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Some combinatorial problems in the theory of symmetric inverse semigroups

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Abstract. Let $X_n = \{1, 2, \cdots, n\}$ and let $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\,\alpha \subseteq X_n$ be a (partial) transformation on X_n. On a partial one-one mapping of X_n the following parameters are defined: the *height* of α is $h(\alpha) = |\,\mathrm{Im}\,\alpha|$, the *right [left] waist* of α is $w^+(\alpha) = \max(\mathrm{Im}\,\alpha)[w^-(\alpha) = \min(\mathrm{Im}\,\alpha)]$, and fix of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |\{x \in X_n : x\alpha = x\}|$. The cardinalities of some equivalences defined by equalities of these parameters on $\mathcal{I}_n$, the semigroup of partial one-one mappings of X_n, and some of its notable subsemigroups that have been computed are gathered together and the open problems highlighted.[1] [2]

1. Introduction and preliminaries

Let $X_n = \{1, 2, \cdots, n\}$. Then a (partial) transformation $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\,\alpha \subseteq X_n$ is said to be *full* or *total* if $\mathrm{Dom}\,\alpha = X_n$; otherwise it is called *strictly* partial. Then the *height* of α is $|\,\mathrm{Im}\,\alpha|$, the *right [left] waist* of α is $w^+(\alpha) = \max(\mathrm{Im}\,\alpha)[w^-(\alpha) = \min(\mathrm{Im}\,\alpha)]$, and *fix* of α is denoted by $f(\alpha)$, and defined by

$$f(\alpha) = |F(\alpha)| = |\{x \in X_n : x\alpha = x\}|.$$

[1]The ideas for this work were formed during a one month stay at Wilfrid Laurier University in the Summer of 2007.

[2]This paper is based on the talk I gave at the 22nd BCC Conference, University of St Andrews, July 2009.

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Of course, other parameters have been defined and many more could still be defined but we shall restrict ourselves to only these, in this paper. It is also well-known that a partial transformation ϵ is *idempotent* ($\epsilon^2 = \epsilon$) if and only if $\operatorname{Im}\epsilon = F(\epsilon)$, and a partial transformation α is *nilpotent* if $\alpha^k = \varnothing$ (the empty or zero map) for some positive integer k. It is worth noting that to define the left (right) waist of a transformation the base set X_n must be totally ordered. The main object of study in this paper is $\mathcal{I}_n$, the semigroup of partial one-one mappings of X_n (more commonly known as the *symmetric inverse semigroup*) and some of its notable subsemigroups. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Many numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers [15, pp. 42 & 96], the factorial [26, 31], the Fibonacci number [13], binomial numbers [14, 10], Catalan numbers [7], Eulerian numbers [18], Schröder numbers [20], Narayana numbers [18], Lah numbers [16, 17], etc., have all featured in these enumeration problems. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [28] but there are also others that are not yet or have just been recorded in [28]. This paper has two main objectives: first, to gather together the various scattered enumeration results; and second, to highlight open problems. Let S be a set of partial one-one transformations on X_n. Next, let

$$F(n;p,m,k) = |\{\alpha \in S : h(\alpha) = p \wedge f(\alpha) = m \wedge w^+(\alpha) = k\}|$$

and, let

$$F(n;p,m) = |\{\alpha \in S : h(\alpha) = p \wedge f(\alpha) = m\}|,$$
$$F(n;p,k) = |\{\alpha \in S : h(\alpha) = p \wedge w^+(\alpha) = k\}|,$$
$$F(n;m,k) = |\{\alpha \in S : f(\alpha) = m \wedge w^+(\alpha) = k\}|.$$

Further, let

$$F(n;k) = |\{\alpha \in S : w^+(\alpha) = k\}|,$$
$$F(n;m) = |\{\alpha \in S : f(\alpha) = m\}|,$$
$$F(n;p) = |\{\alpha \in S : h(\alpha) = p\}|.$$

It is not difficult to see that

$$|S| = \sum_k F(n;k) = \sum_m F(n;m) = \sum_p F(n;p),$$

and any two-variable function can be expressed as a sum of appropriate three-variable functions and so on. Ideally, we would like to compute

$F(n; p, m, k)$ for any finite semigroup of partial one-one transformations but at the moment this seems to be a difficult proposition and so we have to start from the smaller-variable functions to higher-variable functions. It appears that many important integer sequences can be realized as sequences counting these functions in various partial one-one transformation semigroups - akin to Cameron's remark about oligomorphic permutation groups [2]. In $\mathcal{I}_n$ and its subsemigroups of order-preserving/order-reversing, order-decreasing and orientation-preserving/orientation-reversing transformations we have expressions for $|S|$ and most of the two-variable functions and only a few three-variable functions.

