

Perturbations of discrete lattices and almost periodic sets

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ABSTRACT. A discrete set in the p -dimensional Euclidian space is almost periodic, if the measure with the unit masses at points of the set is almost periodic in the weak sense. We propose to construct positive almost periodic discrete set as an almost periodic perturbation of a full rank discrete lattice. Also we prove that each almost periodic discrete set on the real axes is an almost periodic perturbation of some arithmetic progression.

Next, we consider signed almost periodic discrete sets, i.e., when the signed measure with masses $+1$ or -1 at points of a discrete set is almost periodic. We construct a signed discrete set that is not almost periodic, while the corresponding signed measure is almost periodic in the sense of distributions. Also, we construct a signed almost periodic discrete set such that the measure with masses $+1$ at all points of the set is not almost periodic.

The concept of almost periodicity plays an important role in various branches of analysis. In particular, almost periodic discrete sets are used for investigation of zero sets of some holomorphic functions (cf. [8],[5]), in value distribution theory of some classes of meromorphic functions (cf. [3]), as a model of quasicrystals (cf. [7],[9]). Note that in [7] the question (Problem 4.4) was raised if there exist other discrete almost periodic sets in \mathbb{R}^p , besides of the form $L + E$ with a discrete lattice L and a finite set E .

Here we propose several ways to construct almost periodic discrete sets. Next, we introduce signed almost periodic discrete sets. In particular,

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we construct signed discrete set such that its associate measure is almost periodic in the sense of distributions and is not almost periodic in the weak sense.

To formulate our result beforehand we have to recall some known definitions (see, for example, [1]).

A continuous function $f(x)$ in \mathbb{R}^p is *almost periodic*, if for any $\varepsilon > 0$ the set of ε -almost periods of f

$$\{\tau \in \mathbb{R}^p : \sup_{x \in \mathbb{R}^p} |f(x + \tau) - f(x)| < \varepsilon\}$$

is a relatively dense set in \mathbb{R}^p . The latter means that there is $R = R(\varepsilon) < \infty$ such that any ball of radius R contains an ε -almost period of f .

Note that almost periodic functions are uniformly bounded in \mathbb{R}^p . Besides, every almost periodic function is the uniform in $x \in \mathbb{R}^p$ limit of a sequence of exponential polynomials of the form

$$P(x) = \sum_m c_m e^{i\langle x, \lambda_m \rangle}, \quad \lambda_m \in \mathbb{R}^p, \quad c_m \in \mathbb{C}, \quad (1)$$

here $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^p .

A Borel measure μ in \mathbb{R}^p is *almost periodic* if it is almost periodic in the weak sense, i.e., for any continuous function φ in \mathbb{R}^p with a compact support the convolution

$$\int \varphi(x + t) d\mu(t) \quad (2)$$

is an almost periodic function in $x \in \mathbb{R}^p$ (see [10]).

The definition suits to signed measures as well.

Theorem 1 ([10], Theorems 2.1 and 2.7). *For any signed almost periodic measure μ in \mathbb{R}^p there exists $M < \infty$ such that the variation $|\mu|$ satisfies the condition*

$$|\mu|(B(c, 1)) < M \quad \forall c \in \mathbb{R}^p. \quad (3)$$

Besides, there exists uniformly in $x \in \mathbb{R}^p$ a finite limit

$$D(\mu) = \lim_{R \rightarrow \infty} \frac{\mu(B(x, R))}{\omega_p R^p}.$$

Here $B(x, R)$ is an open ball with the center at the point x and radius R , ω_p is the volume of $B(0, 1)$.

Following [7], we will say that a discrete set A is a *Delone set*, if there are $r > 0$ and $R < \infty$ such that each ball of radius r contains at most one element of A , and each ball of radius R contains at least one element of A .

Definition 1 ([7]). *A Delone set A is almost periodic, if its associate measure*

$$\mu_A = \sum_{x \in A} \delta_x, \quad (4)$$

where δ_x is the unit mass at the point x , is almost periodic.

