

Associated prime ideals of weak σ -rigid rings and their extensions

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ABSTRACT. Let R be a right Noetherian ring which is also an algebra over \mathbb{Q} (\mathbb{Q} the field of rational numbers). Let σ be an automorphism of R and δ a σ -derivation of R . Let further σ be such that $a\sigma(a) \in N(R)$ implies that $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R . In this paper we study the associated prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a semiprime right Noetherian ring which is also an algebra over \mathbb{Q} . Let σ and δ be as above. Then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R with $\sigma(U) = U$ and $\delta(U) \subseteq U$ and $P = U[x; \sigma, \delta]$.

We also prove that if R be a right Noetherian ring which is also an algebra over \mathbb{Q} , σ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$ and $\sigma(U) = U$ for all associated prime ideals U of R (viewed as a right module over itself), then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R such that $(P \cap R)[x; \sigma, \delta] = P$ and $P \cap R = U$.

1. Introduction and preliminaries

Notation: All rings are associative with identity. Throughout this paper R denotes a ring with identity $1 \neq 0$. The prime radical of R is denoted by $P(R)$. The set of nilpotent elements of R is denoted by $N(R)$. The fields

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of rational numbers, real numbers and complex numbers are denoted by \mathbb{Q} , \mathbb{R} , \mathbb{C} respectively. For any subset J of a right R -module M , annihilator of J is denoted by $Ann(J)$. The set of prime ideals of R is denoted by $Spec(R)$, the set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$, and the set of minimal prime ideals of R is denoted by $Min.Spec(R)$. Let R be a right Noetherian ring. For any uniform right R -module J , the assassin of J is denoted by $Assas(J)$. Let M be a right R -module. Consider the set

$$\{Assas(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}.$$

We denote this set by $\mathbb{A}(M_R)$.

Remark 1.1. If R is viewed as a right module over itself, we note that $Ass(R_R) = \mathbb{A}(R_R)$ (5Y of Goodearl and Warfield [8]).

For any two ideals I, J of R ; $I \subset J$ means that I is strictly contained in J .

Let K be an ideal of a ring R such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\bigcap_{i=1}^m \sigma^i(K)$ by K^0 .

Ore extensions: Let R be a ring, σ an endomorphism of R and δ a σ -derivation of R ($\delta : R \rightarrow R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$).

For example let σ be an endomorphism of a ring R and $\delta : R \rightarrow R$ any map.

Let $\phi : R \rightarrow M_2(R)$ defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R \text{ be a ring homomorphism.}$$

Then δ is a σ -derivation of R .

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ -stable; i.e. $\sigma(I) = I$ and I is δ -invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R , i.e. $\{\sum_{i=0}^n x^i a_i, a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients of the polynomials on the right as followed in McConnell and Robson [13].

In case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x; \sigma]$ by $S(I)$.

In case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x; \delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [1, 4, 5, 7, 8, 10, 12, 13].

Prime ideals: This article concerns the study of prime ideals of Ore extensions (skew polynomial rings). Regarding associated prime ideals of Ore extension $R[x; \sigma, \delta]$, we have the following from S. Annin [1]:

Definition (2.1) of Annin [1]: Let R be a ring and M_R be a right R -module. Let σ be an endomorphism of R and δ be a σ -derivation of R . M_R is said to be σ -compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. Moreover M_R is said to be δ -compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ implies $m\delta(r) = 0$. If M_R is both σ -compatible and δ -compatible, M_R is said to be $(\sigma - \delta)$ -compatible.

Theorem (2.3) of Annin [1]: Let R be a ring. Let σ be an endomorphism of R and δ a σ -derivation of R and M_R be a right R -module. If M_R is $(\sigma - \delta)$ -compatible, then $Ass(M[x]_S) = \{P[x] \mid P \in Ass(M_R)\}$.

In [12], Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R -module M_R and that of the induced S -module M_S , where $S = R[x; \sigma, \delta]$ (σ is an automorphism and δ is a σ -derivation of a ring R). They have proved the following:

Theorem (5.7) of [12]: Suppose M_R contains enough prime submodules and let for $Q \in Ass(M_S)$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in Ass(M_R)$.

Motivated by these developments, I investigated the nature of associated prime ideals of $R[x; \sigma, \delta]$ over a right Noetherian ring R and their relation with those of the coefficient ring R . In this way I generalized Theorem (2.4) and Theorem (3.7) of Bhat [4] for associated prime ideals case. The minimal prime ideal case has been generalized in Lemma (2.2) of Bhat [5].

