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RESEARCH ARTICLE

Preradicals and submodules

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Abstract. Some collections of submodules of a module defined by certain conditions are studied.

Throughout the whole text, all rings are considered to be associative with unit $1 \neq 0$ and all modules are left unitary.

Let R be a ring. The category of left R-modules will be denoted by $R-$Mod. We shall write $N \leq M$ if N is a submodule of M.

Let $a \in R, I \subseteq R$. Put

$$(I : a) = \{x \in R | xa \in I\}.$$

Let M be an R-module. Let $End(M)$ be the set of all endomorphisms of the R-module M. A submodule N of M is said to be fully invariant in case

$$\forall f \in End(M) : f(N) \leq N.$$

Let $N \leq M$ and $f \in End(M)$. Put

$$(N : f)_M = \{x \in M | f(x) \in N\}.$$

It is clear that $(N : f)_M \leq M$. Put

$$End(M)_N = \{f \in End(M) | f(M) \subseteq N\}.$$

Let $F(M)$ be some non-empty collection of submodules of a left R-module M.

We shall consider the following conditions:

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C1. $L \in F(M), L \le N \le M \Rightarrow N \in F(M)$;
C2. $L \in F(M), f \in End(M) \Rightarrow (L : f)_M \in F(M)$;
C3. $N, L \in F(M) \Rightarrow N \cap L \in F(M)$;
C4. $N \in F(M), N \in Gen(M), L \le N \le M \wedge \forall g \in End(M)_N : (L : g)_M \in F(M) \Rightarrow L \in F(M)$;
C5. $N, K \in F(M), N \in Gen(M) \Rightarrow t_{(K \subseteq M)}(N) \in F(M)$.

Remark 1. Let F be a non-empty set of left ideals of R.
(1) Then F is a preradical filter if and only if F satisfies C1, C2, C3.
(2) Then F is a radical filter if and only if F satisfies C1, C2, C4.

Proof. (1) ($\Rightarrow$)Let F be a preradical filter.
(C1) This is clear.
(C2) Let $f \in End(R)$. Then there exists $a \in R$ such that

$$\forall r \in R : f(r) = ra.$$

Therefore for $L \in F$ we obtain $(L : f)_R = \{x \in R | f(x) \in L\} = \{x \in R | xa \in L\} = (L : a)$. Since F is a preradical filter, $(L : f)_R = (L : a) \in F$.
(C3) This is clear.
($\Leftarrow$) Let F satisfy (C1), (C2), (C3). Then it satisfies (a2) and (a3) [1, p.36].
(a1) (See [1, p.34]. Let $a \in R$ and $L \in F$. Define $f : R \to R$, where $\forall x \in R : f(x) = xa$.
It is easy to see that $f \in End(R)$. Obtain $(L : f)_R = (L : a)$. But $(L : f)_R \in F$. Hence $(L : a) \in F$.
(2) ($\Rightarrow$)Let F be a radical filter.
(C1),(C2) This is clear (see (1) ($\Rightarrow$)).
(C4) Let N be a left ideal of R and $g \in End(R)_N$. Then there exists $a \in R$ such that

$$\forall r \in R : g(r) = ra.$$

It follows from this that $a = 1a = g(1) \in N$. Taking into account (a5) (see [1, p.36]), it is obvious that F satisfies C4.
($\Leftarrow$) Let F satisfy C1, C2, C4. Then it is easy to see that it satisfies (a1) and (a2).
(a5) Let $N \in F, L \le N \le R \wedge \forall a \in N : (L : a) \in F$. Then $N \in Gen(R)$ because R is a generator. Let $g \in End(R)_N$. It means that there exists $a \in N$ such that $\forall r \in R : g(r) = ra$. But $(L : g)_R = (L : a)$. Therefore $(L : g)_R \in F$. By (C4), $L \in F$. Hence F satisfies (a5). □

Remark 2. Let M and H be left R-modules and $q : M \to H$ be an isomorphism, U be a non-empty set of submodules of M and $q(U) = \{q(L) | L \in U\}$. Then $q(U)$ satisfies (Ci) if and only if U satisfies (Ci) for every $i \in \{1, 2, 3, 4, 5\}$.

Proof. It is suffice to verify that $q(U)$ satisfies (Ci) if U satisfies (Ci) for every $i \in \{1,2,3,4,5\}$.