We will consider some generalization of discrete sets, namely multiple discrete sets in \mathbb{R}^p . This means that a number $m(x) \in \mathbb{N}$ corresponds to each point x from a discrete set. We denote this object by $A = \{(x, m(x))\}$ and the corresponding discrete set by $s(A)$. Also, we will write $A = (a_k)$, where every term $a \in s(A)$ appears $m(a)$ times in the sequence (a_k) .

In the case $p = 2$ the definition coincides with the definition of the divisor of an entire function in the complex plane.

The definition of almost periodic Delone sets has an evident generalization to almost periodic multiple discrete sets.

Definition 2. *A multiple discrete set A is almost periodic, if its associate measure*

$$\mu_A = \sum_{x \in s(A)} m(x) \delta_x \quad (5)$$

is almost periodic.

Put

$$\text{card}(A \cap E) = \sum_{x \in s(A) \cap E} m(x)$$

for any $E \subset \mathbb{R}^p$. The following result is a consequence of Theorem 1.

Theorem 2. *For any almost periodic multiple discrete set A there exists $M < \infty$ such that*

$$\text{card}(A \cap B(x, 1)) < M \quad \forall x \in \mathbb{R}^p. \quad (6)$$

Besides, there exists uniformly in $x \in \mathbb{R}^p$ a finite density

$$D(A) = \lim_{R \rightarrow \infty} \frac{\text{card}(A \cap B(x, R))}{\omega_p R^p}. \quad (7)$$

Another proof of Theorem 2 see in [4].

There is a geometric criterium for multiple discrete sets to be almost periodic.

Theorem 3 ([4], Theorem 11). *A multiple discrete set $(a_n) \subset \mathbb{R}^p$ is almost periodic if and only if for each $\varepsilon > 0$ the set of ε -almost periods of (a_n)*

$$\{\tau \in \mathbb{R}^p : \exists \text{ a bijection } \sigma : \mathbb{N} \rightarrow \mathbb{N} \text{ such that } \sup_{n \in \mathbb{N}} |a_n + \tau - a_{\sigma(n)}| < \varepsilon\} \quad (8)$$

is relatively dense in \mathbb{R}^p .

At the first time almost periodic divisors appeared in papers [8] and [11], where only shifts along real axis were considered. The definition of almost periodicity based on the above geometric property. An analog of Theorem 3 was proved in [5].

Almost periodic perturbations of discrete lattices. Let $F(x) = (F_1(x), \dots, F_p(x))$ be a mapping from \mathbb{R}^p to \mathbb{R}^p with almost periodic components $F_j(x)$. For convenience of a reader, prove the following known assertion.

Proposition. *For any $\varepsilon > 0$ the set of common ε -almost periods of F_j with integer coordinates is relatively dense in \mathbb{R}^p .*

Proof. By the well-known Kronecker Theorem, the system of inequalities

$$|\exp\langle \tau, \lambda_n \rangle - 1| < \delta, \quad n = 1, \dots, N, \quad (9)$$

has a relatively dense in \mathbb{R}^p set of solutions τ for any $\delta > 0$ and any $\lambda_n \in \mathbb{R}^p$, $n = 1, \dots, N$. Let $\{e_j\}_{j=1}^p$ be the natural basis in \mathbb{R}^p . Common solutions of (9) and the system

$$|\exp 2\pi\langle \tau, e_j \rangle - 1| < \eta, \quad j = 1, \dots, p, \quad (10)$$

form a relatively dense set too. Whenever τ satisfies (10), there is $r \in \mathbb{Z}^p$ such that $|\tau - r| < p\eta/\pi$. Hence for sufficiently small η there exists a relatively dense set of solutions $r \in \mathbb{Z}^p$ of system (9) with 2δ instead of δ . If each function F_j is an exponential polynomial of form (1), then there exist δ and $\lambda_1, \dots, \lambda_N$ such that these solutions are common ε -almost periods of F_j . In the general case we can approximate the functions $F_j(x)$ by sequences of exponential polynomials. \square