Before we state these known results we require the following notation:

Let R be a right Noetherian ring. We know that $Ass(R_R)$ is finite and $\sigma^j(U) \in Ass(R_R)$ for any $U \in Ass(R_R)$, and for all integers $j \geq 1$, therefore, there exists an integer $m \geq 1$ such that $\sigma^m(U) = U$ for all $U \in Ass(R_R)$. We denote $\bigcap_{i=1}^m \sigma^i(U)$ by U^0 as mentioned in the introduction. Since $Min.Spec(R)$ is also finite, same notation for $Min.Spec(R)$ also.

Theorem (2.4) of [4]: Let R be a right Noetherian ring and σ be an automorphism of R . Then:

1. $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $(P \cap R) = U^0$.

2. $P \in \text{Min.Spec}(S(R))$ if and only if there exists $U \in \text{Min.Spec}(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.

Theorem (3.7) of [4]: Let R be a right Noetherian \mathbb{Q} -algebra and δ be a derivation of R . Then:

1. $P \in \text{Ass}(D(R)_{D(R)})$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Ass}(R_R)$.
2. $P \in \text{Min.Spec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Min.Spec}(R)$.

Before we state the main result, we require the following:

Weak σ -rigid rings:

Let R be a ring and σ be an endomorphism of R . Recall that in [11], σ is called a rigid endomorphism if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$, and R is called a σ -rigid ring.

Example 1.2. Let $R = \mathbb{C}$, and $\sigma : R \rightarrow R$ be the map defined by $\sigma(a + ib) = a - ib$, $a, b \in \mathbb{R}$. Then it can be seen that R is a σ -rigid ring.

Definition 1.3. (Ouyang [14]): Let R be a ring and σ be an endomorphism of R . Then R is said to be a weak σ -rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.

Example 1.4. (Example (2.1) of Ouyang [14]: Let σ be an endomorphism of a ring R such that R is a σ -rigid ring. Let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R . Now σ can be extended to an endomorphism $\bar{\sigma}$ of A by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. The it can be seen that A is a weak $\bar{\sigma}$ -rigid ring.

2. Main results

We now state the main result in the form of the following Theorem:

Theorem A: Let R be a semiprime right Noetherian ring, which is also an algebra over \mathbb{Q} . Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ be a σ -derivation of R . Then $P \in \text{Ass}(O(R)_{O(R)})$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $O(P \cap R) = P$ and $(P \cap R) = U$. This result has been proved in Theorem (2.6).

Towards the proof of the above Theorem, we require the following:

Recall that an ideal of a ring R is said to be completely semiprime if $a^2 \in R$ implies that $a \in R$.

Let R be a Noetherian ring and σ an automorphism of R . We now give a necessary and sufficient condition for R to be a weak σ -rigid ring in the following Theorem:

Theorem 2.1. *Let R be a Noetherian ring. Let σ be an automorphism of R . Then R is a weak σ -rigid ring if and only if $N(R)$ is completely semiprime.*

Proof. First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R . Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$ is an ideal of R . Now I is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ -rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R)$. Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Conversely let $N(R)$ be completely semiprime. We will show that R is a weak σ -rigid ring. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Hence R is a weak σ -rigid ring. \square

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime ideal. We note that a reduced is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [3, 9].

Proposition 2.2. *Let R be a 2-primal right Noetherian ring which is also an algebra over \mathbb{Q} . Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ a σ -derivation of R . Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{Min.Spec}(R)$.*

Proof. Let R be 2-primal weak σ -rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$.

We will now show that $P(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

We next show that $\sigma(U) = U$ for all $U \in \text{Min.Spec}(R)$. Let $U = U_1$ be a minimal prime ideal of R . Let U_2, U_3, \dots, U_n be the other minimal primes of R . Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of

R . Renumber so that $\sigma(U) = U_n$. Let $a \in \bigcap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \bigcap_{i=1}^n U_i = P(R)$. Now $P(R)$ is completely semiprime implies that $a \in P(R)$, and thus $\bigcap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Let now $V = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$.

First of all, we will show that V is an ideal of R . Let $a, b \in V$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1$. Therefore $a - b \in V$. Also it is easy to see that for any $a \in V$ and for any $r \in R$, $ar \in V$ and $ra \in V$. Therefore V is a δ -invariant ideal of R .

