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Automorphisms of finitary incidence rings

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Abstract. Let P be a quasiordered set, R an associative unital ring, $\mathcal{C}(P,R)$ a partially ordered category associated with the pair (P,R) [6], $FI(P,R)$ a finitary incidence ring of $\mathcal{C}(P,R)$ [6]. We prove that the group $\mathrm{Out}FI$ of outer automorphisms of $FI(P,R)$ is isomorphic to the group $\mathrm{Out}\mathcal{C}$ of outer automorphisms of $\mathcal{C}(P,R)$ under the assumption that R is indecomposable. In particular, if R is local, the equivalence classes of P are finite and $P=\bigcup\limits_{i\in I}P_i$ is the decomposition of P into the disjoint union of the connected components, then $\mathrm{Out}FI\cong(H^1(\overline{P},C(R)^*)\rtimes\prod\limits_{i\in I}\mathrm{Out}R)\rtimes\mathrm{Out}P$. Here $H^1(\overline{P},C(R)^*)$ is the first cohomology group of the order complex of the induced poset $\overline{P}$ with the values in the multiplicative group of central invertible elements of R. As a consequences, Theorem 2 [9], Theorem 5 [2] and Theorem 1.2 [8] are obtained.

Introduction

Recall that an incidence algebra $I(P,R)$ of a locally finite poset P over a ring R is the set of formal sums of the form

$$\alpha=\sum_{x\le y}\alpha(x,y)[x,y],$$

where $\alpha(x,y)\in R$, $[x,y]=\{z\in P\mid x\le z\le y\}$ is a segment of the partial order. The study of the automorphism group of an incidence algebra was

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started by Stanley [9]. He showed that the group of outer automorphisms of an incidence algebra of a finite poset P over a field R is isomorphic to the semidirect product $(R^*)^n \rtimes \mathrm{Out}P$ where R^* is the group of invertible elements of the field R, $\mathrm{Out}P$ is the group of outer automorphisms of the poset P and n is such that $(R^*)^n \cong H^1(P, R^*)$. This result was first generalized by Baclawski [2] (P is a locally finite quasiordered set, R is a field), then by Scharlau [8] (P is a finite quasiordered set with 0 or 1, R is a division ring, finite-dimensional over its center) and lately by Drozd and Kolesnik [4] (P is a finite quasiordered set, R is a field). After the notion of the finitary incidence algebra, which generalizes the notion of the incidence algebra to the cases of the arbitrary partially ordered [7] and quasiordered [6] sets, had been introduced, the task to describe the automorphism group of this type of algebras has arisen.

Let $P(\preccurlyeq)$ be a quasiordered set, R an associative unital ring. As in [6], $\mathcal{C}(P, R)$ denotes the preadditive category associated with the pair (P, R), namely:

1. $Ob\,\mathcal{C}(P, R) = \overline{P} = P/_\sim$ with the induced order $\leq$.
2. For any $\bar{x}, \bar{y} \in \overline{P}, \bar{x} \leq \bar{y}$ the set of morphisms $\mathrm{Mor}(\bar{x}, \bar{y}) = M_{\bar{x}\times\bar{y}}(R)$ (if $\bar{x} \not\leq \bar{y}$, then $\mathrm{Mor}(\bar{x}, \bar{y}) = 0_{\bar{x}\bar{y}}$).

Here $M_{\bar{x}\times\bar{y}}(R)$ is the additive group of matrices over R, whose rows and columns are indexed by the elements of the classes $\bar{x}$ and $\bar{y}$, respectively, and each row has only a finite number of nonzero elements. For any two such matrices $\alpha_{\bar{x}\bar{z}} \in \mathrm{Mor}(\bar{x}, \bar{z}), \alpha_{\bar{z}\bar{y}} \in \mathrm{Mor}(\bar{z}, \bar{y})$ the product $\alpha_{\bar{x}\bar{z}}\alpha_{\bar{z}\bar{y}} \in \mathrm{Mor}(\bar{x}, \bar{y})$ is defined and gives the composition of the morphisms $\alpha_{\bar{x}\bar{z}}$ and $\alpha_{\bar{z}\bar{y}}$ in $\mathcal{C}(P, R)$. The category $\mathcal{C}(P, R)$ is a particular case of the so-called partially ordered category (pocategory), which was considered in [6]. For such categories the notion of the finitary incidence ring was introduced [6]. We shall formulate its definition for $\mathcal{C}(P, R)$. Consider the set of formal sums of the form

$$\alpha = \sum_{\bar{x}\leq\bar{y}} \alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}], \tag{1}$$

where $[\bar{x}, \bar{y}]$ is a segment of the partial order, $\alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x}, \bar{y})$. The sum (1) is called a finitary series if for any $[\bar{x}, \bar{y}]$ there exists only a finite number of $[\bar{u}, \bar{v}] \subset [\bar{x}, \bar{y}]$, $\bar{u} < \bar{v}$ such that $\alpha_{\bar{u}\bar{v}} \neq 0_{\bar{u}\bar{v}}$. The set of the finitary series forms a ring under the convolution [6, Theorem 1]. It is denoted by $FI(P, R)$ (in fact $FI(P, R)$ is an algebra over the center of R, but for the most part we are going to use only its ring properties). $FI(P, R)$ has the unity element δ, where $\delta_{\bar{x}\bar{x}}$ is the identity matrix of size $|\bar{x}| \times |\bar{x}|$, $\delta_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}$ for $\bar{x} < \bar{y}$. The finitary series can also be considered as the functions on the set of the segments of P with the values in R, namely: $\alpha(x, y)$ means the element of the matrix $\alpha_{\bar{x}\bar{y}}$, which is situated in the intersection of the x-th row and y-th column.

In this article we study the automorphism group AutFI of the ring $FI(P,R)$ under the assumption that R is indecomposable. In the first section it is proved that the group OutFI = AutFI/InnFI of outer automorphisms of the finitary ring is isomorphic to the group Out$\mathcal{C}$ of outer automorphisms of the category $\mathcal{C}(P,R)$. After that in the second section we prove that under some additional assumptions on R the group Out$\mathcal{C}$ is isomorphic to the semidirect product $\mathrm{Out}_0\mathcal{C} \rtimes \mathrm{Out}P$, where $\mathrm{Out}_0\mathcal{C}$ belongs to the exact sequence

$$1 \to H^1(\overline{P}, C(R)^*) \to \mathrm{Out}_0\mathcal{C} \to \prod_{\bar{x}\in\overline{P}} \mathrm{Out}M_{\bar{x}\times\bar{x}}(R)$$

(here $C(R)^*$ is the multiplicative group of the central invertible elements of the ring R, $\mathrm{Out}M_{\bar{x}\times\bar{x}}(R)$ is the group of outer automorphisms of the ring $M_{\bar{x}\times\bar{x}}(R)$). In particular, if R is a local ring, P is a class finite quasiordered set and $P = \bigcup_{i\in I} P_i$ is the decomposition of P into the disjoint union of the connected components, then $\mathrm{Out}FI \cong (H^1(\overline{P}, C(R)^*) \rtimes \prod_{i\in I} \mathrm{Out}R) \rtimes \mathrm{Out}P$, as proved in the third section. Finally in the last section we investigate the group K-OutFI = K-AutFI/InnFI, where K-AutFI means the subgroup of AutFI consisting of those automorphisms, which agree with the structure of algebra over $K = C(R)$. As the consequences, we obtain the results of Stanley, Scharlau and Baclawski about the automorphism group of incidence algebra.

1. The connection with the automorphisms of $\mathcal{C}(P,R)$

In what follows if no additional information is given $P(\preccurlyeq)$ is meant to be an arbitrary quasiordered set, R an indecomposable associative unital ring.

The restriction of an element $\alpha \in FI(P,R)$ to the equivalence class $\bar{x} \in \overline{P}$ is by definition the series $\alpha_{\bar{x}} = \alpha_{\bar{x}\bar{x}}[\bar{x},\bar{x}]$. *The diagonal* of α is $\alpha_D = \sum_{\bar{x}\in\overline{P}} \alpha_{\bar{x}\bar{x}}[\bar{x},\bar{x}]$. Accordingly α is said to be *diagonal* iff $\alpha_D = \alpha$. Note that $(\alpha\beta)_D = \alpha_D\beta_D$, $(\alpha\beta)_{\bar{x}} = \alpha_{\bar{x}}\beta_{\bar{x}}$ and

$$\alpha_{\bar{x}}\beta\gamma_{\bar{y}} = \alpha_{\bar{x}\bar{x}}\beta_{\bar{x}\bar{y}}\gamma_{\bar{y}\bar{y}}[\bar{x},\bar{y}]. \tag{2}$$

As a consequence, $\alpha_{\bar{x}}\beta\gamma_{\bar{x}} = \alpha_{\bar{x}}\beta_{\bar{x}}\gamma_{\bar{x}}$, $\alpha_{\bar{x}}\beta_{\bar{y}} = 0$ for $x \nsim y$. In particular, $\{\delta_{\bar{x}}\}_{\bar{x}\in\overline{P}}$ is a set of orthogonal idempotents and

$$\delta_{\bar{x}}\alpha\delta_{\bar{y}} = \alpha_{\bar{x}\bar{y}}[\bar{x},\bar{y}], \quad \delta_{\bar{x}}\alpha\delta_{\bar{x}} = \alpha_{\bar{x}}. \tag{3}$$

First of all we shall be interested in the action of the automorphisms of the finitary ring on $\delta_{\bar{x}}$.