Types of bijective transformations	Semigroup
Permutations	$\mathcal{S}_n$
Partial one-one transformations	$\mathcal{I}_n$
Order-preserving	$\mathcal{IO}_n$
Order-preserving or order-reversing	$\mathcal{PODI}_n$
Order-decreasing	$\mathcal{ID}_n$
Order-preserving or order-decreasing	$\mathcal{IC}_n$
Orientation-preserving	$\mathcal{POPI}_n$
Orientation-preserving or orientation-reversing	$\mathcal{PORI}_n$

Table 1

We shall present the known results by means of tables and exhibit/ explain some of the techniques that have been used to obtain these results, as well as explore some of the open problems. In the next section, (Section 2) we consider the symmetric group $\mathcal{S}_n$, and the symmetric inverse semigroup, $\mathcal{I}_n$. In Section 3 we consider the order-preserving/order-reversing subsemigroups of $\mathcal{I}_n$ and in Section 4, we consider its order-decreasing version, while in Section 5 we consider its order-preserving and order-decreasing version. In Section 6 we consider the orientation-preserving/ orientation-reversing subsemigroups of $\mathcal{I}_n$. Concluding remarks form the contents of Section 7.

2. The symmetric inverse semigroup

For more detailed studies of the symmetric inverse semigroup, $\mathcal{I}_n$ we refer the reader to the books [24, 15, 9] and the papers [12, 16]. First, note that $k = w^+(\alpha)$ is undefined when $p = 0$. Due to the presence of the empty map, it seems reasonable to define $k = 0$ if $p = 0$; and $F(n; k) = F(n; p, k) = 1$ if $k = p = 0$. This, and other observations we record in the following lemma, which will be used implicitly whenever needed.

Lemma 2.1. *Let $X_n = \{1, 2, \cdots, n\}$ and $P = \{p, m, k\}$, where for a given $\alpha \in \mathcal{I}_n$, we set $p = h(\alpha), m = f(\alpha)$ and $k = w^+(\alpha)$. We also define $F(n; k) = F(n; p, k) = 1$ if $k = p = 0$. Then we have the following:*

1. $n \geq k \geq p \geq m \geq 0$;
2. $k = 1 \implies p = 1$;
3. $p = 0 \Leftrightarrow k = 0$

The following proposition is easy to prove, nevertheless, we include its proof to demonstrate the technique rather than because it is new.

Proposition 2.2. *Let $S = \mathcal{I}_n$. Then*

$$F(n; p, k) = \binom{n}{p}\binom{k-1}{p-1}p!, (n \geq k \geq p \geq 0).$$

Proof. First observe that the p elements of $\operatorname{Dom}\alpha$ can be chosen from X_n in $\binom{n}{p}$ ways, and since k is the maximum element in $\operatorname{Im}\alpha$ then the remaining $p-1$ elements of $\operatorname{Im}\alpha$ can be chosen from $\{1, 2, \cdots, k-1\}$ in $\binom{k-1}{p-1}$ ways. Finally, observe that the p elements of $\operatorname{Dom}\alpha$ can be tied to the p images in a one-one fashion, in $p!$ ways. The result now follows. □

The following corollaries can easily be deduced:

Corollary 2.3. *Let $S = \mathcal{I}_n$. Then*

$$F(n; p) = \binom{n}{p}^2 p!, (n \geq p \geq 0).$$

Corollary 2.4. *Let $S = \mathcal{I}_n$. Then*

$$F(n; k) = \sum_{p=0}^{k}\binom{n}{p}\binom{k-1}{p-1}p!, (n \geq k \geq 0).$$

Now let $i = a_i = a$, for all $a \in \{p, m, k\}$, and $0 \leq i \leq n$.

