

## Thin systems of generators of groups

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**ABSTRACT.** A subset  $T$  of a group  $G$  with the identity  $e$  is called  $k$ -thin ( $k \in \mathbb{N}$ ) if  $|A \cap gA| \leq k$ ,  $|A \cap Ag| \leq k$  for every  $g \in G$ ,  $g \neq e$ . We show that every infinite group  $G$  can be generated by some 2-thin subset. Moreover, if  $G$  is either Abelian or a torsion group without elements of order 2, then there exists a 1-thin system of generators of  $G$ . For every infinite group  $G$ , there exist a 2-thin subset  $X$  such that  $G = XX^{-1} \cup X^{-1}X$ , and a 4-thin subset  $Y$  such that  $G = YY^{-1}$ .

For a group  $G$  we denote by  $\mathcal{F}_G$  the family of all finite subsets of  $G$ . A subset  $A$  of an infinite group  $G$  with the identity  $e$  is said to be

- *left (right) large* if there exists  $F \in \mathcal{F}_G$  such that  $G = FA$  ( $G = AF$ );
- *large* if  $A$  is left and right large;
- *left (right) small* if  $G \setminus FA$  ( $G \setminus AF$ ) is left (right) large for every  $F \in \mathcal{F}_G$ ;
- *small* if  $A$  is left and right small;
- *left (right) P-small* if there exists an injective sequence  $(g_n)_{n \in \omega}$  in  $G$  such that the subsets  $\{g_n A : n \in \omega\}$  ( $\{Ag_n : n \in \omega\}$ ) are pairwise disjoint;
- *P-small* if  $A$  is left and right P-small;
- *left (right) k-thin* for  $k \in \mathbb{N}$  if  $|gA \cap A| \leq k$  ( $|Ag \cap A| \leq k$ ) for every  $g \in G$ ,  $g \neq e$ ;

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- *k*-thin, if  $A$  is left and right *k*-thin.

For the relationships between these types of subsets see [3]. In particular, every *k*-thin subset is small, but a small subset could be much more big than every *k*-thin subsets. For example, every *k*-thin subset  $T$  is a universal zero, i.e.  $\mu(T) = 0$  for every Banach measure  $\mu$  on  $G$ . On the other hand, for every countable amenable group  $G$  and every  $\varepsilon > 0$ , there exist a small subset  $S$  and Banach measure  $\mu$  on  $G$  such that  $\mu(S) > 1 - \varepsilon$ . We note also that a subset  $A$  is left *k*-thin if and only if  $A^{-1}$  is right *k*-thin.

Answering a question from [4], I. V. Protasov [5] (see also [6, Theorem 13.1]) proved that every infinite group  $G$  can be generated by some small subset. Moreover, there exists a small and P-small generating subset of  $G$  [2].

In this paper we show (Theorem 1) that every infinite group  $G$  can be generated by some 2-thin subset. Moreover, if  $G$  is either Abelian or torsion group with no elements of order 2, then  $G$  can be generated by some 1-thin subset. By Theorem 2, for every infinite group  $G$ , there exists a 2-thin subset  $X$  such that  $G = XX^{-1} \cup X^{-1}X$ . Since every *k*-thin subset is small, this is an answer to the Question 13.2 from [6]. We show also that, in every infinite group  $G$ , there is a 4-thin subset  $X$  such that  $G = XX^{-1}$ .

Given a subset  $X$  of a group  $G$ , we denote by  $\langle X \rangle$  the subgroup of  $G$  generated by  $X$ .

**Theorem 1.** *Every infinite group  $G$  has a 2-thin system of generators. Moreover, if  $G$  has no elements of order 2 and  $G$  is either Abelian or a torsion group, then there exists a 1-thin system of generators of  $G$ .*

*Proof.* Let  $|G| = \kappa$ . We construct inductively an increasing system  $\{G_\alpha : \alpha < \kappa\}$  of subgroups of  $G$  and a subset  $X = \{x_\alpha : \alpha < \kappa\}$  such that

- (i)  $G_0 = \langle e \rangle$ ,  $G = \bigcup_{\alpha < \kappa} G_\alpha$ ;
- (ii)  $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$  for every limit ordinal  $\beta < \kappa$ ;
- (iii)  $G_{\alpha+1} = \langle G_\alpha, x_\alpha \rangle$  for every  $\alpha < \kappa$ .