Let F be the same as above, and L be an arbitrary discrete full rank lattice in \mathbb{R}^p . Rewrite it in the form $L = \{k\Gamma, k \in \mathbb{Z}^p\}$, where Γ is a non-generated $p \times p$ matrix. If $r \in \mathbb{Z}^p$ is a common ε -almost period of components of F , then $\tau = r\Gamma$ is an $p\varepsilon$ -almost period of the discrete multiple set

$$A = (a_k), \quad a_k = k\Gamma + F(k), \quad k \in \mathbb{Z}^p, \quad (11)$$

where the bijection $\sigma : \mathbb{Z}^p \rightarrow \mathbb{Z}^p$ in (8) has the form $\sigma(k) = k + r$. Whenever all components F are almost periodic, Theorem 3 implies that A is an almost periodic multiple discrete set. Also, note that in the case of sufficiently small $\sup_{\mathbb{R}^p} |F(x)|$ we obtain an almost periodic Delone set.

It is easy to construct an almost periodic set in \mathbb{R}^p without any periods. Take $F(x) = \frac{1}{3\sqrt{p}}(\sin x_1, \dots, \sin x_p)$ for $x = (x_1, \dots, x_p)$, and put $A = k + F(k)$, $k \in \mathbb{Z}^p$. If $\tau \in \mathbb{R}^p \setminus \{0\}$ is a period of A , then $k + \tau + F(k) = k' + F(k')$ for all $k \in \mathbb{Z}^p$ and some $k' = k'(k, \tau)$. Clearly, $\tau = \tau^{(1)} + \tau^{(2)}$, where $\tau^{(1)} = (\tau_1^{(1)}, \dots, \tau_p^{(1)}) \in \mathbb{Z}^p$ and modula of all components of $\tau^{(2)} = (\tau_1^{(2)}, \dots, \tau_p^{(2)})$ is less than $1/2$. Therefore, $k' = k + \tau^{(1)}$ and $\sin(k_j + \tau_j^{(1)}) = \tau_j^{(2)} + \sin k_j$ for all $k = (k_1, \dots, k_p) \in \mathbb{Z}^p$ and $j = 1, \dots, p$, that is impossible.

Clearly, all vectors from a lattice L are periods of every almost periodic set of the form $L + E$ with a finite set E . Therefore we obtain an answer on the question raised in [7].

Note that in [2] and [6] we have got representation (11) with a square lattice L and a *bounded* mapping $F(x)$ for a wide class of multiple discrete sets in \mathbb{R}^p , in particular, for every almost periodic multiple discrete sets. We do not know if each almost periodic set has representation (11) with some lattice L and a mapping $F(x)$ with almost periodic coordinates F_j . But this is true for almost periodic sets in the real axis.

Theorem 4. *Let $A = (a_k)_{k \in \mathbb{Z}}$, where $a_k \leq a_{k+1}$ for all k , be an almost periodic multiple discrete set in \mathbb{R} with the density D . Then $a_k = Dk + f(k)$ with an almost periodic function f .*

In [8] a similar result was obtained for real parts of zeros of almost periodic entire functions from some special class.

Proof. Without loss of generality we may suppose that density D of the set A is equal to 1. Also we suppose that $0 \in A$ and $a_0 = 0$. Take arbitrary $x, y, h \in \mathbb{R}$, $x < y$, $h > 0$. Using Theorem 3, take $L > 2$ such that any interval $i \subset \mathbb{R}$ of length L contains a 1-almost period κ of A . Since $\bar{i} - \kappa \subset (-L, L)$, we get

$$\text{card}(A \cap \bar{i}) \leq M, \quad \text{where} \quad M = \text{card}(A \cap (-L - 1, L + 1)).$$

Take a 1-almost period κ of the set A such that $y < x + \kappa < y + L$. By definition, there is a bijection ρ between all points of the set $A \cap (x + \kappa, x + \kappa + h]$ and some points of the set $A \cap (x - 1, x + h + 1]$. Moreover, the

same ρ is a bijection between some points of the set $A \cap (x + \kappa, x + \kappa + h]$ and all points of the set $A \cap (x + 1, x + h - 1]$. Therefore we have

$$\begin{aligned} & |\text{card}(A \cap (x + \kappa, x + \kappa + h]) - \text{card}(A \cap (x, x + h])| \\ & \leq \text{card}(A \cap ((x - 1, x + 1] \cup (x + h - 1, x + h + 1])) \leq 2M. \end{aligned}$$