(1) Let U satisfy (C1). Consider $L \in q(U), L \leq N \leq H$. Hence $q^{-1}(L) \in U, q^{-1}(L) \leq q^{-1}(N) \leq M$. By (C1), $q^{-1}(N) \in U$. Whence $N = q(q^{-1}(N)) \in q(U)$.

(2) Let U satisfy (C2). Consider $L \in q(U), f \in End(H)$. Hence $q^{-1}fq \in End(M), q^{-1}(L) \in U$. By (C2), $q^{-1}(L : f)_H = (q^{-1}(L) : q^{-1}fq)_M \in U$. Hence $(L : f)_H \in q(U)$.

(3) Let U satisfy (C3). Consider $N, L \in q(U)$. Hence $q^{-1}(N), q^{-1}(L) \in U$. By (C3), $q^{-1}(N \cap L) = q^{-1}(N) \cap q^{-1}(L) \in U$. Therefore $N \cap L \in q(U)$.

(4) Let U satisfy (C4) and let
$N \in q(U), N \in Gen(H), L \leq N \leq H \wedge \forall g \in End(H)_N : (L : g)_H \in q(U)$.

Then $q^{-1}(N) \in U, q^{-1}(N) \in Gen(M), q^{-1}(L) \leq q^{-1}(N) \leq M$. Let $f \in End(M)_{q^{-1}(N)}$. Hence $qfq^{-1} \in End(H)_N$. Since $\forall g \in End(H)_N : (L : g)_H \in q(U)$,

$$q(q^{-1}(L) : f)_M = (L : qfq^{-1})_H \in q(U).$$

Hence $(q^{-1}(L) : f)_M \in U$. By (C4), $q^{-1}(L) \in U$. Hence $L \in q(U)$.

(5) Let U satisfy (C5) and $N, K \in q(U), N \in Gen(H)$. Hence

$$q^{-1}(N), q^{-1}(K) \in U, q^{-1}(N) \in Gen(M).$$

By (C5), $t_{(q^{-1}(K) \subseteq M)}(q^{-1}(N)) \in U$. Hence $q(t_{(q^{-1}(K) \subseteq M)}(q^{-1}(N))) \in q(U)$. But

$$q(t_{(q^{-1}(K) \subseteq M)}(q^{-1}(N))) = q\left(\sum_{g \in End(M)_{q^{-1}(N)}} g(q^{-1}(K))\right) =$$
$$= \sum_{g \in End(M)_{q^{-1}(N)}} q(g(q^{-1}(K))) = \sum_{g \in End(M)_{q^{-1}(N)}} (qgq^{-1})(K) =$$
$$= \sum_{f \in End(H)_N} f(K) = t_{(K \subseteq H)}(N).$$ Therefore $t_{(K \subseteq H)}(N) \in q(U)$. □

Remark 3. Let M be a left R-module. Then the sets

$$\{M\} \text{ and } \{L | L \leq M\}$$

satisfy (C1), (C2), (C3), (C4), (C5).

Proposition 1. *If F(M) satisfies (C1), (C2), (C4), then it satisfies (C5).*

Proof. Let $N, K \in F(M), N \in Gen(M)$. Then

$$t_{(K \subseteq M)}(N) = \sum_{g \in End(M)_N} g(K)$$

(see [3, p.40]).

It is easy to see that

$$\forall h \in End(M)_N : K \le (t_{(K\subseteq M)}(N) : h)_M.$$

By (C1), it follows from this that

$$\forall h \in End(M)_N : (t_{(K\subseteq M)}(N) : h) \in F(M).$$

It is obvious that $t_{(K\subseteq M)}(N) \le N$. Therefore

$$N \in F(M), N \in Gen(M), t_{(K\subseteq M)}(N) \le N \le M \wedge$$

$$\wedge \forall h \in End(M)_N : (t_{(K\subseteq M)}(N) : h)_M \in F(M).$$

Taking into consideration (C4), it follows from this that $t_{(K\subseteq M)}(N) \in F(M)$. □

Put

$$\ker F(M) := \bigcap_{L\in F(M)} L.$$

Proposition 2. *If $F(M)$ satisfies (C2), then* $\ker F(M)$ *is a fully invariant submodule of M.*