Clearly,  $G = \langle X \rangle$ . We suppose that  $X$  is not left 2-thin and choose  $g \in G$ ,  $g \neq e$ , distinct ordinals  $\alpha_1, \alpha_2, \alpha_3$  such that  $gx_{\alpha_1}, gx_{\alpha_2}, gx_{\alpha_3} \in X$ . Let  $gx_{\alpha_1} = x_{\beta_1}$ ,  $gx_{\alpha_2} = x_{\beta_2}$ ,  $gx_{\alpha_3} = x_{\beta_3}$ . By the pigeonhole principle, there exist distinct  $k, l \in \{1, 2, 3\}$  such that either  $\alpha_k < \beta_k$ ,  $\alpha_l < \beta_l$  or  $\alpha_k > \beta_k$ ,  $\alpha_l > \beta_l$ . Let  $\alpha_k < \beta_k$ ,  $\alpha_l < \beta_l$ . Then  $x_{\beta_k}x_{\alpha_k}^{-1} \in G_{\beta_k+1} \setminus G_{\beta_k}$ ,  $x_{\beta_l}x_{\alpha_l}^{-1} \in G_{\beta_l+1} \setminus G_{\beta_l}$  and  $g = x_{\beta_k}x_{\alpha_k}^{-1} = x_{\beta_l}x_{\alpha_l}^{-1}$ , which is impossible

because  $(G_{\beta_{k+1}} \setminus G_{\beta_k}) \cap (G_{\beta_{l+1}} \setminus G_{\beta_l}) = \emptyset$ . Hence,  $X$  is left 2-thin. The same arguments show that  $X$  is right 2-thin.

To prove the second statement, we assume that the constructed above subset  $X$  is not left 1-thin. Then there exist distinct  $\alpha, \beta < \kappa$  and  $g \neq e$  such that  $gx_\alpha, gx_\beta \in X$ . Let  $gx_\alpha = x_{\alpha'}$ ,  $gx_\beta = x_{\beta'}$ . We choose the minimal  $\lambda < \kappa$  such that  $\alpha, \beta, \alpha', \beta' < \lambda$ . Clearly,  $\lambda = \gamma + 1$  for some  $\gamma < \kappa$ . Replacing  $g$  by  $g^{-1}$ , we may suppose that  $\alpha = \gamma$  so  $x_\alpha \in G_{\gamma+1} \setminus G_\gamma$ . Since  $x_{\alpha'} = gx_\alpha$  and  $\alpha' < \alpha$  then  $g \in G_{\alpha+1} \setminus G_\alpha$ . It follows that  $gx_\beta = x_\alpha$  and

$$(*) \quad g^2 x_\beta = x_{\alpha'};$$

$$(**) \quad x_\alpha^2 = x_{\alpha'} x_\beta \text{ if } G \text{ is Abelian.}$$

Let  $G$  be a torsion group with no elements of order 2. Then  $(*)$  is impossible because  $g \in G_{\alpha+1} \setminus G_\alpha$  and  $g^2 \in G_\alpha$ . It follows that  $X$  is left 1-thin. The same arguments show that  $X$  is right 1-thin.

Let  $G$  be an Abelian group with no elements of order 2. We choose a system  $\{G_\alpha : \alpha < \kappa\}$  of subgroups of  $G$  satisfying (i), (ii), (iii) and

$$(iv) \quad G_{\alpha+1}/G_\alpha \simeq \mathbb{Z} \text{ or } G_{\alpha+1}/G_\alpha \simeq \mathbb{Z}_p \text{ for some prime number } p.$$

We construct  $X = \{x_\alpha : \alpha < \kappa\}$  inductively by the following rule. If  $G_{\alpha+1}/G_\alpha$  is not isomorphic to  $\mathbb{Z}_2$ , we choose an arbitrary element  $x_\alpha \in G_{\alpha+1} \setminus G_\alpha$ . Let  $G_{\alpha+1}/G_\alpha \simeq \mathbb{Z}_2$  and  $G_{\alpha+1} = \langle G_\alpha, y_\alpha \rangle$ . If  $y_\alpha^2 \neq x_{\alpha'} x_\beta$  for all distinct  $\alpha', \beta < \alpha$ , we put  $x_\alpha = y_\alpha$ . If  $y_\alpha^2 = x_{\alpha'} x_\beta$  for some distinct  $\alpha', \beta < \alpha$ ,  $\beta < \alpha'$ , we put  $x_\alpha = y_\alpha x_\beta^{-1}$ . Then  $x_\alpha^2 = x_{\alpha'} x_\beta^{-1}$ . If  $x_\alpha^2 = x_{\alpha''} x_{\beta'}$  for some distinct  $\alpha'', \beta' < \alpha$ ,  $\beta' < \alpha''$  then  $x_{\alpha'} x_\beta^{-1} = x_{\alpha''} x_{\beta'}$ . Since  $\beta < \alpha'$  and  $\beta' < \alpha''$ , we have  $\alpha' = \alpha''$ . Hence,  $x_\beta^{-1} = x_{\beta'}$ , but it is impossible, so  $x_\alpha^2 \neq x_{\alpha''} x_{\beta'}$  for all distinct  $\alpha'', \beta' < \alpha$ . If  $X$  is not 1-thin, by  $(**)$ , we get a contradiction with construction of  $X$ .  $\square$