Lemma 1. *Let $\Phi \in \mathrm{Aut}FI$. Then the image $\Phi(\delta_{\bar{x}})$ is the conjugate of $\delta_{\varphi(\bar{x})}$ for some order preserving bijection $\varphi : \overline{P} \to \overline{P}$.*

Proof. By [6, Theorem 3] it is sufficient to prove that there is an order preserving bijection $\varphi : \overline{P} \to \overline{P}$, such that

$$\Phi(\delta_{\bar{x}})_D = \delta_{\varphi(\bar{x})}. \tag{4}$$

Consider an idempotent $\delta_x \in FI(P,R)$, which is defined for any $x \in P$ as follows:

$$\delta_x(u,v) \quad = \quad \begin{cases} 1, & \text{if } u = v = x, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\delta_x = (\delta_x)_{\bar{x}}$, $(\delta_x \alpha \delta_y)(x,y) = \alpha(x,y)$. By the indecomposability of R all δ_x are primitive. Indeed, if $\delta_x = \alpha + \beta$, where α and β are the orthogonal idempotents, then $\alpha = \delta_x \alpha = \alpha \delta_x = \delta_x \alpha \delta_x$, i. e. $\alpha(u,v) = \alpha(x,x)\delta_x(u,v)$. Since $\alpha(x,x)$ is an idempotent in R, $\alpha(x,x)$ equals 0 or 1 because R is indecomposable. Then either α coincides with δ_x, or it is equal to zero. Take an equivalence class $\bar{x} \in \overline{P}$ and choose an arbitrary element $x' \in \bar{x}$. The image $\Phi(\delta_{x'})$ is the primitive idempotent and by [6, Theorem 3] it is the conjugate of $\Phi(\delta_{x'})_D$. Since the restrictions of $\Phi(\delta_{x'})$ to the different classes are the orthogonal idempotents, the primitivity of $\Phi(\delta_{x'})_D$ implies that there exists $\bar{y} \in \overline{P}$, such that $\Phi(\delta_{x'})_D$ coincides with $\Phi(\delta_{x'})_{\bar{y}}$. Note that $\delta_{x'}$ and $\delta_{x''}$ are the conjugates iff $x' \sim x''$. Hence the class $\bar{y}$ does not depend on the choice of the representative $x' \in \bar{x}$. Thus Φ induces the mapping $\varphi : \overline{P} \to \overline{P}$, such that $\Phi(\delta_{x'})$ is the conjugate of $\Phi(\delta_{x'})_{\varphi(\bar{x})}$. Similarly we can consider Φ^{-1} and build $\psi : \overline{P} \to \overline{P}$. Show that they are mutually inverse. Let $\varphi(\bar{x}) \neq \bar{v}$. Then for each $x' \in \bar{x}$: $\delta_v \Phi(\delta_{x'})_{\varphi(\bar{x})} = 0$, i. e. $\delta_v \beta \Phi(\delta_{x'})\beta^{-1} = 0$ for some invertible β. Therefore, $\Phi^{-1}(\delta_v)\Phi^{-1}(\beta)\delta_{x'} = 0$. This means that $(\Phi^{-1}(\delta_v)\Phi^{-1}(\beta))_{\bar{x}} = 0$, thus $\psi(\bar{v}) \neq \bar{x}$. The implication $\psi(\bar{v}) \neq \bar{x} \Rightarrow \varphi(\bar{x}) \neq \bar{v}$ is proved similarly. So, $\psi = \varphi^{-1}$.

Consider $\Phi(\delta_{\bar{x}})$ and prove that its diagonal coincides with the restriction on $\varphi(\bar{x})$. Suppose that there are $v', v'' \in \bar{v} \neq \varphi(\bar{x})$, such that $\delta_{v'}\Phi(\delta_{\bar{x}})\delta_{v''} \neq 0$. Then $\Phi^{-1}(\delta_{v'})\delta_{\bar{x}}\Phi^{-1}(\delta_{v''}) \neq 0$. But $\Phi^{-1}(\delta_{v'})$ and $\Phi^{-1}(\delta_{v''})$ are the conjugates of $\Phi^{-1}(\delta_{v'})_{\varphi^{-1}(\bar{v})}$ and $\Phi^{-1}(\delta_{v''})_{\varphi^{-1}(\bar{v})}$ respectively. This means that there are invertible β and γ, such that (see (2))

$$\Phi^{-1}(\delta_{v'})_{\varphi^{-1}(\bar{v})}\beta^{-1}\delta_{\bar{x}}\gamma\Phi^{-1}(\delta_{v''})_{\varphi^{-1}(\bar{v})} = (\Phi^{-1}(\delta_{v'})\beta^{-1}\delta_{\bar{x}}\gamma\Phi^{-1}(\delta_{v''}))_{\varphi^{-1}(\bar{v})}$$

is different from zero. Therefore $(\delta_{\bar{x}})_{\varphi^{-1}(\bar{v})} \neq 0$, i. e. $\varphi^{-1}(\bar{v}) = \bar{x}$, which contradicts the supposition. Hence $\Phi(\delta_{\bar{x}})_D = \Phi(\delta_{\bar{x}})_{\varphi(\bar{x})}$. Similarly $\Phi^{-1}(\delta_{\varphi(\bar{x})})_D = \Phi^{-1}(\delta_{\varphi(\bar{x})})_{\bar{x}}$. Using (3) we obtain that $\Phi^{-1}(\delta_{\varphi(\bar{x})})$ is the conjugate of $\delta_{\bar{x}}\Phi^{-1}(\delta_{\varphi(\bar{x})})\delta_{\bar{x}}$. Therefore $\delta_{\varphi(\bar{x})}$ is the conjugate of $\Phi(\delta_{\bar{x}})\delta_{\varphi(\bar{x})}\Phi(\delta_{\bar{x}})$. Since $\Phi(\delta_{\bar{x}})$ is an idempotent and the diagonal of $\delta_{\bar{x}}$ is stable under the conjugation, we conclude that $\Phi(\delta_{\bar{x}})_D = \delta_{\varphi(\bar{x})}$.

Prove that φ preserves the partial order. Let $\bar{x} \leq \bar{y}$. Consider $\alpha = \alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}]$ for some nonzero $\alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x}, \bar{y})$. Then $\alpha = \delta_{\bar{x}}\alpha\delta_{\bar{y}}$ by (3). So $\Phi(\alpha) = \Phi(\delta_{\bar{x}})\Phi(\alpha)\Phi(\delta_{\bar{y}}) = \beta\delta_{\varphi(\bar{x})}\beta^{-1}\Phi(\alpha)\gamma\delta_{\varphi(\bar{y})}\gamma^{-1}$ for some invertible $\beta, \gamma \in FI(P, R)$. Hence

$$\beta^{-1}\Phi(\alpha)\gamma = \delta_{\varphi(\bar{x})}\beta^{-1}\Phi(\alpha)\gamma\delta_{\varphi(\bar{y})} = (\beta^{-1}\Phi(\alpha)\gamma)_{\varphi(\bar{x})\varphi(\bar{y})}[\varphi(\bar{x}), \varphi(\bar{y})].$$

Since $\alpha \neq 0$, we have $\beta^{-1}\Phi(\alpha)\gamma \neq 0$, and therefore $(\beta^{-1}\Phi(\alpha)\gamma)_{\varphi(\bar{x})\varphi(\bar{y})} \neq 0$ by the previous equality. Thus $\varphi(\bar{x}) \leq \varphi(\bar{y})$. □

Remark 1. The lemma implies that the correspondence $\Phi \mapsto \varphi$ agrees with the composition of the mappings. In particular, $\Phi^{-1} \mapsto \varphi^{-1}$, and φ^{-1} preserves the partial order.

Let $X \subset \overline{P}$. Denote by δ_X the diagonal finitary series $\sum\limits_{\bar{x}\in X} \delta_{\bar{x}\bar{x}}[\bar{x}, \bar{x}]$.

We shall need the following technical lemma.

Lemma 2. *Let $\Phi \in \mathrm{Aut}FI$, $\varphi : \overline{P} \to \overline{P}$ be the bijection defined by (4), $x \preccurlyeq y$, $Z \subset \overline{P}$. Then*

1. $\Phi(\delta_{\bar{x}})_{\varphi(\bar{x})\varphi(\bar{y})} = 0_{\varphi(\bar{x})\varphi(\bar{y})} \Leftrightarrow \Phi^{-1}(\delta_{\varphi(\bar{y})})_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}$.
2. $\Phi(\delta_Z)_{\varphi(\bar{x})\varphi(\bar{y})} = \Phi(\delta_{Z'})_{\varphi(\bar{x})\varphi(\bar{y})}$, *where Z' consists of those $\bar{z} \in Z$, for which $\Phi(\delta_{\bar{z}})_{\varphi(\bar{x})\varphi(\bar{z})} \neq 0_{\varphi(\bar{x})\varphi(\bar{z})}$ and $\Phi(\delta_{\bar{z}})_{\varphi(\bar{z})\varphi(\bar{y})} \neq 0_{\varphi(\bar{z})\varphi(\bar{y})}$.*

Proof. Prove the first statement. Write $\bar{u} = \varphi(\bar{x}), \bar{v} = \varphi(\bar{y})$ for short. Let $\Phi(\delta_{\bar{x}})_{\bar{u}\bar{v}} = 0_{\bar{u}\bar{v}}$. By (3) this is equivalent to the equality

$$\delta_{\bar{u}}\Phi(\delta_{\bar{x}})\delta_{\bar{v}} = 0 \tag{5}$$

in the ring $FI(P, R)$. Apply Φ^{-1} to this equality. By the Remark 1 there are invertible $\beta, \gamma \in FI(P, R)$, such that

$$\Phi^{-1}(\delta_{\bar{u}}) = \beta\delta_{\bar{x}}\beta^{-1}, \quad \Phi^{-1}(\delta_{\bar{v}}) = \gamma\delta_{\bar{y}}\gamma^{-1}. \tag{6}$$

Then it follows from (5) that $\delta_{\bar{x}}\beta^{-1}\delta_{\bar{x}}\gamma\delta_{\bar{y}} = 0$, which is equivalent to $(\beta^{-1})_{\bar{x}\bar{x}}\gamma_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}$ (see (3)). According to [6, Theorem 2], $(\beta^{-1})_{\bar{x}\bar{x}}$ and $(\gamma^{-1})_{\bar{y}\bar{y}}$ are the invertible elements of the rings $M_{\bar{x}\times\bar{x}}(R)$ and $M_{\bar{y}\times\bar{y}}(R)$

respectively, hence $\gamma_{\bar{x}\bar{y}}(\gamma^{-1})_{\bar{y}\bar{y}} = 0_{\bar{x}\bar{y}}$. This means that $(\gamma\delta_{\bar{y}}\gamma^{-1})_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}$, i. e. $\Phi^{-1}(\delta_{\bar{v}})_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}$ by (6).

Let us turn to the proof of the second statement. Instead of $\Phi(\delta_Z)$ we consider $\delta_{\bar{u}}\Phi(\delta_Z)\delta_{\bar{v}}$ (by (3) this series has the same value at the segment $[\bar{u}, \bar{v}]$ as the initial one). Using (6) we see that its preimage under Φ is equal to $\beta\delta_{\bar{x}}\beta^{-1}\delta_Z\gamma\delta_{\bar{y}}\gamma^{-1}$. It is sufficient to prove that in this product Z can be replaced by Z'. According to (3) and the definition of the convolution, the product $\delta_{\bar{x}}\beta^{-1}\delta_Z\gamma\delta_{\bar{y}}$ depends only on those $\bar{z} \in Z$, for which $(\beta^{-1})_{\bar{x}\bar{z}} \neq 0_{\bar{x}\bar{z}}$ and $\gamma_{\bar{z}\bar{y}} \neq 0_{\bar{z}\bar{y}}$. By the finitarity of β^{-1} and γ there is a finite number of such $\bar{z}$. Note that the first inequality is equivalent to $(\beta\delta_{\bar{x}}\beta^{-1})_{\bar{x}\bar{z}} \neq 0_{\bar{x}\bar{z}}$, i. e. $\Phi^{-1}(\delta_{\bar{u}})_{\bar{x}\bar{z}} \neq 0_{\bar{x}\bar{z}}$. Similarly the second one means that $\Phi^{-1}(\delta_{\bar{v}})_{\bar{z}\bar{y}} \neq 0_{\bar{z}\bar{y}}$. Applying the first statement of the lemma to Φ^{-1}, we obtain the required inequalities. □

For an arbitrary invertible $\beta \in FI(P, R)$ denote by $\tau_\beta \in \mathrm{Inn}FI$ the conjugation by the element β. If $\Phi \in \mathrm{Aut}FI$, then, as it is mentioned above, for each $\bar{x} \in \overline{P}$ there is β, such that $(\tau_\beta\Phi)(\delta_{\bar{x}}) = \delta_{\varphi(\bar{x})}$. It turns out that such a β can be chosen independently of the class $\bar{x}$.