Corollary 2.5. *Let $S = \mathcal{I}_n$. Then $F(n; k_n) = n|I_{n-1}|$, $(n \geq 2)$.*

Corollary 2.6. *Let $S = \mathcal{I}_n$. Then $F(n; k_{n-1}) = n|N(I_{n-1})|$, where $N(T)$ is the set of nilpotents in T.*

S	$\lvert S\rvert$	$\lvert E(S)\rvert$	$\lvert N(S)\rvert$
$\mathcal{S}_n$	$n!$	1	0
$\mathcal{I}_n$	$\sum_{p=0}^{n}\binom{n}{p}^2 p! = a_n$ [26] or [9, Theorem 2.5.1, p.22]	2^n	$\sum_{p=0}^{n}\binom{n}{p}\binom{n-1}{p}p!$ $= \sum_{p=0}^{n-1}\lvert L_{n,n-p}\rvert = u_n$ [17], [8] or [9, Theorem 2.8.5, p.30]
$\mathcal{I}_n \setminus \mathcal{S}_n$	$a_n - n!$	$2^n - 1$	same as in above cell

Table 2

$a_n = 2na_{n-1} - (n-1)^2 a_{n-2},\ a_0 = 1, a_1 = 2$ [1] :
$1, 2, 7, 34, 209, 1546, 13327, 130922, \cdots, (A002720)$;
$u_n = (2n-1)u_{n-1} - (n-1)(n-2)u_{n-2},\ u_0 = 1 = u_1$:
$1, 1, 3, 13, 73, 501, 4051, 37633, \cdots, (A000262)$;
$|L_{n,n-p}|$ is the triangle of signless transpose Lah numbers ($A089231$).

The main statement proved above is by direct combinatorial arguments; however, this approach does not always work. Finding recurrences and guessing a closed formula which can then be proved by induction is another approach effectively used in [18, 19, 20, 21, 22]. The success of this approach depends heavily on enumerating methods and techniques that can be found in [3, 25, 29] and identities that can be found in [27]. We (in [23]) are currently using generating functions to investigate some of the unknown cases and it looks very promising.

	$\mathcal{S}_n$	$\mathcal{I}_n$
$F(n;p)$	$n!$ (if $p = n$) and 0 (if $p \neq n$)	$\binom{n}{p}^2 p!$ [26]
$F(n;m)$	$\binom{n}{m} d_{n-m}$	$\frac{n!}{m!} \sum_{i=0}^{n-m} \frac{(-1)^i}{i!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{1}{j!} = \binom{n}{m} f(n-m;0)$ [22]
$F(n;k)$	$n!$ (if $k = n$) and 0 (if $k \neq n$)	$\sum_{p=0}^{k} \binom{n}{p} \binom{k-1}{p-1} p!$ (Corollary 2.4)
$F(n;p,m)$	$F(n;m)$ (if $p = n$) and 0 (if $p \neq n$)	$\frac{n!}{m!(n-p)!} \sum_{j=0}^{p-m} \binom{n-m-j}{p-m-j} \frac{(-1)^j}{j!}$ [22]
$F(n;p,k)$	$F(n;p)$ (if $k = n$) and 0 (if $k \neq n$)	$\binom{n}{p} \binom{k-1}{p-1} p!$ (Proposition 2.2)
$F(n;m,k)$	$F(n;m)$ (if $k = n$) and 0 (if $k \neq n$)	?
$F(n;p,m,k)$	$F(n;m)$ (if $k = p = n$) and 0 (otherwise)	?

Table 3

$d_n = 1, 0, 1, 2, 9, 44, 265, 1854, 14833, \cdots$, derangements ($A000166$)
$(n-m)F(n;m) = n(2n-2m-1)F(n-1;m) -$
$n(n-1)(n-m-3)F(n-2;m) - n(n-1)(n-2)F(n-3;m)$ ($A144088$)
$f(n;0) : 1, 1, 4, 18, 108, 780, 6600, \cdots$, partial derangements ($A144085$).

3. Order-preserving or order-reversing partial one-one transformations

A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving (order-reversing)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ x \leq y \implies x\alpha \leq y\alpha\ (x\alpha \geq y\alpha)$. The semigroups of order-preserving and order-preserving or order-reversing partial one-one transformations of X_n will be denoted by $\mathcal{IO}_n$ and $\mathcal{PODI}_n$, respectively.