**Question 1.** Let  $G$  be an infinite group with no elements of order 2. Does there exist a 1-thin system of generators of  $G$ ?

**Theorem 2.** For every infinite group  $G$ , there exists a 2-thin subset  $X$  such that  $G = XX^{-1} \cup X^{-1}X$ .

*Proof.* Let  $|G| = \kappa$ ,  $\{g_\alpha : \alpha < \kappa\}$  be a numeration of  $G$ . We construct inductively a family  $\{X_\alpha : \alpha < \kappa\}$  of 2-thin subsets of  $G$  of the form  $X_\alpha = \{x_\beta, y_\beta x_\beta : \beta < \alpha\}$  so that  $\{g_\beta : \beta < \alpha\} \subseteq X_\alpha X_\alpha^{-1}$  and put  $X = \bigcup_{\alpha < \kappa} X_\alpha$ .

We put  $X_0 = \{e, g_0\}$  and assume that we have chosen the 2-thin subsets  $X_\alpha$  for all  $\alpha < \gamma$ . Let  $\gamma = \beta + 1$ . We find the first element  $g$  in the

numeration  $\{g_\alpha : \alpha < \kappa\}$  such that  $g \notin X_\beta X_\beta^{-1} \cup X_\beta^{-1} X_\beta$  and put  $y_\beta = g$ . To choose  $x_\beta$ , we use the following observation.

Let  $A$  be a subset of  $G$ ,  $g \in G$ . If  $|A| < \kappa$  and  $g \notin A$  then  $|\{x \in G : x^{-1}gx \notin A\}| = \kappa$ . Indeed,  $|\{x^{-1}gx : x \in G\}| = |G : Z_g|$ , where  $Z_g = \{x \in G : x^{-1}gx = g\}$ , and either  $|Z_g| = \kappa$  or  $|G : Z_g| = \kappa$ .

We choose  $x_\beta$  to satisfy the following conditions

- (i)  $x_\beta^{-1}y_\beta x_\beta \notin X_\beta^{-1}X_\beta$ ;
- (ii)  $\{x_\beta, y_\beta x_\beta\} \cap X_\beta X_\beta^{-1}X_\beta = \emptyset$ ;
- (iii)  $\{y_\beta, y_\beta^{-1}\}\{x_\beta, y_\beta x_\beta\} \cap X_\beta = \emptyset$ .

Suppose that  $X_{\beta+1} = X_\beta \cup \{x_\beta, y_\beta x_\beta\}$  is not left 2-thin and choose  $g \in G$ ,  $g \neq e$  and distinct  $a, b, c \in X_{\beta+1}$  such that  $ga, gb, gc \in X_{\beta+1}$ . If  $g \in X_\beta X_\beta^{-1}$  then, by (ii) and the choice of  $y_\beta$ ,  $\{a, b, c\} \subseteq X_\beta$  and  $\{ga, gb, gc\} \subseteq X_\beta$  which is impossible because  $X_\beta$  is left 2-thin. Let  $g \notin X_\beta X_\beta^{-1}$ . Replacing if necessary  $a, b, c$  to  $ga, gb, gc$  and  $g$  to  $g^{-1}$ , we may suppose that  $a = x_\beta$ ,  $b = y_\beta x_\beta$ ,  $c \in X_\beta$ . If  $ga \in X_\beta$  and  $gb \in X_\beta$  then  $X_\beta x_\beta^{-1} \cap X_\beta x_\beta^{-1} y_\beta^{-1} \neq \emptyset$  so we get a contradiction with (i). Thus,  $g \in \{y_\beta, y_\beta^{-1}\}$  and  $gc \in \{x_\beta, y_\beta x_\beta\}$ . Hence,  $\{y_\beta, y_\beta^{-1}\} \cap \{x_\beta, y_\beta x_\beta\} X_\beta^{-1} \neq \emptyset$  and we get a contradiction with (iii).