Lemma 3. *Let $\Phi \in \mathrm{Aut}FI$, $\varphi : \overline{P} \to \overline{P}$ be the bijection defined by (4). Then there is $\tau_\beta \in \mathrm{Inn}FI$, such that*

$$(\tau_\beta\Phi)(\delta_{\bar{x}}) = \delta_{\varphi(\bar{x})} \tag{7}$$

for all $\bar{x}$.

Proof. Define β by the formal equality

$$\beta = \sum_{\bar{u} \leq \bar{v}} \Phi(\delta_{\varphi^{-1}(\bar{u})})_{\bar{u}\bar{v}}[\bar{u}, \bar{v}]. \tag{8}$$

Obviously, $\delta_{\varphi(\bar{x})}\beta = \delta_{\varphi(\bar{x})}\Phi(\delta_{\bar{x}})$ for each $\bar{x} \in \overline{P}$. Consider the product $\beta\Phi(\delta_{\bar{x}})$. According to (8) and the definition of the convolution:

$$(\beta\Phi(\delta_{\bar{x}}))_{\bar{u}\bar{v}} = \sum_{\bar{u} \leq \bar{w} \leq \bar{v}} \Phi(\delta_{\varphi^{-1}(\bar{u})})_{\bar{u}\bar{w}}\Phi(\delta_{\bar{x}})_{\bar{w}\bar{v}} = (\Phi(\delta_{\varphi^{-1}(\bar{u})})\Phi(\delta_{\bar{x}}))_{\bar{u}\bar{v}}.$$

Since $\{\delta_{\bar{x}}\}_{\bar{x}\in\overline{P}}$ is a family of orthogonal idempotents in $FI(P, R)$ and Φ is an isomorphism, we obtain that $(\beta\Phi(\delta_{\bar{x}}))_{\bar{u}\bar{v}} = \Phi(\delta_{\bar{x}})_{\varphi(\bar{x})\bar{v}}$ if $\bar{u} = \varphi(\bar{x})$ and 0 otherwise. Thus, $\beta\Phi(\delta_{\bar{x}}) = \delta_{\varphi(\bar{x})}\Phi(\delta_{\bar{x}})$, i. e.

$$\beta\Phi(\delta_{\bar{x}}) = \delta_{\varphi(\bar{x})}\beta \tag{9}$$

for an arbitrary $\bar{x} \in \overline{P}$. Note that $\beta_{\bar{x}} = \Phi(\delta_{\varphi^{-1}(\bar{x})})_{\bar{x}} = \delta_{\bar{x}}$ by (4). To prove the lemma it is sufficient to establish the finitarity of β. Indeed, then by [6, Theorem 2] β will be invertible and therefore $\beta\Phi(\delta_{\bar{x}})\beta^{-1} = \delta_{\bar{x}}$ from (9).

Suppose that the set $[\bar{u}_s, \bar{v}_s]_{s\in S}$, $\bar{u}_s < \bar{v}_s$ of all different nontrivial subsegments of some fixed segment $[\bar{u}, \bar{v}] \subset \overline{P}$, for which $\beta_{\bar{u}_s\bar{v}_s} \neq 0_{\bar{u}_s\bar{v}_s}$, is infinite. By the definition of β this means that

$$\Phi(\delta_{\varphi^{-1}(\bar{u}_s)})_{\bar{u}_s\bar{v}_s} \neq 0_{\bar{u}_s\bar{v}_s}. \tag{10}$$

According to the Lemma 2

$$\Phi^{-1}(\delta_{\bar{v}_s})_{\varphi^{-1}(\bar{u}_s)\varphi^{-1}(\bar{v}_s)} \neq 0_{\varphi^{-1}(\bar{u}_s)\varphi^{-1}(\bar{v}_s)}. \tag{11}$$

It follows from (10) that for each $\bar{u}_0 \in \overline{P}$ there is only a finite number of $\bar{u}_s$, which coincide with $\bar{u}_0$. Indeed, if $\bar{u}_s = \bar{u}_0 \in [u, v]$ for some set of indexes $S_0 \subset S$, then $\Phi(\delta_{\varphi^{-1}(\bar{u}_0)})_{\bar{u}_0\bar{v}_s} \neq 0_{\bar{u}_0\bar{v}_s}$ for this set of indexes by (10). Since $\Phi(\delta_{\varphi^{-1}(\bar{u}_0)})$ is a finitary series and $[\bar{u}_0, \bar{v}_s]$ are the different nontrivial subsegments of the segment $[\bar{u}, \bar{v}]$, S_0 must be finite. Similarly only a finite number of $\bar{v}_s$ can coincide with some $\bar{v}_0 \in \overline{P}$ by (11) and the Remark 1. Consider an arbitrary segment $[\bar{u}_1, \bar{v}_1]$ from $\{[\bar{u}_s, \bar{v}_s]\}$. According to our remark, there is only a finite number of segments in $\{[\bar{u}_s, \bar{v}_s]\}$, one of whose end points coincides with one of the end points of $[\bar{u}_1, \bar{v}_1]$, i. e. $\{\bar{u}_s, \bar{v}_s\} \cap \{\bar{u}_1, \bar{v}_1\} \neq \emptyset$. Throw away all such segments except $[\bar{u}_1, \bar{v}_1]$. Then among the remaining segments choose $[\bar{u}_2, \bar{v}_2] \neq [\bar{u}_1, \bar{v}_1]$. Repeat the procedure for this segment, i. e. throw away all $[\bar{u}_s, \bar{v}_s] \neq [\bar{u}_2, \bar{v}_2]$, for which $\{\bar{u}_s, \bar{v}_s\} \cap \{\bar{u}_2, \bar{v}_2\} \neq \emptyset$ (there is a finite number of such segments). Note that $[\bar{u}_1, \bar{v}_1]$ will remain because $\{\bar{u}_1, \bar{v}_1\} \cap \{\bar{u}_2, \bar{v}_2\} = \emptyset$ by the result of the previous step. Again, chose some $[\bar{u}_3, \bar{v}_3] \neq [\bar{u}_1, \bar{v}_1], [\bar{u}_2, \bar{v}_2]$ and so on. By iterating this process, we finally obtain the infinite set $\{[\bar{u}_i, \bar{v}_i]\}_{i=1}^{\infty}$ of segments, for which (10) and (11) are fulfilled, and, moreover, for each i there is a unique segment with the left end point $\bar{u}_i$ and a unique segment with the right end point $\bar{v}_i$ (and there are no segments with the right end point $\bar{u}_i$ or with the left end point $\bar{v}_i$).

Take $X = \{\varphi^{-1}(\bar{u}_i)\}$ and consider the finitary series δ_X. According to the second statement of the Lemma 2, the value of $\Phi(\delta_X)_{\bar{u}_i\bar{v}_i}$ must coincide with $\Phi(\delta_{X'})_{\bar{u}_i\bar{v}_i}$, where X' consists of those $\bar{u}_j$, for which $\Phi(\delta_{\varphi^{-1}(\bar{u}_j)})_{\bar{u}_i\bar{u}_j} \neq 0_{\bar{u}_i\bar{u}_j}$ and $\Phi(\delta_{\varphi^{-1}(\bar{u}_j)})_{\bar{u}_j\bar{v}_i} \neq 0_{\bar{u}_j\bar{v}_i}$. In our case the only possibility for j is to be equal to i. Thus, $\Phi(\delta_X)_{\bar{u}_i\bar{v}_i} = \Phi(\delta_{\varphi^{-1}(\bar{u}_i)})_{\bar{u}_i\bar{v}_i} \neq 0_{\bar{u}_i\bar{v}_i}$ for all i. This contradicts the finitarity of $\Phi(\delta_X)$. □

Remark 2. The series β from the previous lemma is determined up to the multiplication by the diagonal series.

Proof. Obviously, we need to prove that if $\tau_\gamma(\delta_{\bar{x}}) = \delta_{\bar{x}}$ for all $\bar{x}$, then γ is diagonal. Indeed, $\gamma\delta_{\bar{x}} = \delta_{\bar{x}}\gamma$ means that $\gamma_{\bar{x}\bar{y}} = 0_{\bar{x}\bar{y}}, \gamma_{\bar{z}\bar{x}} = 0_{\bar{z}\bar{x}}$ for all $\bar{y}, \bar{z} \neq \bar{x}$. Since this is true for all $\bar{x}$, γ is diagonal. □

Denote by Aut$\mathcal{C}$ the automorphism group of the category $\mathcal{C}(P,R)$. An automorphism $\varphi \in$ Aut$\mathcal{C}$ is called *inner* if there is a diagonal invertible series $\beta \in FI(P,R)$, such that for each $\alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x},\bar{y})$ we have $\varphi(\alpha_{\bar{x}\bar{y}}) = \beta_{\bar{x}}\alpha_{\bar{x}\bar{y}}\beta_{\bar{y}}^{-1}$. The set of inner automorphisms forms a normal subgroup of Aut$\mathcal{C}$, which is denoted by Inn$\mathcal{C}$. Accordingly, Out$\mathcal{C}$ = Aut$\mathcal{C}$/Inn$\mathcal{C}$ denotes the group of outer automorphisms of the category $\mathcal{C}(P,R)$.

The following theorem is the main result of this section.

Theorem 1. *The group* OutFI *is isomorphic to* Out$\mathcal{C}$.

Proof. We shall build an epimorphism $f : \mathrm{Aut}FI \to \mathrm{Out}\mathcal{C}$ and prove that its kernel coincides with InnFI.