The general study of $\mathcal{IO}_n$ was initiated in [11] while $\mathcal{PODI}_n$ first appeared in [5].

Remark 3.1. For $p = 0, 1$ the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p \geq 2$.

Now we announce some new results of the author (and his coauthor), whose detail proofs are going to appear in [23].

Proposition 3.2. *Let $S = \mathcal{PODI}_n$. Then $F(n; p, k) = 2\binom{n}{p}\binom{k-1}{p-1}$, (if $n \geq k \geq p \geq 2$) and equals n (if $k = 1$ or $p = 1$).*

Corollary 3.3. *Let $S = \mathcal{PODI}_n$. Then $F(n; p) = 2\binom{n}{p}^2, (n \geq p > 1)$ and equals n^2 $(if\ p = 1)$.*

Corollary 3.4. *Let $S = \mathcal{PODI}_n$. Then $F(n; k) = 2\binom{n+k-1}{k} - n, (n \geq k \geq 1)$.*

S	$\|S\|$	$\|E(S)\|$	$\|N(S)\|$
$\mathcal{IO}_n$	$\binom{2n}{n}$ [11]	2^n	b_n[23]
$\mathcal{PODI}_n$	$2\binom{2n}{n} - n^2 - 1$ [5]	2^n	?

Table 4

$(n+1)b_{n+1} = 2(4n+1)b_n - 3(5n-3)b_{n-1} - 2(2n-1)b_{n-2},\ b_0 = 1 = b_1$: $1, 1, 3, 9, 29, 97, 333, 1165, 4135, \cdots$ (A081696).

	$\mathcal{IO}_n$	$\mathcal{PODI}_n$
$F(n; p)$	$\binom{n}{p}^2$ [11]	$2\binom{n}{p}^2$ (if $p > 1$) and n^2 (if $p = 1$) (Corollary 3.3)
$F(n; m)$	$\sum_{j=m}^{n} F(j-1; m-1)b_{n-j}$ [23]	?
$F(n; k)$	$\binom{n+k-1}{k}$ [23]	$2\binom{n+k-1}{k} - n$ (Corollary 3.4)
$F(n; p, m)$	?	?
$F(n; p, k)$	$\binom{n}{p}\binom{k-1}{p-1}$ [23]	$2\binom{n}{p}\binom{k-1}{p-1}$ (if $p > 1$) and n (if $k = 1$ or $p = 1$) (Proposition 3.2)
$F(n; m, k)$	?	?
$F(n; p, m, k)$	?	?

Table 5

4. Order-decreasing partial one-one transformations

A transformation α in $\mathcal{I}_n$ is said to be *order-decreasing (increasing)* if $(\forall x \in \operatorname{Dom}\alpha)\, x\alpha \leq x\, (x\alpha \geq x)$. The two semigroups of order-decreasing and order-increasing partial one-one transformations of X_n are isomorphic [31]. The semigroup of order-decreasing partial one-one transformations of X_n will be denoted by $\mathcal{ID}_n$. The general study of this class of semigroups was initiated in [30].

The following result easily follows from [31, Theorem 4.2].

Proposition 4.1. *Let $S = \mathcal{ID}_n$. Then $F(n;m) = \binom{n}{m} B_{n-m}$, where B_n is the n-th Bell's number.*

Proof. Let $\alpha \in \mathcal{ID}_n$ and let $x_1, x_2, \cdots, x_m$ be the fixed points of α. Since α is one-one and order-decreasing, it follows that for $x \in (X_n \setminus \{x_1, \cdots, x_m\}) \cap \operatorname{Dom}\alpha$ we have $x\alpha \in X_n \setminus \{x_1, \cdots, x_m\}$ and $x\alpha < x$. Therefore the restriction of α to $X_n \setminus \{x_1, \cdots, x_m\}$ is well defined and is a nilpotent element of $\mathcal{I}(X_n \setminus \{x_1, \cdots, x_m\}$. The number of nilpotents that can be formed by these $n - m$ elements (after relabelling) is B_{n-m} (by [31, Theorem 4.2]), and the result now follows. □

Conjecture 4.2. *Let $S = \mathcal{ID}_n$. Then $F(n;k) = \binom{n}{k} B_k$.*

S	$\lvert S\rvert$	$\lvert E(S)\rvert$	$\lvert N(S)\rvert$
$\mathcal{ID}_n$	B_{n+1} [1]	2^n	B_n [31]

Table 6

	$\mathcal{ID}_n$
$F(n;p)$	$S(n, n-p)$ [1]
$F(n;m)$	$\binom{n}{m} B_{n-m}$ (Proposition 4.1)
$F(n;k)$	$\binom{n}{k} B_k$ (Conjecture 4.2)
$F(n;p,m)$	?
$F(n;p,k)$	?
$F(n;m,k)$	?
$F(n;p,m,k)$	?