Proof. Let $f \in End(M), m \in \ker F(M)$. By (C2),

$$\bigcap_{L\in F(M)} L \subseteq \bigcap_{L\in F(M)} (L : f)_M.$$

Then $m \in \bigcap_{L\in F(M)} (L : f)_M$. Hence $f(m) \in \bigcap_{L\in F(M)} L$. Thus

$$f(\ker F(M)) \subseteq \ker F(M).$$

Let M be a left R-module. Let r be a preradical in R-Mod. Put

$$F_r(M) = \{L \le M | M/L \in T(r)\},$$

where $T(r) = \{M \in R - Mod | r(M) = M\}$ (see [1, 3]). □

Lemma 1. *Let $N \le M$ and $f \in End(M)$. Then $f(M) \cap N = f((N : f)_M)$and $f(M)/(f(M) \cap N) \cong M/(N : f)_M$.*

Proof. It is obvious that $f(M)\cap N = f((N:f)_M)$. Let

$$g:\begin{cases} M/(N:f)_M \to f(M)/(f(M)\cap N);\\ m+(N:f)_M \mapsto f(m)+f(M)\cap N.\end{cases}$$

Since

$$m_1+(N:f)_M = m_2+(N:f)_M \Rightarrow m_1-m_2\in (N:f)_M \Rightarrow$$

$$\Rightarrow f(m_1-m_2)\in f(M)\cap N \Rightarrow f(m_1)+f(M)\cap N = f(m_2)+f(M)\cap N,$$

the correspondence g is a mapping.

It is easy to see that g is an epimorphism.

Let $f(m_1)+f(M)\cap N = f(m_2)+f(M)\cap N$. Then $m_1-m_2\in (N:f)_M$. Hence $m_1+(N:f)_M = m_2+(N:f)_M$. Thus g is a monomorphism.

Therefore g is an isomorphism. □

Theorem 1. *Let r be a hereditary preradical in $R-Mod$ and M be a left R-module. Then the system $F_r(M)$satisfies (C1), (C2), (C3).*

Proof. (C1). Let $L\in F_r(M), L\le N\le M$. Since $M/N\cong (M/L)/(N/L)$ [2],$N\in F_r(M)$ because the class $T(r)$ is closed under epimorphic images [3] (see also [1, p.36]).

(C2). Let $L\in F_r(M), f\in End(M)$. Then $M/L\in T(r)$. Since the class $T(r)$ is closed under submodules, $(f(M)+L)/L\in T(r)$. But $(f(M)+L)/L\cong f(M)/(f(M)\cap L)$ [2]. Hence $f(M)/(f(M)\cap L)\in T(r)$. Since $f(M)/(f(M)\cap L)\cong M/(L:f)_M$(See Lemma 1), $(L:f)_M\in F_r(M)$.

(C3). Let $N,L\in F_r(M)$. Then $M/N, M/L\in T(r)$. Hence $M/N\oplus M/L\in T(r)$ because the class $T(r)$ is closed under direct sums [1, 3].

Consider the homomorphism $w:\begin{cases} M\to M/N\oplus M/L,\\ m\mapsto (m+N, m+L).\end{cases}$

Then $im(w)\in T(r)$ because $T(r)$ is closed under submodules [1, 3].

It is clear that $\ker(w) = N\cap L$. But $im(w)\cong M/\ker(w)$ [2]. Therefore $N\cap L\in F_r(M)$ (see also [1, p.36]). □

Theorem 2. *Let r be a hereditary radical in $R-Mod$ and M be a left R-module. Then the system $F_r(M)$ satisfies (C1), (C2), (C3), (C4).*

Proof. Taking into account Theorem 1, we shall prove only (C4).

Let

$$N\in F_r(M), N\in Gen(M), L\le N\le M\wedge$$
$$\wedge\forall g\in End(M)_N : (L:g)_M\in F_r(M).$$

By Lemma 1,

$$\forall g\in End(M)_N : (g(M)+L)/L\cong M/(L:g)_M.$$

It follows from $\forall g \in End(M)_N : (L : g)_M \in F_r(M)$ that

$$\forall g \in End(M)_N : M/(L : g)_M \in T(r).$$

Therefore

$$\forall g \in End(M)_N : (g(M) + L)/L \in T(r).$$

Since the class $T(r)$ is closed under direct sums and factor modules [1, 3],

$$\sum_{g \in End(M)_N} (g(M) + L)/L \in T(r).$$

Since $N \in Gen(M)$,

$$\sum_{g \in End(M)_N} g(M) = N$$

(see [2]).

Thus $N/L = (N + L)/L = \sum_{g \in End(M)_N} (g(M) + L)/L \in T(r)$.

Since the class $T(r)$ is closed under extensions [1, 3], taking into account that $N/L \in T(r), (R/L)/(N/L) \cong R/N \in T(r)$, we have that $R/L \in T(r)$.