Let $\Phi \in \mathrm{Aut}FI$. There is a bijection $\varphi : Ob\,\mathcal{C}(P,R) \to Ob\,\mathcal{C}(P,R)$ given by (4). Define the corresponding mapping of the morphisms $\varphi : \mathrm{Mor}(\bar{x},\bar{y}) \to \mathrm{Mor}(\varphi(\bar{x}),\varphi(\bar{y}))$ (we denote it by the same letter). According to the Lemma 3 there is $\tau_\beta \in \mathrm{Inn}FI$, such that (7) is satisfied. Consider $\alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x},\bar{y})$ and identify it with the series $\varepsilon(\alpha_{\bar{x}\bar{y}})$, where ε is the embedding of the semigroup Mor$\,\mathcal{C}(P,R)$ in the multiplicative semigroup $FI(P,R)$, namely: $\varepsilon(\alpha_{\bar{x}\bar{y}}) = \alpha_{\bar{x}\bar{y}}[\bar{x},\bar{y}]$. Then by (3) we have $\varepsilon(\alpha_{\bar{x}\bar{y}}) = \delta_{\bar{x}}\varepsilon(\alpha_{\bar{x}\bar{y}})\delta_{\bar{y}}$. Therefore, $\Phi\varepsilon(\alpha_{\bar{x}\bar{y}}) = \Phi(\delta_{\bar{x}})\Phi\varepsilon(\alpha_{\bar{x}\bar{y}})\Phi(\delta_{\bar{y}})$. Using (7) we obtain $\beta\Phi\varepsilon(\alpha_{\bar{x}\bar{y}})\beta^{-1} = \delta_{\varphi(\bar{x})}\beta\Phi\varepsilon(\alpha_{\bar{x}\bar{y}})\beta^{-1}\delta_{\varphi(\bar{y})}$. In other words, $\tau_\beta\Phi\varepsilon(\alpha_{\bar{x}\bar{y}}) = \varepsilon((\tau_\beta\Phi\varepsilon(\alpha_{\bar{x}\bar{y}}))_{\varphi(\bar{x})\varphi(\bar{y})})$. Thus,

$$\varphi = \varepsilon^{-1}\tau_\beta\Phi\varepsilon \tag{12}$$

defines the required mapping. Obviously, it is an isomorphism of the abelian groups and $\varphi(\delta_{\bar{x}\bar{x}}) = \delta_{\varphi(\bar{x})\varphi(\bar{x})}$. Moreover, φ agrees with the composition, because ε does. So, there is a mapping $f : \mathrm{Aut}FI \to \mathrm{Out}\mathcal{C}$, namely

$$f(\Phi) = \varphi \cdot \mathrm{Inn}\mathcal{C}. \tag{13}$$

According to the Remark 2, the definition of f is correct. Prove that f is a homomorphism. Consider another automorphism $\Psi \in \mathrm{Aut}FI$, $f(\Psi) = \psi\cdot\mathrm{Inn}\mathcal{C}$. As it was mentioned above, $\Phi(\delta_{\bar{x}}) = \tau_{\beta^{-1}}(\delta_{\varphi(\bar{x})})$. Applying Lemma 3 to Ψ, we obtain

$$\Psi\Phi(\delta_x) = \tau_{\Psi(\beta^{-1})}\Psi(\delta_{\varphi(\bar{x})}) = \tau_{\Psi(\beta^{-1})\gamma^{-1}}(\delta_{\psi\circ\varphi(\bar{x})})$$

for some invertible $\gamma \in FI(P,R)$. Therefore, $\tau_{\gamma\Psi(\beta)}\Psi\Phi(\delta_x) = \delta_{\psi\circ\varphi(\bar{x})}$. Thus, $f(\Psi\Phi) = \chi\cdot\mathrm{Inn}\mathcal{C}$, where χ acts on objects as $\psi\circ\varphi$ and on morphisms as $\varepsilon^{-1}\tau_{\gamma\Psi(\beta)}\Psi\Phi\varepsilon = (\varepsilon^{-1}\tau_\gamma\Psi\varepsilon)(\varepsilon^{-1}\tau_\beta\Phi\varepsilon)$ (see (12)); hence f is a homomorphism.

Conversely, let $\varphi \in \mathrm{Aut}\mathcal{C}$, $\alpha \in FI(P,R)$. Define $\widehat{\varphi}(\alpha)$ as follows:

$$\widehat{\varphi}(\alpha)_{\bar{x}\bar{y}} = \varphi(\alpha_{\varphi^{-1}(\bar{x})\varphi^{-1}(\bar{y})}).$$

Obviously, $\widehat{\varphi}$ is linear. Furthermore, since φ and φ^{-1}, being the functions on $\overline{P}$, preserve the partial order,

$$\widehat{\varphi}(\alpha\beta)_{\bar{x}\bar{y}} = \sum_{\bar{x}\leq\bar{z}\leq\bar{y}} \varphi(\alpha_{\varphi^{-1}(\bar{x})\varphi^{-1}(\bar{z})})\varphi(\beta_{\varphi^{-1}(\bar{z})\varphi^{-1}(\bar{y})}) = (\widehat{\varphi}(\alpha)\widehat{\varphi}(\beta))_{\bar{x}\bar{y}}.$$

Therefore, $\widehat{\varphi} \in \mathrm{Aut}FI$. Obviously, $\widehat{\varphi}(\delta_{\bar{x}}) = \delta_{\varphi(\bar{x})}$ and hence $f(\widehat{\varphi}) = \varphi{\cdot}\mathrm{Inn}\mathcal{C}$.

By (12) and (13) $Ker f$ consists of the automorphisms Φ, for which the image $\varepsilon^{-1}\tau_\beta\Phi\varepsilon(\alpha_{\bar{x}\bar{y}})$ coincides with $\gamma_{\bar{x}}\alpha_{\bar{x}\bar{y}}\gamma_{\bar{y}}^{-1}$ for all $\bar{x} \leq \bar{y}, \alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x}, \bar{y})$ and for some diagonal invertible $\gamma \in FI(P, R)$. This is equivalent to $\tau_{\gamma^{-1}\beta}\Phi(\alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}]) = \alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}]$. In particular, $\tau_{\gamma^{-1}\beta}\Phi(\delta_{\bar{x}}) = \delta_{\bar{x}}$. Denote $\Phi_1 = \tau_{\gamma^{-1}\beta}\Phi$ for short. Then, using (3), for an arbitrary $\alpha \in FI(P, R)$ we have:

$$\Phi_1(\alpha)_{\bar{x}\bar{y}}[\bar{x}, \bar{y}] = \delta_{\bar{x}}\Phi_1(\alpha)\delta_{\bar{y}} = \Phi_1(\delta_{\bar{x}}\alpha\delta_{\bar{y}}) = \Phi_1(\alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}]) = \alpha_{\bar{x}\bar{y}}[\bar{x}, \bar{y}].$$

Thus, $\tau_{\gamma^{-1}\beta}\Phi = \mathrm{id}_{FI(P,R)}$, i. e. $\Phi = \tau_{\beta^{-1}\gamma}$. □

2. The group $\mathrm{Out}\mathcal{C}$

Theorem 1 shows that the study of the group of outer automorphisms of the finitary ring is reduced to the study of the group of outer automorphisms of the category $\mathcal{C}(P, R)$.

Denote by $\mathrm{Aut}_0\mathcal{C}$ the subgroup of $\mathrm{Aut}\mathcal{C}$, consisting of the automorphisms of $\mathcal{C}(P, R)$, which act identically on the objects. Let $\mathrm{Out}_0\mathcal{C}$ denote the image of $\mathrm{Aut}_0\mathcal{C}$ in $\mathrm{Out}\mathcal{C}$.

Theorem 2. *The following sequence of groups is exact:*

$$1 \to \mathrm{Out}_0\mathcal{C} \to \mathrm{Out}\mathcal{C} \to \mathrm{Aut}\overline{P},$$

where $\mathrm{Aut}\overline{P}$ *is the automorphism group of the poset* $\overline{P}$.

Proof. Let $\varphi \in \mathrm{Aut}\mathcal{C}$. Then obviously $\varphi_{Ob} \in \mathrm{Aut}\overline{P}$, where φ_{Ob} is the restriction of φ to the set $Ob\,\mathcal{C} = \overline{P}$. Note that if $\varphi \in \mathrm{Inn}\mathcal{C}$, then $\varphi_{Ob} = \mathrm{id}$. Hence $f : \mathrm{Out}\mathcal{C} \to \mathrm{Aut}\overline{P}$ is defined, namely:

$$f(\varphi \cdot \mathrm{Inn}\mathcal{C}) = \varphi_{Ob}. \tag{14}$$

Obviously, f is a homomorphism and its kernel consists of the cosets $\varphi \cdot \mathrm{Inn}\mathcal{C}$, for which $\varphi(\bar{x}) = \bar{x}$, i. e. $Ker f = \mathrm{Out}_0\mathcal{C}$. □

We are interested in the image of $\mathrm{Out}\mathcal{C}$ in $\mathrm{Aut}\overline{P}$. For this reason suppose that the ring R has the following property:

$$M_{X\times X}(R) \cong M_{Y\times Y}(R) \Rightarrow |X| = |Y|. \tag{15}$$

In particular, commutative rings satisfy (15) for finite X and Y (see [3, Corollary 5.13]); we shall give another class of such rings below.

Let Aut$P$ denote the automorphism group of the quasiordered set P. The image of an arbitrary class $\bar{x} \subset P$ under $\varphi \in \mathrm{Aut}P$ is again a class $\overline{\varphi(x)}$, such that $|\varphi(\bar{x})| = |\bar{x}|$. An automorphism φ is called inner if $\varphi(\bar{x}) = \bar{x}$. The subgroup of inner automorphisms is denoted by Inn$P$, then the group of outer automorphisms is $\mathrm{Out}P = \mathrm{Aut}P/\mathrm{Inn}P$.

Lemma 4. *Under the condition* (15) *the image of the group* Out$\mathcal{C}$ *in* Aut$\overline{P}$ *is isomorphic to the group* Out$P$.

Proof. Taking into account the remark before the lemma, it is easy to show that the group Out$P$ is isomorphic to the subgroup G of Aut$\overline{P}$, consisting of the automorphisms ψ, such that $|\psi(\bar{x})| = |\bar{x}|$ for all $\bar{x} \in \overline{P}$. Therefore, we need to prove that $f(\mathrm{Out}\mathcal{C}) = G$, where f is the homomorphism defined by (14).

Let $\varphi \in \mathrm{Aut}\mathcal{C}$. Since φ is an automorphism, $M_{\bar{x}\times\bar{x}}(R)$ is isomorphic to $M_{\varphi(\bar{x})\times\varphi(\bar{x})}(R)$. Therefore, by (15) $|\varphi(\bar{x})| = |\bar{x}|$ and hence $\varphi_{Ob} \in G$. Conversely, take $\psi \in G$ and extend it arbitrarily to the automorphism of P. Define $\widehat{\psi}(\alpha_{\bar{x}\bar{y}}) \in \mathrm{Mor}(\psi(\bar{x}), \psi(\bar{y}))$ as follows:

$$\widehat{\psi}(\alpha_{\bar{x}\bar{y}})(\psi(x'), \psi(y')) = \alpha_{\bar{x}\bar{y}}(x', y'), \tag{16}$$

where $x' \in \bar{x}$, $y' \in \bar{y}$, $\alpha_{\bar{x}\bar{y}}(x', y')$ is the element of the matrix $\alpha_{\bar{x}\bar{y}}$, corresponding to the pair (x', y'). The definition is correct, because ψ maps bijectively $\bar{x}$ onto $\psi(\bar{x})$ and $\bar{y}$ onto $\psi(\bar{y})$. Moreover, $\widehat{\psi}$ is an isomorphism of the abelian groups $\mathrm{Mor}(\bar{x}, \bar{y})$ and $\mathrm{Mor}(\psi(\bar{x}), \psi(\bar{y}))$ with $\widehat{\psi}(\mathrm{id}_{\bar{x}}) = \mathrm{id}_{\psi(\bar{x})}$. Furthermore, since ψ is an automorphism of P,

$$\widehat{\psi}(\alpha_{\bar{x}\bar{y}}\alpha_{\bar{y}\bar{z}})(\psi(x'), \psi(z')) = (\widehat{\psi}(\alpha_{\bar{x}\bar{y}})\widehat{\psi}(\alpha_{\bar{y}\bar{z}}))(\psi(x'), \psi(z')).$$

Thus, $\widehat{\psi} \in \mathrm{Aut}\mathcal{C}$. Finally, note that $f(\widehat{\psi} \cdot \mathrm{Inn}\mathcal{C}) = \psi$. □

Theorem 3. *Let the ring R satisfy* (15). *Then the group* Out$\mathcal{C}$ *is isomorphic to the semidirect product* $\mathrm{Out}_0\mathcal{C} \rtimes \mathrm{Out}P$.