Table 7

$S(n,r)$ is the Stirling number of the second kind:
$S(n,r) = S(n-1, r-1) + rS(n-1, r)$, $S(n,1) = 1 = S(n,n)$ (A008277).
$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k : 1, 2, 5, 15, 52, 203, 877, 4140, \cdots$ (A000110).

5. Order-preserving and order-decreasing partial one-one transformations

We define $\mathcal{IC}_n = \mathcal{IO}_n \cap \mathcal{ID}_n$ as the semigroup of order-preserving and order-decreasing partial one-one transformations of X_n. Surprisingly, the semigroup $\mathcal{IC}_n$ first appeared in [9] and not much is known about it.

Proposition 5.1. *Let $S = \mathcal{IC}_n$. Then $|N(\mathcal{IC}_n)| = \frac{1}{n}\binom{2n}{n-1} = C_n$, where C_n is the n-th Catalan number.*

Proof. It is similar to the proof of [21, Proposition 2.3]. □

Conjecture 5.2. *Let $S = \mathcal{IC}_n$. Then $F(n;p) = \frac{1}{n-p+1}\binom{n}{p}\binom{n+1}{p+1}$.* (This is known as the Narayana triangle – A001263.)

Conjecture 5.3. *Let $S = \mathcal{IC}_n$. Then $F(n;m) = \frac{m+1}{2n-m+1}\binom{2n-m+1}{n+1}$.* (This is triangle – A033184.)

Conjecture 5.4. *Let $S = \mathcal{IC}_n$. Then $F(n;k) = \frac{n-k+1}{n}\binom{n+k-1}{n-1}$.* (This is known as the Catalan triangle – A009766.)

S	$\|S\|$	$\|E(S)\|$	$\|N(S)\|$
$\mathcal{IC}_n$	C_{n+1} [9, *Theorem* 2.4.8(*ii*)]	2^n	C_n (Proposition 5.1)

Table 8

	$\mathcal{IC}_n$
$F(n;p)$	$\frac{1}{n-p+1}\binom{n}{p}\binom{n+1}{p+1}$ (Conjecture 5.2)
$F(n;m)$	$\frac{m+1}{2n-m+1}\binom{2n-m+1}{n+1}$ (Conjecture 5.3)
$F(n;k)$	$\frac{n-k+1}{n}\binom{n+k-1}{n-1}$ (Conjecture 5.4)
$F(n;p,m)$	?
$F(n;p,k)$	?
$F(n;m,k)$	?
$F(n;p,m,k)$	?

Table 9

6. Orientation-preserving or orientation-reversing partial one-one transformations

Let $a = (a_1, a_2, \cdots, a_t)$ be a sequence of t $(t > 0)$ distinct elements from the chain X_n. We say that a is *cyclic (anti-cyclic)* if there exists no more than one index $i \in \{1, 2, \cdots, t\}$ such that $a_i > a_{i+1}(a_i < a_{i+1})$, where a_{t+1} denotes a_1. For $\alpha \in \mathcal{I}_n$, suppose that Dom $\alpha = \{a_1, a_2, \cdots, a_t\}$, with $t \geq 0$ and $a_1 < a_2 < \cdots < a_t$. We say that α is *orientation-preserving (orientation-reversing)* if $(a_1\alpha, a_2\alpha, \cdots, a_t\alpha)$ is cyclic (anti-cyclic). The semigroups of orientation-preserving and orientation-preserving/ orientation-reversing partial one-one transformations of X_n will be denoted by $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively. The former semigroup first appeared in [4] while the latter in [5].

Remark 6.1. For $p = 0, 1, 2$ the concepts of orientation-preserving and orientation-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p > 2$.

Now we announce new results by the author, whose proofs are given in the preprint [32].

