times = D_F\langle 1, 1\rangle$ and so F has level $s = 2$ (the least integer $n \geq 1$ such that -1 can be expressed as a sum of n squares in F). Now, if we impose $\Gamma \neq 2\Gamma$ it follows that F has $R(F)$-rigid elements. A fact that cannot happen with the usual rigid elements: by [4, Lemma 3.6], if every $x \in F^\times$ is a sum of $s(F)$ squares in F, then no element of F is $(F^\times)^2$-rigid. Therefore we cannot just translate to $R(F)$-rigid elements the properties of rigid elements.

More generally, choose K and H as above and also satisfying $\operatorname{rank}(K^\times/(K^\times)^2) = m$ and $\operatorname{rank}(H^\times/(H^\times)^2) = n$, for some integers $m > 0$ and $n > 1$ and $\Gamma \neq 2\Gamma$. Then we get a pre-2-henselian valued field F such that $\operatorname{rank}(R(F)/(F^\times)^2) = m$ and $\operatorname{rank}(F^\times/R(F)) = n$.

The last statement is nearly the content of [2, Corollary 3.13]. The field presented in [2, Theorem 2.3] satisfies $-1 \in (F^\times)^2$ and $(F^\times : R(F)) = 4$. An easy calculation shows that $B(R(F)) = R(F)$ and consequently there is a pre-2-henselian valuation ring A such that $\Gamma_A \neq 2\Gamma_A$. The same is true for fields constructed using the recipe of [2, Theorem 3.12]. Some other examples of fields with non-trivial Kaplansky radical, i.e., $R(F) \neq (F^\times)^2, F^\times$, like the one due to Gross-Fischer [8, Examples, p 306-307] or the examples [13, (A), (C), and (D)] constructed by Kula, are all pre-2-henselian valued fields. Therefore, our method described above gives a large family of examples, many of them have been construct before using different particular methods.

For the next application let us recall that X_F usually stands for the space of orderings of a field F and it has naturally the structure of a Boolean topological space (see for example [14, Chapter VIII, §6]). If F is formally real, then $R(F) \subset D_F\langle 1, 1\rangle \subset \sum(F^\times)^2$ and so a pre-2-henselian valuation ring A of F is compatible with all orderings of F. This is a property which (F, A) shares with any of its henselizations (F_h, A_h), as one deduces from [15, Theorem 3.16]. Moreover, by applying [15, Theorem 3.10] to (F, A) and (F_h, A_h) we get that the map $P \mapsto P \cap F$ from X_{F^h} to

X_F is bijective. Since this map is always continuous, compactness implies that the map is an homeomorphism. The above comments show that we can also prescribe the space of orderings X_F in constructing examples of pre-2-henselian valued fields. It suffices to choose H having the desired X_H.

A final word, theorems 4.2 and 4.7 have a corresponding formulation in Witt ring language.

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