Proof. Identify Out$P$ with the subgroup G of Aut$\overline{P}$. By the Theorem 2 and the Lemma 4 it is sufficient to build the monomorphism $g : G \to \mathrm{Out}\mathcal{C}$, such that $fg = \mathrm{id}_G$. Fix the numeration of the elements in each $\bar{x} \subset P$. Let $\omega(x)$ denote the number of the element x in the equivalence class $\bar{x}$. We shall say that $\varphi \in \mathrm{Aut}P$ agrees with ω if $\omega(\varphi(x)) = \omega(x)$ for all $x \in P$. Note that in each coset of the subgroup Inn$P$ there is a unique automorphism, which agrees with ω, because an inner automorphism, which agrees with ω, is the identity. Let $\psi \in G$. Extend ψ to the automorphism

ψ_ω of the set P, which agrees with ω. By our remark this can be done uniquely. Then the mapping $g(\psi) = \widehat{\psi_\omega} \cdot \mathrm{Inn}\mathcal{C}$, where $\widehat{\psi_\omega}$ is given by (16), is defined correctly. Obviously, $\widehat{(\psi\eta)_\omega} = \widehat{\psi_\omega}\widehat{\eta_\omega}$. Thus, g is a homomorphism. Suppose that $\widehat{\psi_\omega} \in \mathrm{Inn}\mathcal{C}$. Then, in particular, $\widehat{\psi_\omega}(\alpha_{\bar{x}\bar{x}}) \in \mathrm{Mor}(\bar{x}, \bar{x})$, i. e. $\psi(\bar{x}) = \bar{x}$. Hence, $\psi = \mathrm{id}_{\overline{P}}$ and therefore g is a monomorphism. Finally $(\widehat{\psi_\omega})_{Ob} = \psi$ by (16). This means that $f(g(\psi)) = \psi$. $\square$

Show that the condition (15) is essential.

Example 1. Let R be a ring, such that $R_R^2 \cong R_R^3$ (see [1]). Take P with $\overline{P} = \{\bar{x}, \bar{y}, 1\}$, where $\bar{x} = \{x_1, x_2\}$, $\bar{y} = \{y_1, y_2, y_3\}$, 1 is an one-element class; $\bar{x}$ and $\bar{y}$ are incomparable, $\bar{x}, \bar{y} < 1$. Then $\mathrm{Out}\mathcal{C} \neq \mathrm{Out}_0\mathcal{C} \rtimes \mathrm{Out}P$.

Indeed, it is easy to see that $\mathrm{Out}P = 1$. Therefore, we need to prove that $\mathrm{Out}\mathcal{C} \neq \mathrm{Out}_0\mathcal{C}$, i. e. to find an automorphism φ of the category $\mathcal{C}(P, R)$, such that $\varphi_{Ob} \neq \mathrm{id}$. Note that $\mathrm{Mor}(\bar{x}, \bar{x}) = M_2(R)$, $\mathrm{Mor}(\bar{y}, \bar{y}) = M_3(R)$, $\mathrm{Mor}(1, 1) = R$, $\mathrm{Mor}(\bar{x}, 1) = R_R^2$, $\mathrm{Mor}(\bar{y}, 1) = R_R^3$, $\mathrm{Mor}(\bar{x}, \bar{y}) = 0$ (here $M_n(R)$ denotes the ring of $n \times n$ matrices over R). It is convenient to represent the elements of R_R^2 and R_R^3 by the columns. Then $M_2(R) \cong End(R_R^2)$, $M_3(R) \cong End(R_R^3)$, where a matrix acts on a column by the left multiplication (since the modules are right). Let $f : R_R^2 \to R_R^3$ be an isomorphism. For an arbitrary $A \in M_2(R)$ define $g(A) \in M_3(R)$ by its action on a column $(r_1, r_2, r_3)^T \in R_R^3$:

$$g(A)(r_1, r_2, r_3)^T = fAf^{-1}(r_1, r_2, r_3)^T.$$

Obviously, g is an isomorphism of the rings $M_2(R)$ and $M_3(R)$. Note that $g(A)f(r_1, r_2)^T = fA(r_1, r_2)^T$ for an arbitrary $(r_1, r_2)^T \in R_R^2$. Define the mapping of the morphisms φ as follows: $\varphi|_{\mathrm{Mor}(\bar{x},1)} = f : \mathrm{Mor}(\bar{x}, 1) \to \mathrm{Mor}(\bar{y}, 1)$, $\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})} = g : \mathrm{Mor}(\bar{x}, \bar{x}) \to \mathrm{Mor}(\bar{y}, \bar{y})$, $\varphi|_{\mathrm{Mor}(1,1)} = \mathrm{id}$. By the construction $\varphi \in \mathrm{Aut}\mathcal{C}$ and $\varphi_{Ob}(\bar{x}) = \bar{y}$.

3. The group $\mathrm{Out}_0\mathcal{C}$

In this section we are going to investigate the group $\mathrm{Out}_0\mathcal{C}$. Obviously, the restriction of any automorphism $\varphi \in \mathrm{Aut}_0\mathcal{C}$ to the ring $\mathrm{Mor}(\bar{x}, \bar{x})$ is an automorphism of this ring. Denote by $\mathrm{Aut}_1\mathcal{C}$ the subgroup consisting of those automorphisms φ from $\mathrm{Aut}_0\mathcal{C}$, for which

$$\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})} = \mathrm{id} \tag{17}$$

for all $\bar{x} \in \overline{P}$. Let $\mathrm{Out}_1\mathcal{C}$ be an image of this subgroup in $\mathrm{Out}\mathcal{C}$. We shall first describe $\mathrm{Out}_1\mathcal{C}$.

Recall that the order complex $K(X)$ of a poset X is the simplicial complex, whose n-dimensional faces are the chains of length n in X. Let $C^n(X,A)$, $Z^n(X,A)$, $B^n(X,A)$ and $H^n(X,A)$ denote the groups of n-dimensional cochains, cocycles, coboundaries and cohomologies of the complex $K(X)$ with the values in an abelian group A.

Lemma 5. *The group* $\mathrm{Out}_1\mathcal{C}$ *is isomorphic to* $H^1(\overline{P},C(R)^*)$, *where* $C(R)^*$ *is the multiplicative group of the central invertible elements of the ring* R.

Proof. Prove that $\mathrm{Aut}_1\mathcal{C}\cong Z^1(\overline{P},C(R)^*)$ and $\mathrm{Aut}_1\mathcal{C}\cap\mathrm{Inn}\mathcal{C}$ goes to $B^1(\overline{P},C(R)^*)$ under this isomorphism. Let $\varphi\in\mathrm{Aut}_1\mathcal{C}$, $\bar{x},\bar{y}\in\overline{P}$, $\bar{x}\le\bar{y}$, $x'\sim x$, $y'\sim y$. Consider $\delta_{x'y'}\in\mathrm{Mor}(\bar{x},\bar{y})$, defined as follows:

$$\delta_{x'y'}(u,v) \quad = \quad \begin{cases}1, & \text{if } u=x', v=y',\\ 0, & \text{otherwise.}\end{cases} \tag{18}$$

Note that

$$\delta_{x'x}\alpha_{\bar{x}\bar{y}}\delta_{yy'}=\alpha_{\bar{x}\bar{y}}(x,y)\delta_{x'y'} \tag{19}$$

for each $\alpha_{\bar{x}\bar{y}}\in\mathrm{Mor}(\bar{x},\bar{y})$. In particular, $\delta_{x'x}\delta_{xy}\delta_{yy'}=\delta_{x'y'}$. Apply φ to this equality. Since $\delta_{x'x}\in\mathrm{Mor}(\bar{x},\bar{x})$ and $\delta_{yy'}\in\mathrm{Mor}(\bar{y},\bar{y})$, using (17) we obtain $\delta_{x'x}\varphi(\delta_{xy})\delta_{yy'}=\varphi(\delta_{x'y'})$. Therefore by (19) we have

$$\varphi(\delta_{x'y'})=\sigma(\bar{x},\bar{y})\delta_{x'y'} \tag{20}$$

for some $\sigma(\bar{x},\bar{y})\in R$ and for all $x'\in\bar{x}, y'\in\bar{y}$. Prove that $\sigma(\bar{x},\bar{y})$ belongs to the center of R. Indeed, for an arbitrary $r\in R$ according to (18) and (19) we have

$$\varphi(r\delta_{xy})=\varphi(r\delta_{xx}\delta_{xy})=\varphi(r\delta_{xx})\varphi(\delta_{xy})=r\delta_{xx}\sigma(\bar{x},\bar{y})\delta_{xy}=r\sigma(\bar{x},\bar{y})\delta_{xy}. \tag{21}$$

Similarly $\varphi(r\delta_{xy})=\varphi(\delta_{xy}r\delta_{yy})=\sigma(\bar{x},\bar{y})r\delta_{xy}$. Therefore $r\sigma(\bar{x},\bar{y})=\sigma(\bar{x},\bar{y})r$. Since r is arbitrary, $\sigma(\bar{x},\bar{y})\in C(R)$. Prove that $\sigma(\bar{x},\bar{y})$ is invertible. By (19) $R\delta_{xy}=\delta_{xx}\mathrm{Mor}(\bar{x},\bar{y})\delta_{yy}$. Hence $\varphi(R\delta_{xy})=R\delta_{xy}$. This means that there is $r\in R$, such that $\varphi(r\delta_{xy})=\delta_{xy}$. Then it follows from (21) that $r\sigma(\bar{x},\bar{y})=1$. Since $\sigma(\bar{x},\bar{y})\in C(R)$, $r=\sigma(\bar{x},\bar{y})^{-1}$. So $\sigma\in C^1(\overline{P},C(R)^*)$.