Proposition 6.2. *Let $S = \mathcal{POPI}_n$. Then*

$$F(n; p, k) = \binom{n}{p}\binom{k-1}{p-1}p \ (n \geq k \geq p > 0).$$

Corollary 6.3. *Let $S = \mathcal{POPI}_n$. Then*

$$F(n; p) = \begin{cases} \binom{n}{p}^2 p & (n \geq p > 1), \\ n^2 & (p = 1). \end{cases}$$

Corollary 6.4. *Let $S = \mathcal{POPI}_n$. Then $F(n; k) = n\binom{n+k-2}{k-1}, (n \geq k \geq 1)$.*

Corollary 6.5. *Let $S = \mathcal{POPI}_n$. Then $F(n; k_n) = n\binom{2n-2}{n-1}, (n \geq 1)$.*

Proposition 6.6. *Let $S = \mathcal{PORI}_n$. Then*

$$F(n; p, k) = \begin{cases} 2\binom{n}{p}\binom{k-1}{p-1}p & (n \geq k \geq p > 2), \\ n^2 & (k = 2), \\ 2(k-1)\binom{n}{2} & (p = 2), \\ n & ((k = 1) \vee (p = 1)). \end{cases}$$

Corollary 6.7. *Let $S = \mathcal{PORI}_n$. Then*

$$F(n; p) = \begin{cases} 2\binom{n}{p}^2 p & (n \geq p > 2), \\ \binom{n}{p}^2 p & (p = 1, 2). \end{cases}$$

Corollary 6.8. *Let $S = \mathcal{PORI}_n$. Then*

$$F(n; k) = 2n\binom{n+k-2}{n-1} - n - n(n-1)(k-1), \ (n \geq k > 0).$$

S	$\|S\|$	$\|E(S)\|$	$\|N(S)\|$
$\mathcal{POPI}_n$	$1 + \frac{n}{2}\binom{2n}{n}$ [4]	2^n	?
$\mathcal{PORI}_n$	$1 + n\binom{2n}{n} - n^2(n^2 - 2n + 3)/2$ [5]	2^n	?

Table 10

	$\mathcal{POPI}_n$	$\mathcal{PORI}_n$
$F(n;p)$	$\binom{n}{p}^2 p$ (if $p>1$) n^2 (if $p=1$) (Corollary 6.3)	$2\binom{n}{p}^2 p$ (if $p>2$) and $\binom{n}{p}^2 p$ (if $p=1,2$) (Corollary 6.7)
$F(n;m)$	?	?
$F(n;k)$	$n\binom{n+k-2}{k-1}$ (if $k>0$) (Corollary 6.4)	$2n\binom{n+k-2}{k-1}-n-n(n-1)(k-1)$ (if $k>0$) (Corollary 6.8)
$F(n;p,m)$	?	?
$F(n;p,k)$	$\binom{n}{p}\binom{k-1}{p-1}p$ (if $k\geq p>0$) (Proposition 6.2)	$2\binom{n}{p}\binom{k-1}{p-1}p$ (if $k\geq p>2$) n^2 (if $k=2$) $2(k-1)\binom{n}{2}$ (if $p=2$) n (if $k=1$ or $p=1$) (Proposition 6.6)
$F(n;m,k)$	?	?
$F(n;p,m,k)$	?	?

Table 11

7. Concluding remarks

Remark 7.1 All these combinatorial functions can be computed when restricted to special subsets within a particular semigroup, for example, the set of nilpotents, $N(S)$ [17].

Remark 7.2 We have only considered 7 classes of transformation semigroups: $\mathcal{I}_n$, $\mathcal{IO}_n$, $\mathcal{PODI}_n$, $\mathcal{ID}_n$, $\mathcal{IC}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, however, there are many other classes of transformation semigroups that can be studied from this point of view.

Remark 7.3 When the totally ordered set X_n is replaced by a partially ordered set (poset), for each $n>1$ there are 'several' non-isomorphic posets, each of which gives rise to potentially different combinatorial results.

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References

[1] Borwein, D., Rankin, S. and Renner, L. Enumeration of injective partial transformations. *Discrete Math.* **73** (1989), 291–296.

[2] Cameron, P. J. Sequences realized by oligomorphic permutation groups. *J. Integer Seq.* **3** (2000), 00.1.5, HTML document (electronic).