Since

$$\begin{aligned} (x + \kappa, x + \kappa + h] \setminus (y, y + h] & \subset (y + h, y + h + L), \\ (y, y + h] \setminus (x + \kappa, x + \kappa + h] & \subset (y, y + L), \end{aligned}$$

we obtain

$$\begin{aligned} & |\text{card}(A \cap (x, x + h]) - \text{card}(A \cap (y, y + h])| \\ & \leq 2M + \text{card}(A \cap [(y, y + L) \cup (y + h, y + h + L)]) \leq 4M. \end{aligned}$$

For any $T \in \mathbb{N}$ the half-interval $(x, x + Th]$ is the union of half-intervals $(x + (j - 1)h, x + jh]$, $j = 1, \dots, T$. If we set $y = x + (j - 1)h$, $j = 2, \dots, T$, we get

$$|\text{card}(A \cap (x, x + h]) - T^{-1} \text{card}(A \cap (x, x + Th])| \leq 4M.$$

Taking into account (7) with $D(A) = 1$, we obtain

$$|\text{card}(A \cap (x, x + h]) - h| \leq 4M \quad \forall x \in \mathbb{R}, h > 0. \quad (12)$$

Furthermore, put

$$\begin{aligned} n(t) &= \text{card}(A \cap (0, t]) \quad \text{for } t > 0, \\ n(t) &= -\text{card}(A \cap (t, 0]) \quad \text{for } t < 0, \quad n(0) = 0. \end{aligned}$$

Take $\varepsilon < L/(24M)$. Let $\tau > 2L$ be an ε -almost period of the set A . Clearly, if $x, y, x + \tau, y + \tau$ do not belong to the ε -neighborhood U_ε of the set $s(A)$, then we have

$$n(y + \tau) - n(x + \tau) = n(y) - n(x).$$

Therefore, the function $n(x + \tau) - n(x)$ takes the same number $p \in \mathbb{N}$ for all $x \in \mathbb{R} \setminus (U_\varepsilon \cup (U_\varepsilon - \tau))$.

Next, denote by $E[a]$ the integer part of a real number a . Put $N = E[L/(4M\varepsilon)] + 1$. It is easily shown that

$$\frac{L}{4M\varepsilon} < N < \frac{\tau}{2M\varepsilon(E[\tau/L] + 1)} - 1. \quad (13)$$

Denote by $\text{mes}G$ the Lebesgue measure of the set G . Since any half-interval $(y, y + \tau]$ contains at most $M(E[\tau/L] + 1)$ points of the set A , we get

$$\begin{aligned} \text{mes} \left(\bigcup_{j=0}^N [(U_\varepsilon - j\tau) \cap (0, \tau)] \right) &\leq \sum_{j=0}^N \text{mes}[U_\varepsilon \cap (j\tau, (j+1)\tau)] \\ &\leq (N+1)2\varepsilon M(E[\tau/L] + 1) < \tau. \end{aligned}$$

Hence there is $x \in (0, \tau)$ such that the points $x, x + \tau, \dots, x + N\tau$ do not belong to A_ε . Therefore,

$$n(x + N\tau) - n(x) = \sum_{j=1}^N n(x + j\tau) - n(x + (j-1)\tau) = Np.$$

On the other hand, using (12) with $h = N\tau$, we get

$$|n(x + N\tau) - n(x) - Np| < 4M.$$

Consequently, by (13), we get $|\tau - p| < 4M/N < 16M^2\varepsilon/L$.

Put $\gamma(k) = a_k - k$ for all $k \in \mathbb{Z}$. We shall prove that

$$|\gamma(m+p) - \gamma(m)| < H\varepsilon \quad \forall m \in \mathbb{Z}, \quad (14)$$

with $H = 5M + 16M^2/L$. Suppose the contrary. For example, let $\gamma(m+p) > \gamma(m) + H\varepsilon$ for some $m \in \mathbb{Z}$. This yields that

$$a_{m+p} > a_m + p + H\varepsilon > a_m + \tau + 5M\varepsilon.$$

Since $a_n \leq a_m$ for $n < m$ and $a_n \geq a_{m+p}$ for $n > m+p$, we see that for all $t \in (a_m, a_m + 5M\varepsilon)$ we have

$$n(t + \tau) - n(t) = \text{card}(A \cap (t, t + \tau]) \leq p - 1. \quad (15)$$