Prove that σ is actually a cocycle. Indeed, it is easy to see that $\delta_{xy}\delta_{yz}=\delta_{xz}$ for arbitrary $x\preccurlyeq y\preccurlyeq z$. Hence by (20) $\sigma(\bar{x},\bar{y})\sigma(\bar{y},\bar{z})=\sigma(\bar{x},\bar{z})$. Now determine how φ acts on an arbitrary $\alpha_{\bar{x}\bar{y}}\in\mathrm{Mor}(\bar{x},\bar{y})$. According to (19), $\varphi(\alpha_{\bar{x}\bar{y}})(x,y)\delta_{xy}=\delta_{xx}\varphi(\alpha_{\bar{x}\bar{y}})\delta_{yy}$. By (17) the last product is equal to $\varphi(\delta_{xx}\alpha_{\bar{x}\bar{y}}\delta_{yy})$. But $\delta_{xx}\alpha_{\bar{x}\bar{y}}\delta_{yy}=\alpha_{\bar{x}\bar{y}}(x,y)\delta_{xy}$ and hence by (21) $\varphi(\delta_{xx}\alpha_{\bar{x}\bar{y}}\delta_{yy})=\alpha_{\bar{x}\bar{y}}(x,y)\sigma(\bar{x},\bar{y})\delta_{xy}$. Finally

$$\varphi(\alpha_{\bar{x}\bar{y}})=\sigma(\bar{x},\bar{y})\alpha_{\bar{x}\bar{y}}. \tag{22}$$

Conversely, each $\sigma \in Z^1(\overline{P}, C(R)^*)$ defines an automorphism $\varphi \in \mathrm{Aut}_1\mathcal{C}$ with the help of (22). Obviously, the correspondence $\varphi \leftrightarrow \sigma$ is bijective and agrees with the multiplication in $\mathrm{Aut}_1\mathcal{C}$ and $Z^1(\overline{P}, C(R)^*)$.

Now let $\varphi \in \mathrm{Aut}_1\mathcal{C} \cap \mathrm{Inn}\mathcal{C}$ and $\beta \in FI(P, R)$ be the corresponding diagonal invertible series. Take arbitrary $\bar{x} \in P$, $x', x'' \sim x$. By (17) and the definition of the conjugation $\beta_{\bar{x}\bar{x}}\delta_{x'x'} = \delta_{x'x'}\beta_{\bar{x}\bar{x}}$. If $x' \neq x''$, then the value of the left-hand side of this equality at the segment $[x', x'']$ obviously equals zero, while the value of the right-hand side equals $\beta_{\bar{x}\bar{x}}(x', x'')$. Since x', x'' are the arbitrary elements of the class $\bar{x}$, $\beta_{\bar{x}\bar{x}}$ is a diagonal matrix for each $\bar{x}$. Furthermore, $\beta_{\bar{x}\bar{x}}\delta_{x'x''} = \delta_{x'x''}\beta_{\bar{x}\bar{x}}$ implies $\beta_{\bar{x}\bar{x}}(x', x') = \beta_{\bar{x}\bar{x}}(x'', x'')$. Thus, $\beta_{\bar{x}\bar{x}} = \lambda(\bar{x})\delta_{\bar{x}\bar{x}}$ for some function $\lambda : \overline{P} \to R^*$. Then $\beta_{\bar{x}\bar{x}}\alpha_{\bar{x}\bar{y}}\beta^{-1}_{\bar{y}\bar{y}} = \lambda(\bar{x})\alpha_{\bar{x}\bar{y}}\lambda(\bar{y})^{-1}$. Taking $x = y$ and $\alpha_{\bar{x}\bar{y}} = r\delta_{\bar{x}\bar{x}}$ by (17) we obtain $\lambda(\bar{x})r = r\lambda(\bar{x})$. Therefore, $\lambda(\bar{x}) \in C(R)^*$. Thus, a cocycle, corresponding to φ, satisfies $\sigma(\bar{x}, \bar{y}) = \lambda(\bar{x})\lambda(\bar{y})^{-1}$, i. e. it is a coboundary. Conversely, let $\sigma(\bar{x}, \bar{y}) = \lambda(\bar{x})\lambda(\bar{y})^{-1}$ for some $\lambda \in C^0(\overline{P}, C(R)^*)$. Define $\beta = \sum_{\bar{x}\in\overline{P}} \lambda(\bar{x})\delta_{\bar{x}\bar{x}}[\bar{x}, \bar{x}]$. Then, obviously, the conjugation by β coincides with the action of σ. □

Denote by $\mathrm{Out}M_{\bar{x}\times\bar{x}}(R)$ the group of outer automorphisms of the ring $M_{\bar{x}\times\bar{x}}(R)$.

Theorem 4. *The following sequence of groups is exact:*

$$1 \to H^1(\overline{P}, C(R)^*) \to \mathrm{Out}_0\mathcal{C} \to \prod_{\bar{x}\in\overline{P}} \mathrm{Out}M_{\bar{x}\times\bar{x}}(R). \tag{23}$$

Proof. Define $f : \mathrm{Aut}_0\mathcal{C} \to \prod_{\bar{x}\in\overline{P}} \mathrm{Aut}M_{\bar{x}\times\bar{x}}(R)$ as follows:

$$f(\varphi) = \{\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})}\}_{\bar{x}\in\overline{P}}$$

for an arbitrary $\varphi \in \mathrm{Aut}_0\mathcal{C}$. Obviously, f is a homomorphism, under which $\mathrm{Inn}\mathcal{C}$ goes to $\prod_{\bar{x}\in\overline{P}} \mathrm{Inn}M_{\bar{x}\times\bar{x}}(R)$. Hence a mapping $\bar{f} : \mathrm{Out}_0\mathcal{C} \to \prod_{\bar{x}\in\overline{P}} \mathrm{Out}M_{\bar{x}\times\bar{x}}(R)$ is defined, namely $\bar{f}(\varphi \cdot \mathrm{Inn}\mathcal{C}) = f(\varphi) \cdot \prod_{\bar{x}\in\overline{P}} \mathrm{Inn}M_{\bar{x}\times\bar{x}}(R)$. Moreover, the kernel of $\bar{f}$ consists of those $\varphi \cdot \mathrm{Inn}\mathcal{C}$, for which $\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})} = \mathrm{id}$ for all $\bar{x} \in \overline{P}$. Therefore, $Ker f$ coincides with the group $\mathrm{Out}_1\mathcal{C}$, which is isomorphic to $H^1(\overline{P}, C(R)^*)$ by the previous lemma. □

Corollary 1. *Let P be an arbitrary quasiordered set, R an indecomposable unital ring, such that for any sets X and Y an isomorphism $M_{X\times X}(R) \cong M_{Y\times Y}(R)$ implies $|X| = |Y|$. Then* $\mathrm{Out}FI \cong \mathrm{Out}_0\mathcal{C} \rtimes \mathrm{Out}P$*, where* $\mathrm{Out}_0\mathcal{C}$ *belongs to the exact sequence* (23).

The description of the image of $\mathrm{Out}_0\mathcal{C}$ in $\prod_{\bar{x}\in\overline{P}} \mathrm{Out}M_{\bar{x}\times\bar{x}}(R)$ seems to be difficult in general situation, so we shall restrict ourselves to one special case. Recall that the ring R is called local if $R/\mathrm{Rad}R$ is a division ring. In particular, R is indecomposable. Prove that R satisfies (15) in the case of finite X and Y. Denote by $M_n(R)$ the ring of $n \times n$ matrices over R. Suppose that $M_n(R) \cong M_m(R)$. Consider the matrix units $\delta_{ij} \in M_n(R)$, $i, j = 1, \ldots, n$. By definition $\{\delta_{ii}\}_{i=1}^n$ is the decomposition of the unit of $M_n(R)$. Obviously,

$$\delta_{ii} M_n(R) \delta_{jj} = R\delta_{ij}. \tag{24}$$

In particular, $\delta_{ii} M_n(R) \delta_{ii} \cong R$ is a local ring. Therefore, δ_{ii} is completely primitive [5, p. 59, Definition 2] for all i. If $\varphi : M_n(R) \to M_m(R)$ is an isomorphism, then $\{\varphi(\delta_{ii})\}_{i=1}^n$ is the decomposition of the unit of $M_m(R)$, consisting of the completely primitive idempotents. But $M_m(R)$ already has such decomposition of the unit of cardinality m. Then by [5, p. 59, Theorem 2] $n = m$.

Let $\psi \in \mathrm{Aut}R$, $\alpha \in M_n(R)$. Define

$$(\widehat{\psi}(\alpha))_{ij} = \psi(\alpha_{ij}). \tag{25}$$

Obviously, $\widehat{\psi} \in \mathrm{Aut}M_n(R)$. It turns out that in the case when R is local, each automorphism of the ring $M_n(R)$ can be represented as $\widehat{\psi}$ up to conjugacy.

Lemma 6. *Let R be a local ring, $\varphi \in \mathrm{Aut}M_n(R)$. Then there is a unique up to conjugacy in R automorphism $\psi \in \mathrm{Aut}R$ and an invertible matrix $\beta \in M_n(R)$, such that $\varphi = \tau_\beta \widehat{\psi}$, where τ_β is the conjugation by β and $\widehat{\psi}$ is defined by* (25).

Proof. Let $\{\delta_{ij}\}_{i,j=1}^n$ be matrix units of the ring $M_n(R)$. According to the reasoning before the theorem, $\{\delta_{ii}\}_{i=1}^n$ and $\{\varphi(\delta_{ii})\}_{i=1}^n$ are two decompositions of the unit consisting of the completely primitive idempotents. By [5, p. 59, Theorem 2] there is an invertible matrix $\beta_1 \in M_n(R)$, such that $\tau_{\beta_1}\varphi(\delta_{ii}) = \delta_{ii}$. Therefore, by (24) there are $\varphi_i \in \mathrm{Aut}R$, such that

$$\tau_{\beta_1}\varphi(r\delta_{ii}) = \varphi_i(r)\delta_{ii} \tag{26}$$

for an arbitrary $r \in R$. Since it is easy to show that β_1 is determined up to the diagonal multiplier, each φ_i is determined up to the inner automorphism of the ring R. According to (24) denote by σ_{ij} the element of the ring R, such that

$$\tau_{\beta_1}\varphi(\delta_{ij}) = \sigma_{ij}\delta_{ij}. \tag{27}$$

In particular, $\sigma_{ii} = 1$. Furthermore $\delta_{ij}\delta_{jk} = \delta_{ik}$ implies $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$. Therefore, σ_{ij} is invertible and

$$\sigma_{ij} = \sigma_{i1}\sigma_{j1}^{-1}. \tag{28}$$

Now take an arbitrary $r \in R$ and consider $r\delta_{i1}$. By the definition of the matrix units $r\delta_{ii}\delta_{i1} = \delta_{i1}r\delta_{11}$ and hence $\varphi_i(r)\sigma_{i1} = \sigma_{i1}\varphi_1(r)$. Therefore,

$$\varphi_i = \tau_{\sigma_{i1}}\varphi_1. \tag{29}$$

Consider a diagonal matrix $\beta_2 = \sum\limits_{i=1}^{n} \sigma_{i1}\delta_{ii}$. The equalities (26), (27), (28) and (29) imply

$$\tau_{\beta_1}\varphi(r\delta_{ij}) = (\tau_{\beta_1}\varphi(r\delta_{ii}))(\tau_{\beta_1}\varphi(\delta_{ij})) = \varphi_i(r)\delta_{ii}\sigma_{ij}\delta_{ij} =$$
$$\sigma_{i1}\varphi_1(r)\sigma_{i1}^{-1}\sigma_{i1}\sigma_{j1}^{-1}\delta_{ij} = \sigma_{i1}\varphi_1(r)\sigma_{j1}^{-1}\delta_{ij} = \tau_{\beta_2}(\varphi_1(r)\delta_{ij}).$$

Hence $\tau_\beta\varphi(r\delta_{ij}) = \varphi_1(r)\delta_{ij}$, where $\beta = \beta_2^{-1}\beta_1$. Take an arbitrary matrix $\alpha \in M_n(R)$. Since $\delta_{ii}\alpha\delta_{jj} = \alpha_{ij}\delta_{ij}$, we have

$$\delta_{ii}(\tau_\beta\varphi(\alpha))\delta_{jj} = \tau_\beta\varphi(\delta_{ii}\alpha\delta_{jj}) = \tau_\beta\varphi(\alpha_{ij}\delta_{ij}) = \varphi_1(\alpha_{ij})\delta_{ij}$$

and hence

$$(\tau_\beta\varphi(\alpha))_{ij} = \varphi_1(\alpha_{ij}).$$

Thus, $\varphi = \tau_{\beta^{-1}}\widehat{\varphi_1}$. □

Corollary 2. *Let R be a local ring. Then* $\mathrm{Out}M_n(R) \cong \mathrm{Out}R$.