[3] Comtet, L. *Advanced Combinatorics: the art of finite and infinite expansions*, D. Reidel Publishing Company, Dordrecht, Holland, 1974.

[4] Fernandes, V. H. The monoid of all injective orientation-preserving partial transformations on a finite chain. *Comm. Algebra* **32** (2000), 3401–3426.

[5] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. Presentations for some monoids of injective partial transformations on a finite chain. *Southeast Asian Bull. Math.* **28** (2004), 903–918.

[6] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. The cardinal and idempotent number of various monoids of transformations on a finite chain. (submitted).

[7] Ganyushkin, E. and Mazorchuk, V. On the structure of $\mathcal{IO}_n$. *Semigroup Forum* **66** (2003), 455–483.

[8] Ganyushkin, O. and Mazorchuk, V. Combinatorics of nilpotents in symmetric inverse semigroups. *Ann. Comb.* **8 (2)** (2004), 161–175.

[9] Ganyushkin, O. and Mazorchuk, V. *Classical Finite Transformation Semigroups: An Introduction*, Springer, London, 2009.

[10] Garba, G. U. On the nilpotent ranks of certain semigroups of transformations. *Glasgow Math. J.* **36** (1994), 1–9.

[11] Garba, G. U. Nilpotents in semigroups of partial one-to-one order-preserving mappings. *Semigroup Forum*, **48** (1994), 37–49.

[12] Gomes, G. M. S. and Howie, J. M. Nilpotents in finite symmetric inverse semigroups. *Proc. Edinburgh Math. Soc.*, **30** (1987), 383–395.

[13] Howie, J. M. Products of idempotents in certain semigroups of transformations. *Proc. Edinburgh Math. Soc.* **17** (1971), 223–236.

[14] Howie, J. M., Lusk, E. L. and McFadden, R. B. Combinatorial results relating to product of idempotents in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh Sect. A* **115** (1990), 289–299.

[15] Howie, J. M. *Fundamentals of semigroup theory.* Oxford: Clarendon Press, 1995.

[16] Janson, S. and Mazorchuk, V. Some remarks on the combinatorics of $\mathcal{IS}_n$. *Semigroup Forum*, **109** (2005), 391–405.

[17] Laradji, A. and Umar, A. On the number of nilpotents in the partial symmetric semigroup. *Comm. Algebra*, **32** (2004), 3017–3023.

[18] Laradji, A. and Umar, A. On certain finite semigroups of order-decreasing transformations I. *Semigroup Forum*, **69** (2004), 184–200.

[19] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving partial transformations. *Journal of Algebra*, **278** (2004), 342–359.

[20] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-decreasing partial transformations. *J. Integer Seq.*, **7** (2004), 04.3.8.

[21] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving full transformations. *Semigroup Forum*, **72** (2006), 51–62.

[22] Laradji, A. and Umar, A. Combinatorial results for the symmetric inverse semigroup. *Semigroup Forum*, **75** (2007), 221–236.

[23] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving or order-reversing subpermutations (in preparation).

[24] Limpscomb, S. *Symmetric Inverse Semigroups*, Mathematical Surveys of The American mathematical Society, no. 46, Providence, R. I., 1996.

[25] LIU, C. L. *Introduction to combinatorial mathematics*, McGraw Hill Company, New York, 1968.

[26] MUNN, W. D. The characters of the symmetric inverse semigroup. *Proc. Cambridge Philos. Soc.*, **53** (1957), 13–18

[27] RIORDAN, J. *Combinatorial Identities*, John Wiley and Sons, New York, 1968.

[28] SLOANE, N. J. A. *The On-Line Encyclopedia of Integer Sequences*, @http://www.research.att.com/ ∼ njas/sequences/.

[29] STANLEY, R. P. *Enumerative Combinatorics* Vol. I, Cambridge University Press, 1997.

[30] UMAR, A. *Semigroups of order-decreasing transformations*, Ph. D Thesis, University of St Andrews, 1992.

[31] UMAR, A. On the semigroups of partial one-to-one order-decreasing finite transformations, *Proc. Roy. Soc. Edinburgh*, **123A** (1993), 355–363.

[32] UMAR, A. Some combinatorial results for the semigroups of orientation-preserving partial transformations (submitted).

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