On the other hand,

$$\begin{aligned} &\text{mes}((a_m, a_m + L) \cap [U_\varepsilon \cup (U_\varepsilon - \tau)]) \\ &\leq 2\varepsilon \text{card}([a_m, a_m + L] \cap [A \cup (A - \tau)]) \leq 4M\varepsilon < 5M\varepsilon. \end{aligned}$$

Since the left-hand side of (15) is equal to p for all $t \in (a_m, a_m + L) \setminus [U_\varepsilon \cup (U_\varepsilon - \tau)]$, we obtain a contradiction. In the same way we prove that the case $\gamma(m+p) < \gamma(m) - H\varepsilon$ is impossible as well. Hence (14) is valid for all $m \in \mathbb{Z}$. If we continue the function γ as a linear function to each interval $(m, m+1)$, we obtain the continuous function f on \mathbb{R} such that

$$|f(x+p) - f(x)| < H\varepsilon, \quad \forall x \in \mathbb{R}.$$

Since the number p with this property exists in the $(16M^2\varepsilon/L)$ -neighborhood of each ε -almost period τ of the set A , we see that f is an almost periodic function. Theorem is proved. \square

Signed multiple discrete sets. Now we will consider some generalization of discrete sets, namely signed multiple discrete sets in \mathbb{R}^p . This means that a number $m(x) \in \mathbb{Z} \setminus \{0\}$ corresponds to each point x from a discrete set. As above, we denote this object by $A = \{(x, m(x))\}$ and the corresponding discrete set by $s(A)$. Equality (5) define the associate measure μ_A of A . Also, put

$$\begin{aligned} A^+ &= \{(x, m(x)), x \in s(A), m(x) > 0\}, \\ A^- &= \{(x, m(x)), x \in s(A), m(x) < 0\}. \end{aligned}$$

In the case $p = 2$ the definition coincides with the definition of the divisor of a meromorphic function in the complex plane.

Definition 3. *A signed multiple discrete set A is almost periodic, if its associate measure μ_A is almost periodic.*

Note that each continuous function with a compact support can be approximated by a sequence of functions from C^∞ with supports in a fixed ball. Therefore, if a signed measure μ satisfies (3), we can take only functions $\varphi \in C^\infty$ in definition (2). Next, take a positive function $\varphi \in C^\infty$ such that $\varphi(x) \equiv 1$ for $|x| \leq 1$ and $\varphi(x) \equiv 0$ for $|x| \geq 2$ in (2). Since almost periodic functions are bounded in \mathbb{R}^p , we see that every almost periodic in the sense of distributions *positive* measure satisfies (3). Hence the class of positive multiple discrete sets with almost periodic in the sense of distributions associate measures coincides with the class of positive almost periodic multiple discrete sets. But this assertion does not valid for signed multiple discrete sets.

Theorem 5. *There is a signed multiple discrete set such that its associate measure is almost periodic in the sense of distributions and is not almost periodic in the weak sense.*

Proof. Let $\alpha(n)$, $n \in 2\mathbb{Z} \setminus \{0\}$, be the greatest $k \in \mathbb{N}$ such that 2^k is a divisor of n . Put

$$a_n^+ = n + 1/(\alpha(n) + 1)^2, \quad a_n^- = n - 1/(\alpha(n) + 1)^2, \quad n \in 2\mathbb{Z} \setminus \{0\}.$$

Define the signed multiple discrete set $A = A^+ \cup A^-$, where

$$A^+ = \{(a_n^+, \alpha(n))\}_{n \in 2\mathbb{Z} \setminus \{0\}}, \quad A^- = \{(a_n^-, \alpha(n))\}_{n \in 2\mathbb{Z} \setminus \{0\}}.$$

The measure μ_A does not satisfy (3), therefore it is not almost periodic. Let us show that μ_A is almost periodic in the sense of distributions.