Thus $L \in F_r(M)$. □

Corollary 1. *Let r be a hereditary radical in $R - Mod$ and M be a left R-module. Then the system $F_r(M)$satisfies (C5).*

Proof. By Theorem 2, Proposition 1. □

Corollary 2. *Let r be a hereditary preradical in $R - Mod$ and M be a left R-module. Then $\ker F_r(M)$ is a fully invariant submodule of M.*

Proof. By Theorem 1, Proposition 2. □

Proposition 3. *Let M be a left R-module and K be a fully invariant submodule of M. Then $U = \{L|K \leq L \leq M\}$ satisfies (C1), (C2), (C3).*

Proof. (C1) This is clear.

(C2) Let $L \in U, f \in End(M)$. Since K is a fully invariant submodule of M, $f(K) \leq K \leq L$. Therefore $K \leq (L : f)_M$. By (C1), $(L : f)_M \in U$.

(C3) This is clear. □

Theorem 3. *Let M be a left R-module and K be a fully invariant submodule of M such that $t_{(K \subseteq M)}(K) = K$. Then $U = \{L|K \leq L \leq M\}$ satisfies (C1), (C2), (C3), (C4).*

Proof. By Proposition 3, U satisfies (C1), (C2), (C3).

(C4) Let

$$N \in U \wedge L \leq N \leq M \wedge (\forall g \in End(M)_N : (L : g)_M \in U)$$

and k be an arbitrary element of K. Since $t_{(K \subseteq M)}(K) = K$,

$$k = p_1(k_1) + p_2(k_2) + ... + p_s(k_s),$$

where $s \in \{1, 2, ...\}, p_1, p_2, ..., p_s \in End(M)_K, k_1, k_2, ..., k_s \in K$. Since $K \leq N$, $End(M)_K \leq End(M)_N$. Thus $p_1, p_2, ..., p_s \in End(M)_N$. Now we obtain that $\forall j \in \{1, 2, ..., s\} : (L : p_j)_M \in U$. Hence $\forall j \in \{1, 2, ..., s\} : p_j(K) \leq L$. It means that $\forall j \in \{1, 2, ..., s\} : p_j(k_j) \in L$. Taking this into consideration, we have

$$k = p_1(k_1) + p_2(k_2) + ... + p_s(k_s) \in L.$$

Therefore $K \leq L$. It follows from this that $L \in U$. □

Theorem 4. *Let F be a field and V be a vector space over F with $\dim_F V < \infty$. If $U \neq \{V\}$ is a non-empty set of subspaces of V satisfying (C1), (C2), (C3), then $U = \{L | L \leq V\}$.*

Proof. By Proposition 2, $\ker U$ is a fully invariant subspace of V. It is easy to see that every fully invariant subspace of V is either $\{0\}$ or V. Since $U \neq \{V\}$, $\ker U = \{0\}$. Let $P = \{\dim_F \left(\bigcap_{L \in D} L \right) | D \subseteq U \& |D| < \infty\}$. Hence $\emptyset \neq P \subseteq \{0, 1, 2, ...\}$. Therefore there exists $t = \min P \in \{0, 1, 2, ...\}$. It follows from this that there exists $D_0 \subseteq U$ such that $|D_0| < \infty$ and $\dim_F \left(\bigcap_{L \in D_0} L \right) = t$. Hence

$$\forall B \in U : t \leq \dim_F \left(\left(\bigcap_{L \in D_0} L \right) \cap B \right) \leq \dim_F \left(\bigcap_{L \in D_0} L \right) = t.$$

Hence $\forall B \in U : \dim_F \left(\left(\bigcap_{L \in D_0} L \right) \cap B \right) = \dim_F \left(\bigcap_{L \in D_0} L \right) \&$

$\& \left(\bigcap_{L \in D_0} L \right) \cap B \leq \bigcap_{L \in D_0} L$. It follows from this that $\forall B \in U : \left(\bigcap_{L \in D_0} L \right) \cap$

$\cap B = \left(\bigcap_{L \in D_0} L \right)$. Then we obtain $\forall B \in U : \bigcap_{L \in D_0} L \subseteq B$. It means that $\bigcap_{L\in D_0} L \subseteq \ker U$. But $\ker U \subseteq \bigcap_{L\in D_0} L$. Therefore $\ker U = \bigcap_{L\in D_0} L$. Now we have that $\{0\} = \bigcap_{L\in D_0} L$. Taking into account $|D_0| < \infty$, by (C3), $\{0\} = \bigcap_{L\in D_0} L \in U$. Now apply (C1). □

Example 1. Let F be a field and V be a vector space over F with $\dim_F V = k_0$ and k be a non-finite cardinal number such that $k \leq k_0$. Then

$$U_k = \{L | L \leq V, \dim_F(V/L) < k\}$$

satisfies (C1), (C2), (C3), (C4), (C5).