Suppose that  $X_\beta$  is not right 2-thin and choose  $g \in G$ ,  $g \neq e$  and distinct  $a, b, c \in X_{\beta+1}$  such that  $ag, bg, cg \in X_{\beta+1}$ . Let  $g \in X_\beta^{-1} X_\beta$ . If either  $a = x_\beta$  or  $a = y_\beta x_\beta$  then, by (ii), either  $g = x_\beta^{-1} y_\beta x_\beta$  or  $g = x_\beta^{-1} y_\beta^{-1} x_\beta$ , and in both cases we get a contradiction with (i). Hence,  $a, b, c \in X_\beta$  and  $ag, bg, cg \in X_\beta$  so  $X_\beta$  is not right 2-thin. Let  $g \notin X_\beta^{-1} X_\beta$ . Replacing if necessary  $a, b, c$  to  $ag, bc, cg$  and  $g$  to  $g^{-1}$ , we may suppose that  $a = x_\beta$ ,  $b = y_\beta x_\beta$ ,  $c \in X_\beta$ . If  $ag \in X_\beta$  and  $bg \in X_\beta$  then  $y_\beta \in X_\beta X_\beta^{-1}$  contradicting the choice of  $y_\beta$ . Thus, we have

$$\{x_\beta, y_\beta x_\beta\}g \cap \{x_\beta, y_\beta x_\beta\} \neq \emptyset,$$

$$X_\beta g \cap \{x_\beta, y_\beta x_\beta\} \neq \emptyset.$$

It follows that

$$\{x_\beta, y_\beta x_\beta\}^{-1} \{x_\beta, y_\beta x_\beta\} \cap X_\beta^{-1} \{x_\beta, y_\beta x_\beta\} \neq \emptyset,$$

so  $\{y_\beta, y_\beta^{-1}, e\} \{x_\beta, y_\beta x_\beta\} \cap X_\beta \neq \emptyset$  and we get a contradiction with (i) and (ii).

□

**Corollary 1.** *For every infinite Abelian group  $G$ , there exists a 2-thin subset  $X$  such that  $G = XX^{-1}$ .*

**Remark 1.** Let a group  $G$  be defined to have a *small square roots* if for any subset  $A \subseteq G$  with  $|A| < |G|$  the set  $\sqrt{A} = \{x \in G : x^2 \in A\}$  has cardinality  $|\sqrt{A}| < |G|$ . Taras Banach proved that if an infinite group  $G$  with identity  $e$  has small square roots, then it contains a 1-thin subset  $X$  such that  $G = \sqrt{\{e\}} \cup XX^{-1} \cup X^{-1}X$ . By this theorem, for every Abelian group  $G$  with no elements of order 2 there exists a 1-thin subset  $X$  such that  $G = XX^{-1}$ .

By the Chou's lemma [1], for every infinite group  $G$  there exists a 4-thin subset  $X$  such that  $|X| = |G|$ .

**Corollary 2.** *For every infinite group  $G$ , there exists a 2-thin subset  $X$  such that  $|X| = |G|$ .*

**Theorem 3.** *For every infinite group  $G$ , there exists a 4-thin subset  $X$  such that  $G = XX^{-1}$ .*

*Proof.* Let  $|G| = \kappa$ ,  $\{g_\alpha : \alpha < \kappa\}$  be a numeration of  $G$ . We construct inductively a family  $\{X_\alpha : \alpha < \kappa\}$  of 4-thin subsets of  $G$  of the form  $X_\alpha = \{x_\beta, y_\beta x_\beta : \beta < \alpha\}$ . Also we demand the fulfilment of the condition  $|X_\alpha \cap X_\alpha g| \leq 2$  for all  $g \notin X_\alpha X_\alpha^{-1}$ . Observe that  $\{y_\beta : \beta < \alpha\} \subseteq X_\alpha X_\alpha^{-1}$  and put  $X = \bigcup_{\alpha < \kappa} X_\alpha$ .

We put  $X_0 = \{e, g_0\}$  and assume that we have chosen subsets  $X_\alpha$  for all  $\alpha < \gamma$  such that

- (1)  $|X_\alpha \cap gX_\alpha| \leq 4$  for all  $g \in G \setminus \{e\}$ ;
- (2)  $|X_\alpha \cap X_\alpha g| \leq 2$  for  $g \notin X_\alpha X_\alpha^{-1} \cup \{e\}$ ;
- (3)  $|X_\alpha \cap X_\alpha g| \leq 4$  for  $g \in X_\alpha X_\alpha^{-1} \setminus \{e\}$ .