We will now show that $V \in \text{Spec}(R)$. Suppose $V \notin \text{Spec}(R)$. Let $a \notin V, b \notin V$ be such that $aRb \subseteq V$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq V$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $V \in \text{Spec}(R)$. Now $V \subseteq U$, so $V = U$ as $U \in \text{Min.Spec}(R)$. Hence $\delta(U) \subseteq U$. \square

Corollary 2.3. *Let R be a 2-primal right Noetherian ring which is also an algebra over \mathbb{Q} . Let σ be an automorphism of R such that $\sigma(U) = U$ for all $U \in \text{Min.Spec}(R)$. Let δ be a σ -derivation of R . Then $\delta(U) \subseteq U$.*

Lemma 2.4. *Let R be a right Noetherian ring which is also an algebra over \mathbb{Q} . Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ a σ -derivation of R . Then*

1. *If U is a minimal prime ideal of R , then $O(U)$ is a minimal prime ideal of $O(R)$ and $O(U) \cap R = U$.*
2. *If P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R .*

Proof. (1) Let U be a minimal prime ideal of R . Then by Proposition (2.2) $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now on the same lines as in Theorem (2.22) of Goodearl and Warfield [8] we have $O(U) \in \text{Spec}(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R , a contradiction. Therefore $O(U) \in \text{Min.Spec}(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now the proof follows on the same lines as in Theorem (2.22) of Goodearl and Warfield [8] using Lemma (2.1) and Lemma (2.2) of Bhat [2] and Proposition (2.2). \square

Theorem 2.5 (Hilbert Basis Theorem). *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

Proof. See Theorem (2.6) of Goodearl and Warfield [8]. \square

With this we now state and prove Theorem A:

Theorem 2.6. *Let R be a semiprime right Noetherian ring, which is also an algebra over \mathbb{Q} . Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ be a σ -derivation of R . Then $P \in \text{Ass}(O(R)_{O(R)})$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. $O(R)$ is right Noetherian by Theorem (2.5). Let $P \in \text{Ass}(O(R)_{O(R)})$. Now by Remark (1.1) $\text{Ass}(O(R)_{O(R)}) = \mathbb{A}(O(R)_{(R)})$. Let $P = \text{Ann}(I) = \text{Assas}(I)$ for some ideal I of $O(R)$ such that I is uniform as a right $O(R)$ -module. Choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Let $U = \text{Ann}(a_n R) = \text{Assas}(a_n R)$. Now R is right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$, and since R is semiprime, $U \in \text{Min.Spec}(R)$ by Proposition (2.2.14) of McConnell and Robson [13]. Now R is a weak σ -rigid ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$, i.e. $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$. Now R is right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$, and since R is semiprime, $U \in \text{Min.Spec}(R)$ by Proposition (2.2.14) of McConnell and Robson [13]. Now R is a weak σ -rigid ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now it can be easily seen that $O(U) = \text{Ann}(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = \text{Ann}(cO(R)) = \text{Assas}(cO(R))$. \square

Example 2.7. 1. R as in Example 1.2 is a semiprime weak σ -rigid ring, but R being a field has no ideals and is therefore a trivial example.

2. Let τ be the conjugacy map on \mathbb{C} . Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

Define $\sigma : R \rightarrow R$ by $\sigma((a_{ij})) = (\tau(a_{ij}))$. Then it can be seen that σ is an endomorphism of R and R is a weak σ -rigid ring.

Now for any $s \in R$, define $\delta_s : R \rightarrow R$ by $\delta_s(a) = as - \sigma(a)$, for $a \in R$. Then δ_s is a σ -derivation of R .

Let

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \in \text{Ass}(R_R).$$

In fact $U = \text{Ann}(I) = \text{Assas}(I)$, where $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{C} \right\}$ is a right ideal of R . Now we note that $\sigma(I) = I$, $\delta_s(I) \subseteq I$, Then it can be seen that σ is an endomorphism of R and $\sigma(U) \subseteq U$. and $\delta_s(U) \subseteq U$. Also $O(U) \in \text{Ass}(O(R)_{O(R)})$. In fact $O(U) = \text{Ann}(O(I)) = \text{Assas}(O(I))$.

Example 2.8. Now let $R = F \times F$, F a field and $\sigma : R \rightarrow R$ defined by $\sigma((u, v)) = (v, u)$ for $u, v \in F$. Then σ is an automorphism of R . But R is not a weak σ -rigid ring as for any $0 \neq a \in F$, we have $(a, 0)\sigma((a, 0)) = (0, 0) \in N(R)$, but $(a, 0) \notin N(R)$.