Proof. (C1) Let $L \in U_k, L \leq N \leq V$. It is obvious that there exists an epimorphism $\pi : V/L \to V/N$. Hence $V/L = H \oplus T$ for some subspaces $H \cong V/N, T$ of V/L. It follows from this that $\dim_F(V/N) = \dim_F(H) \leq \dim_F(H \oplus T) = \dim_F(V/L) < k$. Whence $\dim_F(V/N) < k$. Now we obtain $N \in U_k$.

(C2) Let $L \in U_k, f \in End(V)$. By Lemma 1, $V/(L : f)_V \cong$ $\cong f(V)/(f(V \cap L)$. By Corollary 3.7 (3) [2, p.46], $f(V)/(f(V) \cap L) \cong$ $\cong (f(V) + L)/L$. It follows from this that $V/(L : f)_V \cong (f(V) + L)/L$. Since $(f(V) + L)/L \leq V/L \& \dim_F(V/L) < k$, $\dim_F(V/(L : f)_V) < k$. Thus $(L : f)_V \in U_k$.

(C3) Let $L, M \in U_k$. Hence $\dim_F(V/L) < k \& \dim_F(V/M) < k$. It is easy to see that

$$\dim_F(V/(L \cap M)) \leq \dim_F(V/L) + \dim_F(V/M).$$

Therefore $\dim_F(V/(L\cap M)) < k + k = k$ [4, p.417]. Thus $L\cap M \in U_k$.

(C4) Let $N \in U_k \wedge L \leq N \leq V \wedge (\forall g \in End(V)_N : (L : g)_V \in U_k)$. Thus $V = N \oplus W$ for some subspace W. Consider the following homomorphism:

$$g : V \to V, g(n + w) = n, (n \in N, w \in W).$$

Then, by Lemma 1, $V/(L : g)_V \cong g(V)/(g(V) \cap L) = N/L$. Hence

$$\dim_F(N/L) = \dim_F(V/(L : g)_V) < k.$$

It is obvious that $V/L = N/L \oplus K$ for some subspace K. But $K \cong$ $\cong (V/L)/(N/L) \cong V/N$. Since $N \in U_k$, $\dim_F K = \dim_F(V/N) < k$. Hence $\dim_F(V/L) = \dim_F(N/L) + \dim_F K < k + k = k$. Thus $L \in U_k$.

(C5) Now apply Proposition 1. □

Definition 1. *A non-empty collection $F(M)$ of submodules of a left R-module M satisfying (C1), (C2), (C3) is said to be a (preradical) filter of M.*

Example 2. Let M be a left R-module and $f \in End(M)$ such that $f^n(M)$ is a fully invariant submodule of M for any $n \in \{1, 2, ...\}$. Then $\{L \leq M | \exists n \in \{1, 2, ...\} : f^n(M) \subseteq L\}$ is a collection satisfying (C1), (C2), (C3), (C4), (C5).

Proof. (C1) This is clear.

(C2) Let $f^n(M) \subseteq L$ and $g \in End(M)$. Then $g(f^n(M)) \subseteq f^n(M)$. Hence $g(f^n(M)) \subseteq L$. Therefore $f^n(M) \subseteq (L : g)_M$.

(C3) Let $f^n(M) \subseteq L$, $f^m(M) \subseteq N$, and $n \leq m$. $f^m(M) \subseteq f^n(M)$. Hence $f^m(M) \subseteq L \cap N$.

(C4) Let $f^m(M) \subseteq N$, $N \in Gen(M)$, $L \leq N \leq M$, and

$$\forall g \in End(M)_N \exists n(g) : f^{n(g)}(M) \subseteq (L : g)_M.$$

Then it is easily seen that $f^m \in End(M)_N$. Put $n_0 := n(f^m)$. Hence $f^{n_0}(M) \subseteq (L : f^m)_M$. Therefore $f^{n_0+m}(M) \subseteq L$. (C5) Apply Proposition 1. □

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