Proof. Let $\varphi \in \mathrm{Aut}M_n(R)$. According to the previous lemma, $\varphi = \tau_\beta\widehat{\psi}$ and ψ is defined up to conjugacy in R. Put $f(\varphi \cdot \mathrm{Inn}M_n(R)) = \psi \cdot \mathrm{Inn}R$. Obviously, f is defined correctly and it is a homomorphism of the groups $\mathrm{Out}M_n(R)$ and $\mathrm{Out}R$. Prove that f is a monomorphism. Indeed, if ψ is an inner automorphism of R, then $\widehat{\psi}$ is a conjugation in $M_n(R)$ by a scalar matrix and $\varphi \in \mathrm{Inn}M_n(R)$. The surjectivity of f is obvious, because $f(\widehat{\psi} \cdot \mathrm{Inn}M_n(R)) = \psi \cdot \mathrm{Inn}R$. □

Theorem 5. *Let R be a local ring, P a quasiordered set whose classes are finite, $P = \bigcup\limits_{i\in I} P_i$ the decomposition of P into the disjoint union of the connected components. Then the group $\mathrm{Out}_0\mathcal{C}$ is isomorphic to the semidirect product $H^1(\overline{P}, C(R)^*) \rtimes \prod\limits_{i\in I} \mathrm{Out}R$.*

Proof. Let $\varphi \in \mathrm{Aut}_0\mathcal{C}$. Applying Lemma 6 to $\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})}$ for each $\bar{x} \in \overline{P}$, we obtain the representation of $\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})}$ as $\tau_{\beta_{\bar{x}\bar{x}}}\widehat{\varphi_{\bar{x}}}$, where $\beta_{\bar{x}\bar{x}} \in M_{\bar{x}\times\bar{x}}(R)$ is an invertible matrix, $\varphi_{\bar{x}} \in \mathrm{Aut}R$ and $\widehat{\varphi_{\bar{x}}}$ is given by (25). Let $\beta = \sum\limits_{\bar{x}\in\overline{P}} \beta_{\bar{x}\bar{x}}[\bar{x},\bar{x}]$. Then $\beta \in FI(\mathcal{C})$ is a diagonal invertible series, such that $(\tau_{\beta^{-1}}\varphi)|_{\mathrm{Mor}(\bar{x},\bar{x})} = \widehat{\varphi_{\bar{x}}}$ for all $\bar{x} \in \overline{P}$ (here $\tau_{\beta^{-1}}$ means the inner automorphism of $\mathcal{C}(P,R)$ corresponding to β^{-1}). Thus, we can assume up to the conjugation by β that

$$\varphi(\alpha_{\bar{x}\bar{x}})(x',x'') = \varphi_{\bar{x}}(\alpha_{\bar{x}\bar{x}}(x',x'')) \tag{30}$$

for any $x',x'' \in \bar{x}$, $\alpha_{\bar{x}\bar{x}} \in \mathrm{Mor}(\bar{x},\bar{x})$. Take $x \prec y$ and show that $\varphi_{\bar{x}}$ differs from $\varphi_{\bar{y}}$ by an inner automorphism of the ring R. Since by (30) $\varphi(\delta_{x'x''}) = \delta_{x'x''}$ and $\varphi(\delta_{y'y''}) = \delta_{y'y''}$ for any $x',x'' \in \bar{x}$, $y',y'' \in \bar{y}$, we obtain as in the proof of the Lemma 5 that $\varphi(\delta_{x'y''}) = \sigma(\bar{x},\bar{y})\delta_{x'y''}$ for some constant $\sigma(\bar{x},\bar{y}) \in R$, which depends only on the classes $\bar{x}$ and $\bar{y}$. Therefore, $\varphi_{\bar{x}}(r)\sigma(\bar{x},\bar{y})\delta_{xy} = \varphi(r\delta_{xy}) = \sigma(\bar{x},\bar{y})\varphi_{\bar{y}}(r)\delta_{xy}$ for any $r \in R$. These equalities guarantee the invertibility of $\sigma(\bar{x},\bar{y})$, because it follows from (30) that $\varphi(R\delta_{xy}) = R\delta_{xy}$ (it is sufficient to take r, such that $\varphi(r\delta_{xy}) = \delta_{xy}$). Thus, $\varphi_{\bar{x}} = \tau_{\sigma(\bar{x},\bar{y})}\varphi_{\bar{y}}$.

Choose $x_i \in P_i$ for each $i \in I$ and put $g(\varphi\cdot\mathrm{Inn}\mathcal{C}) = \{\varphi_{\bar{x}_i}\cdot\mathrm{Inn}R\}_{i\in I}$. Our reasoning shows that g is defined correctly and it is a homomorphism of the groups $\mathrm{Out}_0\mathcal{C}$ and $\prod\limits_{i\in I}\mathrm{Out}R$ with the kernel $\mathrm{Out}_1\mathcal{C}$ which is isomorphic to $H^1(\overline{P},C(R)^*)$. It remains only to build and embedding $h : \prod\limits_{i\in I}\mathrm{Out}R \to \mathrm{Out}_0\mathcal{C}$, such that $gh = \mathrm{id}_{\prod\limits_{i\in I}\mathrm{Out}R}$. Let $\varphi_i \in \mathrm{Aut}R$, $i \in I$. For arbitrary $\bar{x} \le \bar{y}$, $x,y \in P_i$ and $\alpha_{\bar{x}\bar{y}} \in \mathrm{Mor}(\bar{x},\bar{y})$ define $\widehat{\varphi}(\alpha_{\bar{x}\bar{y}}) \in \mathrm{Mor}(\bar{x},\bar{y})$ as follows:

$$\widehat{\varphi}(\alpha_{\bar{x}\bar{y}})(x',y') = \varphi_i(\alpha_{\bar{x}\bar{y}}(x',y')), \tag{31}$$

where $x' \in \bar{x}$, $y' \in \bar{y}$. It is easy to see that $\widehat{\varphi} \in \mathrm{Aut}_0\mathcal{C}$, moreover, if all φ_i are inner, then $\widehat{\varphi}$ is also inner. Therefore, $h(\{\varphi_i\cdot\mathrm{Inn}R\}_{i\in I}) = \widehat{\varphi}\cdot\mathrm{Inn}\mathcal{C}$ is defined. Obviously, h is a homomorphism. Suppose that $\widehat{\varphi} \in \mathrm{Inn}\mathcal{C}$, i. e. $\widehat{\varphi}(\alpha_{\bar{x}\bar{y}}) = \gamma_{\bar{x}\bar{x}}\alpha_{\bar{x}\bar{y}}\gamma_{\bar{y}\bar{y}}^{-1}$ for any $\bar{x} \le \bar{y}$. Since $\widehat{\varphi}(\delta_{x'x''}) = \delta_{x'x''}$ for all $x',x'' \in \bar{x}$, $\gamma_{\bar{x}\bar{x}}$ is a scalar matrix, similarly so is $\gamma_{\bar{y}\bar{y}}$. Furthermore, since $\widehat{\varphi}(\delta_{x'y'})$ is by definition equal to $\delta_{x'y'}$, $\gamma_{\bar{x}\bar{x}}(x',x') = \gamma_{\bar{y}\bar{y}}(y',y')$. Therefore, $\gamma_{\bar{x}\bar{x}} = s_i\delta_{\bar{x}\bar{x}}, \gamma_{\bar{y}\bar{y}} = s_i\delta_{\bar{y}\bar{y}}$ for some $s_i \in R^*$. Thus, $\widehat{\varphi}(\alpha_{\bar{x}\bar{y}})(x',y') = \tau_{s_i}(\alpha_{\bar{x}\bar{y}}(x',y'))$, where s_i depends only on the connected component P_i, which contains x and y. This means that h is a monomorphism. Obviously, $g(\widehat{\varphi}\cdot\mathrm{Inn}\mathcal{C}) = \{\varphi_i\cdot\mathrm{Inn}R\}_{i\in I}$. □

Corollary 3. *Let R be a local ring, P a quasiordered set whose classes are finite, $P = \bigcup\limits_{i\in I} P_i$ the decomposition of P into the disjoint union* *of the connected components. Then the group* $\mathrm{Out}FI$ *is isomorphic to* $(H^1(\overline{P}, C(R)^*) \rtimes \prod_{i\in I} \mathrm{Out}R) \rtimes \mathrm{Out}P$.

Recall that the restriction (15) on R was imposed in order to assert that the isomorphism $M_{\varphi(\bar{x})\times\varphi(\bar{x})}(R) \cong M_{\bar{x}\times\bar{x}}(R)$, where $\varphi \in \mathrm{Aut}\mathcal{C}$, implies the equality $|\varphi(\bar{x})| = |\bar{x}|$. Suppose that P is partially ordered. Then $x \sim y$ iff $x = y$, i. e. all the equivalence classes under $\sim$ are one-element, $\overline{P} = P$ and $\mathrm{Out}P = \mathrm{Aut}P$. Therefore, we don't need to require (15). Furthermore, since $M_{\bar{x}\times\bar{x}}(R) = R$ for all $\bar{x}$, $\varphi|_{\mathrm{Mor}(\bar{x},\bar{x})} \in \mathrm{Aut}R$ and the Theorem 5 can be proved without using the Lemma 6. Thus, in the case of the partial order we can refuse the locality of R.

Remark 3. Let P be a partially ordered set, R an indecomposable ring, $P = \bigcup_{i\in I} P_i$ the decomposition of P into the disjoint union of the connected components. Then the group $\mathrm{Out}FI$ is isomorphic to $(H^1(P, C(R)^*) \rtimes \prod_{i\in I} \mathrm{Out}R) \rtimes \mathrm{Aut}P$.