Proposition 2.9. *Let R be a Noetherian \mathbb{Q} -algebra. Let σ be an automorphism of R and δ a σ -derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $U \in \text{Min.Spec}(R)$ with $\sigma(U) = U$ implies that $\delta(U) \subseteq U$.*

Proof. See Lemma (2.6) of Bhat [6]. □

We now prove the following Theorem:

Theorem 2.10. *Let R be a right Noetherian ring which is also an algebra over \mathbb{Q} , σ be an automorphism of R and δ a σ -derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$ and $\sigma(U) = U$ for all $U \in \mathbb{A}(R_R)$. Then $P \in \text{Ass}(O(R)_{O(R)})$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. $O(R)$ is right Noetherian by Theorem (2.5). Let $J \in \text{Ass}(O(R)_{O(R)})$. Now by Remark (1.1) $\text{Ass}(O(R)_{O(R)}) = \mathbb{A}(O(R)_{(R)})$. Let $P = \text{Ann}(I) = \text{Assas}(I)$ for some ideal I of $O(R)$ such that I is uniform as a right $O(R)$ -module. Choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Let $U = \text{Ann}(a_n R) = \text{Assas}(a_n R)$. Now R is right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$. Now by hypothesis $\sigma(U) = U$, and therefore, Proposition (2.9) implies that $\delta(U) \subseteq U$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$. So $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$. Now R is right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$. Now by hypothesis $\sigma(U) = U$, and therefore, Proposition (2.9) implies that $\delta(U) \subseteq U$. Now it can be easily seen that $O(U) = \text{Ann}(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = \text{Ann}(cO(R)) = \text{Assas}(cO(R))$. □

Example 2.11. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. Then

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \in \text{Ass}(R_R).$$

In fact $U = Ann(I) = Assas(I)$, where $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{R} \right\}$ is a right ideal of R .

Let $\sigma : R \rightarrow R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and $\sigma(U) \subseteq U$. For any $s \in R$, define $\delta_s : R \rightarrow R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ -derivation of R . Also we see that $\sigma(\delta_s(u)) = \delta_s(\sigma(u))$ for all $u \in R$. For let $u = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $s = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix}$. Then

$$\sigma(\delta_s(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \delta_s(\sigma(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we note that $\sigma(I) = I$, $\delta_s(I) \subseteq I$ and $\delta_s(U) \subseteq U$. Also $O(U) \in Ass(O(R)_{O(R)})$. In fact $O(U) = Ann(O(I)) = Assas(O(I))$.

Example 2.12. Let $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. Then $P = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} \in Ass(R_R)$.

In fact $P = Ann(I)$ where $I = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{R} \end{pmatrix}$ is a right ideal of R . Let

$\sigma : R \rightarrow R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and $\sigma(P) \subseteq P$.

For any $s \in R$, define $\delta_s : R \rightarrow R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ -derivation of R . But we see that $\sigma(\delta_s(u)) \neq \delta_s(\sigma(u))$ for all $u \in R$. Let $u = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$. Then

$$\sigma(\delta_s(u)) = \begin{pmatrix} 0 & pb + qc - aq \\ 0 & 0 \end{pmatrix} \text{ and } \delta_s(\sigma(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 2.13. Let $R = \mathbb{R} \times \mathbb{R}$, $\sigma : R \rightarrow R$ defined by $\sigma((a, b)) = (b, a)$ for $a, b \in \mathbb{R}$. Then σ is an automorphism of R . Let now $r \in \mathbb{R}$. Define $\delta_r : R \rightarrow R$ by $\delta_r((a, b)) = (a, b)r - r\sigma((a, b))$ for $a, b \in \mathbb{R}$. Then δ is a σ -derivation. Now for any $(a, b) \in R$,

$$\begin{aligned} \sigma(\delta_r((a, b))) &= \sigma((u, v)r - r\sigma((u, v))) = \\ &= \sigma((u, v)r - r(v, u)) = \sigma((ur, vr) - \sigma(vr, ur)) = (vr, ur) - (ur, vr). \end{aligned}$$

Also

$$\begin{aligned} \delta_r(\sigma((u, v))) &= \delta_r(v, u) = (v, u)r - r\sigma((v, u)) = \\ &= (v, u)r - r(u, v) = (vr, ur) - (ur, vr). \end{aligned}$$

Therefore $\sigma(\delta((u, v))) = \delta(\sigma((u, v)))$ for all $(u, v) \in R$. We see that $U = 0 \times \mathbb{R} \in Ass(R_R)$. In fact $U = Ann(\mathbb{R} \times \{0\}) = Assas(\mathbb{R} \times \{0\})$. But we note that $\sigma(U) \neq U$.

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