If  $\gamma$  is a limit ordinal, we put  $X_\gamma = \bigcup_{\alpha < \gamma} X_\alpha$ . Let  $\gamma = \beta + 1$ . We find the first element  $g$  in the numeration  $\{g_\alpha : \alpha < \kappa\}$  such that  $g \notin X_\beta X_\beta^{-1}$  and put  $y_\beta = g$ . Then we choose  $x_\beta$  to satisfy the following conditions

- (i)  $\{x_\beta, y_\beta x_\beta\} \cap X_\beta X_\beta^{-1} X_\beta = \emptyset$ ;
- (ii)  $\{e, y_\beta, y_\beta^{-1}\} \{x_\beta, y_\beta x_\beta\} \{e, y_\beta, y_\beta^{-1}\} \cap X_\beta = \emptyset$ ;
- (iii)  $x_\beta^{-1} y_\beta x_\beta \notin (X_\beta^{-1} X_\beta \cup X_\beta X_\beta^{-1}) \setminus \{y_\beta, y_\beta^{-1}\}$ .

We put  $X_{\beta+1} = X_\beta \cup \{x_\beta, y_\beta x_\beta\}$ . Now it is necessary to show the fulfilment of (1)–(3) for  $\alpha = \beta + 1$ . First we show that  $|X_{\beta+1} \cap gX_{\beta+1}| \leq 4$  for all  $g \in G \setminus \{e\}$ . Since  $X_{\beta+1} = X_\beta \cup \{x_\beta, y_\beta x_\beta\}$ , for every  $g \in G \setminus \{e\}$ , we have

$$X_{\beta+1} \cap gX_{\beta+1} = (X_\beta \cup \{x_\beta, y_\beta x_\beta\}) \cap (gX_\beta \cup g\{x_\beta, y_\beta x_\beta\}) =$$

$$= (X_\beta \cap gX_\beta) \cup Y_1 \cup Y_2 \cup Y_3,$$

where  $Y_1 = X_\beta \cap \{gx_\beta, gy_\beta x_\beta\}$ ,  $Y_2 = \{x_\beta, y_\beta x_\beta\} \cap gX_\beta$ ,  $Y_3 = \{x_\beta, y_\beta x_\beta\} \cap \{gx_\beta, gy_\beta x_\beta\}$ . We consider two cases:

*Case 1:*  $g \in X_\beta X_\beta^{-1}$ . By (i),  $Y_1 = \emptyset$  and  $Y_2 = \emptyset$ . Since  $y_\beta \notin X_\beta X_\beta^{-1}$ ,  $Y_3 = \emptyset$ . Then  $X_{\beta+1} \cap gX_{\beta+1} = X_\beta \cap gX_\beta$  and, by the inductive assumption,  $|X_{\beta+1} \cap gX_{\beta+1}| \leq 4$ .

*Case 2:*  $g \notin X_\beta X_\beta^{-1}$ . Then  $X_\beta \cap gX_\beta = \emptyset$ . Since  $Y_1 \cup Y_2 \cup Y_3 \subseteq \{x_\beta, y_\beta x_\beta, gx_\beta, gy_\beta x_\beta\}$ , we have  $|X_{\beta+1} \cap gX_{\beta+1}| = |Y_1 \cup Y_2 \cup Y_3| \leq 4$ .

Now we show that  $|X_{\beta+1} \cap X_{\beta+1}g| \leq 2$  for all  $g \notin X_{\beta+1}X_{\beta+1}^{-1}$  and  $|X_{\beta+1} \cap X_{\beta+1}g| \leq 4$  for all  $g \in G \setminus \{e\}$ . Since  $X_{\beta+1} = X_\beta \cup \{x_\beta, y_\beta x_\beta\}$ , for every  $g \in G \setminus \{e\}$ , we have

$$\begin{aligned} X_{\beta+1} \cap X_{\beta+1}g &= (X_\beta \cup \{x_\beta, y_\beta x_\beta\}) \cap (X_\beta g \cup \{x_\beta g, y_\beta x_\beta g\}) = \\ &= (X_\beta \cap X_\beta g) \cup Z_1 \cup Z_2 \cup Z'_3 \cup Z''_3, \end{aligned}$$

where  $Z_1 = \{x_\beta g, y_\beta x_\beta g\} \cap X_\beta$ ,  $Z_2 = \{x_\beta, y_\beta x_\beta\} \cap X_\beta g$ ,  $Z'_3 = \{x_\beta\} \cap \{y_\beta x_\beta g\}$ ,  $Z''_3 = \{y_\beta x_\beta\} \cap \{x_\beta g\}$ . We consider three cases.