4. $C(R)$-automorphisms of the ring $FI(P,R)$

Let A be a unital algebra over a commutative ring K, $\mathrm{Aut}A$ denote the group of its ring automorphisms, $\mathrm{Inn}A$ be a subgroup of inner automorphisms, $\mathrm{Out}A = \mathrm{Aut}A/\mathrm{Inn}A$. We say that an automorphism $\varphi \in \mathrm{Aut}A$ is a K-*automorphism*, if it agrees with the structure of K-algebra. K-automorphisms form a subgroup of $\mathrm{Aut}A$, which we denote by K-$\mathrm{Aut}A$. Note that an automorphism φ belongs to K-$\mathrm{Aut}A$ iff $\varphi(k\cdot 1) = k\cdot 1$ for all $k \in K$. Since K is commutative, $\mathrm{Inn}A \subset K$-$\mathrm{Aut}A$ and hence the group K-$\mathrm{Out}A = K$-$\mathrm{Aut}A/\mathrm{Inn}A$ is defined.

Now let P be a quasiordered set, R an arbitrary associative unital ring. Put $K = C(R)$. Then both R and $FI(P,R)$ are K-algebras. Therefore, the groups K-$\mathrm{Aut}R$, K-$\mathrm{Out}R$, K-$\mathrm{Aut}FI := K$-$\mathrm{Aut}FI(P,R)$, K-$\mathrm{Out}FI := K$-$\mathrm{Out}FI(P,R)$ are defined. By the Theorem 1 we can identify K-$\mathrm{Out}FI$ with the subgroup of $\mathrm{Out}\mathcal{C}$, which we shall denote by K-$\mathrm{Out}\mathcal{C}$. It is easy to see that K-$\mathrm{Out}\mathcal{C}$ consists of the cosets $\varphi\cdot\mathrm{Inn}\mathcal{C}$, where $\varphi(k\delta_{\bar{x}\bar{x}}) = k\delta_{\varphi(\bar{x})\varphi(\bar{x})}$ for all $\bar{x} \in \overline{P}$ and $k \in K$ (recall that $\delta_{\bar{x}\bar{x}}$ denotes the identity matrix in the ring $M_{\bar{x}\times\bar{x}}(R) = \mathrm{Mor}(\bar{x},\bar{x})$). In this section we describe K-$\mathrm{Out}FI$ in the case when the classes of P are finite and R is local.

Lemma 7. *Let R be a local ring, P a quasiordered set whose classes are finite, $P = \bigcup_{i\in I} P_i$ the decomposition of P into the disjoin union of the connected components, $f : (H^1(\overline{P}, K^*) \rtimes \prod_{i\in I} \mathrm{Out}R) \rtimes \mathrm{Out}P \to \mathrm{Out}\mathcal{C}$ the isomorphism from the previous section. Then*

1. $f(H^1(\overline{P},K^*)) \subset K\text{-Out}\mathcal{C}$,

2. $f(\prod\limits_{i\in I} \text{Out}R) \cap K\text{-Out}\mathcal{C} = f(\prod\limits_{i\in I} K\text{-Out}R)$,

3. $f(\text{Out}P) \subset K\text{-Out}\mathcal{C}$.

Proof. Let $\sigma \in Z^1(\overline{P},K^*)$. Then $f(\sigma\cdot B^1(\overline{P},K^*)) = \widehat{\sigma}\cdot\text{Inn}\mathcal{C}$, where for an arbitrary $\alpha_{\bar{x}\bar{y}} \in \text{Mor}(\bar{x},\bar{y})$ the value of $\widehat{\sigma}(\alpha_{\bar{x}\bar{y}})$ is defined by the right-hand side of (22). Since $\sigma(\bar{x},\bar{x}) = 1$ for any $\bar{x}\in\overline{P}$, $\widehat{\sigma}(k\delta_{\bar{x}\bar{x}}) = k\delta_{\bar{x}\bar{x}}$ for all $k\in K$. Thus, $f(H^1(\overline{P},K^*)) \subset K\text{-Out}\mathcal{C}$.

Now let $\{\varphi_i\cdot\text{Inn}R\}_{i\in I} \in \prod\limits_{i\in I}\text{Out}R$. Then $f(\{\varphi_i\cdot\text{Inn}R\}_{i\in I}) = \widehat{\varphi}\cdot\text{Inn}\mathcal{C}$, where $\widehat{\varphi}$ is given by means of (31). Therefore, $\widehat{\varphi}(k\delta_{\bar{x}\bar{x}}) = \varphi_i(k)\delta_{\bar{x}\bar{x}}$ for any $x\in P_i$, $k\in K$. Hence $\widehat{\varphi}\cdot\text{Inn}\mathcal{C}\in K\text{-Out}\mathcal{C}$ iff $\varphi_i\cdot\text{Inn}R \in K\text{-Out}R$ for all $i\in I$. In other words, $f(\prod\limits_{i\in I}\text{Out}R)\cap K\text{-Out}\mathcal{C} = f(\prod\limits_{i\in I} K\text{-Out}R)$.

Consider $\psi\in\text{Aut}P$. An image $f(\psi\cdot\text{Inn}P)$ is a coset $\widehat{\psi}\cdot\text{Inn}\mathcal{C}$, where $\widehat{\psi}$ is defined by the equation (16). Take $k\in K$ and note that $\widehat{\psi}(k\delta_{\bar{x}\bar{x}})(x',x'') = k\delta(\psi^{-1}(x'),\psi^{-1}(x''))$, where $x',x''\in\bar{x}$. If $x'=x''$, then $\widehat{\psi}(k\delta_{\bar{x}\bar{x}})(x',x'') = k$, otherwise, $\widehat{\psi}(k\delta_{\bar{x}\bar{x}})(x',x'') = 0$. Therefore, $\widehat{\psi}(k\delta_{\bar{x}\bar{x}}) = k\delta_{\widehat{\psi}(\bar{x})\widehat{\psi}(\bar{x})}$ and $f(\text{Out}P)\subset K\text{-Out}\mathcal{C}$. □

Theorem 6. *Let R be a local ring, P a quasiordered set whose classes are finite, $P = \bigcup\limits_{i\in I} P_i$ the decomposition of P into the disjoin union of the connected components. Then the group K-OutFI is isomorphic to the semidirect product $(H^1(\overline{P},K^*)\rtimes\prod\limits_{i\in I} K\text{-Out}R)\rtimes\text{Out}P$.*

Proof. Identify K-OutFI with $K\text{-Out}\mathcal{C}\subset\text{Out}\mathcal{C}$. Let $f:(H^1(\overline{P},K^*)\rtimes\prod\limits_{i\in I}\text{Out}R)\rtimes\text{Out}P\to\text{Out}\mathcal{C}$ be the isomorphism from the previous section. Recall that the image of $H^1(\overline{P},K^*)\rtimes\prod\limits_{i\in I}\text{Out}R$ under f coincides with $\text{Out}_0\mathcal{C}$. Denote the subgroup $\text{Out}_0\mathcal{C}\cap K\text{-Out}\mathcal{C}$ by $K\text{-Out}_0\mathcal{C}$. We shall prove that $K\text{-Out}\mathcal{C} = K\text{-Out}_0\mathcal{C}\rtimes f(\text{Out}P)$ and $K\text{-Out}_0\mathcal{C} = f(H^1(\overline{P},K^*))\rtimes f(\prod\limits_{i\in I} K\text{-Out}R))$.

Consider an arbitrary $\chi\in\text{Out}\mathcal{C}$. Then $[\chi] = [\widehat{\sigma}\widehat{\varphi}\widehat{\psi}]$, where $\widehat{\sigma},\widehat{\varphi},\widehat{\psi}\in\text{Aut}\mathcal{C}$ are the isomorphisms from the previous lemma, namely $[\widehat{\sigma}]\in f(H^1(\overline{P},K^*))$, $[\widehat{\varphi}]\in f(\prod\limits_{i\in I}\text{Out}R)$, $[\widehat{\psi}]\in f(\text{Out}P)$ (here and below the square brackets mean that we consider the coset of the subgroup $\text{Inn}\mathcal{C}$). By the previous lemma $[\widehat{\sigma}],[\widehat{\psi}]\in K\text{-Out}\mathcal{C}$. Therefore, $\chi(k\delta_{\bar{x}\bar{x}}) = \widehat{\varphi}(k\delta_{\bar{x}\bar{x}})$. Hence $[\chi]\in K\text{-Out}\mathcal{C}$ iff $[\widehat{\varphi}]\in K\text{-Out}\mathcal{C}$. According to the second statement of the previous lemma this is equivalent to $[\widehat{\varphi}]\in f(\prod\limits_{i\in I} K\text{-Out}R)$.

Thus, $K\text{-Out}\mathcal{C} = f(H^1(\overline{P}, K^*))f(\prod_{i\in I} K\text{-Out}R)f(\text{Out}P)$. Similar reasoning shows that $K\text{-Out}_0\mathcal{C} = f(H^1(\overline{P}, K^*))f(\prod_{i\in I} K\text{-Out}R)$. Furthermore, note that the intersection of $f(H^1(\overline{P}, K^*))$ and $f(\prod_{i\in I} K\text{-Out}R)$ is trivial, because $f(H^1(\overline{P}, K^*)) \cap f(\prod_{i\in I} \text{Out}R) = \{1\}$, and similarly $K\text{-Out}_0\mathcal{C} \cap f(\text{Out}P) = \{1\}$. Hence it is sufficient to prove that $f(H^1(\overline{P}, K^*))$ is normal in $K\text{-Out}_0\mathcal{C}$ and $K\text{-Out}_0\mathcal{C}$ is normal in $K\text{-Out}\mathcal{C}$. The first assertion is obvious: since $f(H^1(\overline{P}, K^*))$ is normal in $\text{Out}_0\mathcal{C}$, it will be normal in its subgroup $K\text{-Out}_0\mathcal{C}$. For the proof of the second assertion consider $[\varphi] \in K\text{-Out}_0\mathcal{C}$ and conjugate it by $[\psi] \in K\text{-Out}\mathcal{C}$. The result of the conjugation belongs to $\text{Out}_0\mathcal{C} \cap K\text{-Out}\mathcal{C}$, because $\text{Out}_0\mathcal{C}$ is normal in $\text{Out}\mathcal{C}$. But by definition $\text{Out}_0\mathcal{C} \cap K\text{-Out}\mathcal{C} = K\text{-Out}_0\mathcal{C}$ and hence $K\text{-Out}_0\mathcal{C}$ is normal in $K\text{-Out}\mathcal{C}$. □

If P is partially ordered, then, as in the Remark 3, it is sufficient to require that R is indecomposable.

Remark 4. Let P be a partially ordered set, R an indecomposable ring, $P = \bigcup_{i\in I} P_i$ the decomposition of P into the disjoint union of the connected components. Then the group $K\text{-Out}FI$ is isomorphic to $(H^1(P, K^*) \rtimes \prod_{i\in I} K\text{-Out}R) \rtimes \text{Aut}P$.

If R is a simple algebra, finite-dimensional over its center, then by Skolem-Noether theorem $K\text{-Out}R = 1$ and hence $K\text{-Out}FI$ is isomorphic to $H^1(\overline{P}, K^*) \rtimes \text{Out}P$. Thus we obtain a generalization of [9, Theorem 2], [2, Theorem 5] and [4, Theorem 4]. In the case when P has 0 or 1, $H^1(\overline{P}, K^*) = 1$ and $K\text{-Out}FI \cong \text{Out}P$. This generalizes [8, Theorem 1.2].

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