Take a function $\varphi \in C^\infty$ such that $\text{supp } \varphi \subset (-1/2, 1/2)$. Suppose that $\tau = 2^p k$ for some $p \in \mathbb{N}$, $k \in \mathbb{Z}$. If $|x - n| \geq 3/4$ for all $n \in 2\mathbb{Z}$, then the same is valid for the point $x + \tau$, therefore,

$$(\varphi * \mu_A)(x + \tau) = (\varphi * \mu_A)(x) = 0.$$

If $|x - n| < 3/4$ for $n \in \{2\mathbb{Z} : \alpha(n) \geq p\} \cup \{0\}$, then either $(\varphi * \mu_A)(x) = 0$, or

$$|\varphi * \mu_A(x)| = \alpha(n) |\varphi(a_n^+ + x) - \varphi(a_n^- + x)| \leq \alpha(n) M |a_n^+ - a_n^-| < 2M/p,$$

where $M = \sup_{\mathbb{R}} |\varphi'(x)|$. Moreover, if this is the case, then also $n + \tau \in \{2\mathbb{Z} : \alpha(n) \geq p\} \cup \{0\}$. Hence the same bound is valid for the value $(\varphi * \mu_A)(x + \tau)$. We obtain

$$|\varphi * \mu_A(x) - \varphi * \mu_A(x + \tau)| < 4M/p.$$

Finally, if $|x - n| < 3/4$ for $n \in \{2\mathbb{Z} : \alpha(n) < p\}$, then $\alpha(n + \tau) = \alpha(n)$. Therefore, we have $a_{n+\tau}^\pm = a_n^\pm + \tau$, and

$$\begin{aligned} (\varphi * \mu_A)(x + \tau) &= \alpha(n + \tau) [\varphi(x + \tau - a_{n+\tau}^+) - \varphi(x + \tau - a_{n+\tau}^-)] \\ &= \alpha(n) [\varphi(x - a_n^+) - \varphi(x - a_n^-)] = (\varphi * \mu_A)(x). \end{aligned}$$

Consequently, all multiples of 2^p are $(4M/p)$ -almost periods of the function $(\varphi * \mu_A)(x)$. Thus μ_A is almost periodic in the sense of distributions. Theorem is proved. \square

Note that we can check almost periodicity of function (2) only for positive continuous functions φ with an arbitrary small diameter of its support. Hence if A is a signed almost periodic multiple discrete set and

$$\inf\{|x - y| : x \in s(A^+), y \in s(A^-)\} > 0,$$

then A^+ and A^- are almost periodic multiple discrete sets as well. But this is false in the general case.

Theorem 6. *There is a signed almost periodic set A such that A^+ and A^- are not almost periodic.*

Proof. Put

$$A^+ = \{(a_n^+, 1)\}_{n \in 2\mathbb{Z}}, \quad A^- = \{(a_n^-, -1)\}_{n \in 2\mathbb{Z}}, \quad A = A^+ \cup A^-,$$

where points a_n^\pm are the same as in the proof of Theorem 5.

Following the proof of Theorem 5, we can assure that μ_A is almost periodic in the sense of distributions. Since the measure μ_A satisfies condition (3), we get that A is a signed almost periodic multiple discrete set.

We will use Theorem 3 for proving that A^+ is not almost periodic. Clearly, the distance between any two points of A^+ has the form $2m + \beta$, $m \in \mathbb{N}$, $|\beta| < 1/4$. Hence whenever τ is an ε -almost period of A^+ , $\varepsilon < 1/4$, we have $\tau = 2n_0 + \gamma$, $n_0 \in \mathbb{Z}$, $|\gamma| < 1/2$. But $0 \notin A^+$, hence the distance between the point $2n_0 + (\alpha(2n_0) + 1)^{-2} - \tau$ and any point of A^+ is more than 1. We obtain a contradiction. Consequently, A^+ is not almost periodic. Analogously, A^- is not almost periodic as well. Theorem is proved. \square

Clearly, the measure $|\mu_A| = 2\mu_{A^+} - \mu_A$ is not almost periodic as well. Therefore we obtain

Corollary. *The positive discrete set $\{a_n^+\} \cup \{a_n^-\}$ is not almost periodic.*

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