*Case 1:*  $g \in X_\beta X_\beta^{-1}$ . By (i),  $Z_1 = \emptyset$  and  $Z_2 = \emptyset$ . Since  $g \in X_\beta X_\beta^{-1}$  and  $y_\beta \notin X_\beta X_\beta^{-1}$  then  $g \in (X_\beta^{-1}X_\beta \cup X_\beta X_\beta^{-1}) \setminus \{y_\beta, y_\beta^{-1}\}$ . So, by (iii),  $Z'_3 = \emptyset$  and  $Z''_3 = \emptyset$ . Hence,  $X_{\beta+1} \cap X_{\beta+1}g = X_\beta \cap X_\beta g$  and required inequalities hold by inductive hypothesis.

*Case 2:*  $g \in \{y_\beta, y_\beta^{-1}\}$ . By (ii),  $Z_1 = \emptyset$  and  $Z_2 = \emptyset$ . Hence,  $X_{\beta+1} \cap X_{\beta+1}g = (X_\beta \cap X_\beta g) \cup Z'_3 \cup Z''_3$ . Since  $g \notin X_\beta X_\beta^{-1}$  then  $|X_\beta \cap X_\beta g| \leq 2$ . Since  $|Z'_3| \leq 1$  and  $|Z''_3| \leq 1$  then  $|X_{\beta+1} \cap X_{\beta+1}g| \leq 4$ . Observe that  $g \in X_{\beta+1}X_{\beta+1}^{-1}$ , so we do not need to check the condition (2).

*Case 3:*  $g \notin X_\beta X_\beta^{-1} \cup \{y_\beta, y_\beta^{-1}\}$ . Since  $g \notin X_\beta X_\beta^{-1}$  then, by inductive hypothesis,  $|X_\beta \cap X_\beta g| \leq 2$ . Since  $y_\beta \notin X_\beta X_\beta^{-1}$  then  $|Z_1| \leq 1$  and  $|Z_2| \leq 1$ . We consider two subcases.

*Subcase 3.1:*  $g \in X_\beta^{-1}X_\beta$ . By (i),  $Z_1 = \emptyset$  and  $Z_2 = \emptyset$ . By (iii),  $Z'_3 = \emptyset$  and  $Z''_3 = \emptyset$ . Hence,  $X_{\beta+1} \cap X_{\beta+1}g = X_\beta \cap X_\beta g$  and required inequalities hold by inductive hypothesis.

*Subcase 3.2:*  $g \notin X_\beta^{-1}X_\beta$ . Then  $X_\beta \cap X_\beta g = \emptyset$ , so  $X_{\beta+1} \cap X_{\beta+1}g = Z_1 \cup Z_2 \cup Z'_3 \cup Z''_3$ . By (ii), if  $Z'_3 \neq \emptyset$  then  $Z_2 = \emptyset$ , and if  $Z''_3 \neq \emptyset$  then  $Z_1 = \emptyset$ . Taking into account the inequalities  $|Z_1| \leq 1$ ,  $|Z_2| \leq 1$ ,  $|Z'_3| \leq 1$  and  $|Z''_3| \leq 1$  we obtain  $|X_{\beta+1} \cap X_{\beta+1}g| \leq 2$ .

So the inequalities (1)–(3) hold for  $\alpha = \beta + 1$ . Note that  $y_\beta \in X_{\beta+1}X_{\beta+1}^{-1}$ . We put  $X = \bigcup_{\alpha < \kappa} X_\alpha$  and observe that, by the choice of  $y_\beta$ ,  $G = XX^{-1}$  and  $X$  is 4-thin. □

**Question 2.** Which is a minimal number  $k_{th}$  such that, for every infinite group  $G$ , there exists a  $k_{th}$ -thin subset  $X$  such that  $G = XX^{-1}$ ?

**Question 3.** Which is a minimal number  $k_{lth}$  such that, for every infinite group  $G$ , there exists a left  $k_{lth}$ -thin subset  $X$  such that  $G = XX^{-1}$ ?

An infinite group  $G$  of period 2 shows that  $k_{th} \geq 2$ ,  $k_{lth} \geq 2$ . By Theorem 3,  $k_{th} \leq 4$ ,  $k_{lth} \leq 4$ .

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