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On some aspects of the theory of modules over group rings

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Dedicated to Professor Leonid A.Kurdachenko in occasion of his 60th birthday

Abstract. The authors discuss some recent developments in the theory of modules over group rings.

Introduction

One of the foundations of the theory of modules over group rings is group theory. Modules evolved in group theory in a natural way. Let G be a group. Suppose that G includes an abelian normal subgroup A. If $H = G/A$, then H acts on A in the following way: if $h = gA \in H$ and $a \in$A, then we define $a^h = a^g = g^{-1}ag$. Since A is an abelian subgroup, this definition depends only on h. If $n \in \mathbf{Z}$, then we put $a^{nh} = (a^n)^h = (a^h)^n$. Every element $x \in \mathbf{Z}H$ can be expressed as a sum

$$x = n_1y_1 + ... + n_ky_k$$

where $n_j \in \mathbf{Z}$, $y_j = g_jA$, $j = 1, ..., k$. Then we put

$$a(n_1y_1 + ... + n_ky_k) = (a^{g_1})^{n_1}...(a^{g_k})^{n_k}.$$

It is easy to see that this transforms A into a $\mathbf{Z}H$-module. If A is a periodic group, then quite often we may replace A by one of its primary p-components. This allows us to assume that A is a p-subgroup where p is a prime. Thus we arrive to a p-module over the ring $\mathbf{Z}H$. In this case, the structure of the lower layer

$$P_1 = \Omega_1(A) = \{a \in A|pa = 0\}$$

of A significantly influences the structure of A. Since P_1 is an elementary abelian p-subgroup, we may think of P_1 as a module over the ring $\mathbf{F_p}H$ where $\mathbf{F_p}$ is a prime field of order p. If A is a torsion-free group, then we can consider a divisible envelope $E = A \otimes_{\mathbf{Z}} \mathbf{Q}$ of $\mathbf{Z}H$-module A. Naturally, the action of H on A can be uniquely extended to the action of H on E. In this case, we come to a module FG where F is a field. This shows that the case when a scalar ring of the group ring is a field is a major one. The described approach allows to employ module- and ring- theoretical methods for characterization of considered groups. This relatively old idea has shown itself to be very efficient in finite groups. The progress made in finite groups naturally led to the implementation of this approach to infinite groups that are closely related to finite groups, namely, to infinite groups with some finiteness conditions. It is well known that many significant results of ring theory are related to finiteness conditions, especially the classical minimal and maximal conditions. That is why, artinian and noetherian rings form two largest and richest branches of ring theory.

Naturally, a progress in study of noetherian and artinian modules stimulated investigations of similar objects in other fields of algebra. Thus, intensive investigations of infinite groups with the minimal and maximal conditions have been initiated by S.N. Chernikov and R. Baer whose researches play a key role there. The study of groups with the minimal and maximal conditions is carried out by methods of group theory. This study does not require implementation of other methods. However investigation of soluble groups with the maximal and minimal condition on normal subgroups (Max-n and Min-n) was impossible without serious implementation of module theory. The classical papers of P. Hall ([10], [11], [12]) played a key role in the employing of both module- and ring- theoretical methods in study of soluble groups. The investigation of abelian-by-nilpotent groups satisfying the maximal condition for normal subgroups led P. Hall to consideration of noetherian modules over a ring of the form $\mathbf{Z}H$ where H is a finitely generated nilpotent group. Main basic connections between groups, rings and modules are established due to the following result of P. Hall: if R is a noetherian ring and G is a polycyclic-by-finite group, then the group ring RG is likewise noetherian. This result stimulated the further development of the theory of group rings of polycyclic-by-finite groups as well as the theory of modules over polycyclic-by-finite groups (see, for example, the following surveys and books D.R. Farkas [7], K.W.Gruenberg [9], D. Passman ([38], [39], [40]), J.E. Roseblade [43]). Investigation of soluble groups with the minimal condition for normal subgroups began significantly later. This investigation requires the study of artinian modules over a ring of the type $\mathbf{Z}H$

where H is a Chernikov group. B. Hartley and D. McDougal [14] obtained a description of such modules and metabelian groups satisfying the minimal condition for normal subgroups. This paper was also an origin for investigation of artinian modules over group rings. It is very important to note that a group ring of a Chernikov group is not artinian. This fact essentially complicates the consideration of modules over group rings of Chernikov groups. Therefore, properties of group rings don't play here such a significant role as in the theory of modules over polycyclic-by-finite groups. First and foremost, some other properties come out here. These properties pertaining to the existence of complements, the structure of injective envelopes, and others. The theory of modules over group rings has its own themes for research. We discuss some important recent development in this area.

§1. Just infinite modules

Consider first just infinite modules. Investigation of these modules is particularly important in connection with generalized soluble groups with the weak minimal and maximal conditions for normal subgroups [20]. These modules arise in the following way. Let A be an infinite noetherian module over a ring R, $\mathcal{M} = \{C | C$ is an R - submodule of A such that A/C is infinite$\}$. Then $\mathcal{M}$ has a maximal element M. Put $V = A/M$. If U is a non- zero submodule of V then V/U is finite. The following two cases are possible here:

(i) V includes a non-zero simple submodule S;

(ii) the intersecton of the family of all non-zero submodules of V is zero.

Consider the first case. The submodule S is infinite and V/S is finite. Therefore, it could be reduced to the cases of infinite simple modules and finite modules. Moreover, for some types of rings R the module V is exactly a simple module; for example, if $R = \mathbf{Z}G$, where G is a hypercentral (even an FC-hypercentral) group. Hence the second case here is the main one.

Let R be a ring. An R-module A is said to be just infinite if it satisfies the following conditions:

(JI 1) for every submodule B of A the factor-module A/B is finite;

(JI 2) the intersection of the family of all non-zero submodules of A is zero.

The term "just infinite modules" belongs to D. J. S. Robinson and J. S. Wilson [42]. These modules have appeared under the name "the minimal infinite modules" in [50]. Further researches have shown that these modules play an important role in the study of groups whose proper factor-groups belong to some class $\mathcal{X}$ of groups (just non-$\mathcal{X}$-groups). Just infinite modules play a crucial role in the study of generalized soluble groups with the weak maximal and minimal conditions for normal subgroups and in some other important investigations. Furthemore, in study of different types of noetherian modules the reduction to just infinite modules allows us to obtain their description and establish their important properties.

The following important structural theorem is the basic theorem for just infinite modules over many important natural types of groups. In the case when a scalar ring R is a principal ideal domain, it was obtained in the paper [15]. In other papers, the main case was often the case when a scalar ring R is a principal ideal domain. Just infinite modules were studied in ([8], [24], [28], [29], [30], [31], [36], [50]). In [25] all results were etanded to the case when a scalar ring R is a Dedekind domain. Here we adduce the results in a more general form.

An infinite Dedekind domain D is said to be **a Dedekind $\mathbf{Z_0}$-domain**, if for every maximal ideal P of D the factor-ring D/P is finite.

Theorem 1.1. *Let D be a Dedekind Z_0-domain, G be a group, A be a just infinite DG-module, which is a D-torsion-free, $C_G(A) =< 1 >$. Then D is embedded in a principal ideal domain J and DG-module A is embedded in a JG-module V with the following properties:*

(1) V is a free J-module;

(2) V is a finitely generated module as J-module;

(3) $C_G(V) =< 1 >$;

(4) V is a just infinite JG-module.

This theorem makes possible a reduction to the case of a simple module. The following corollary justifies this fact.

Corollary 1.2. *Let D be a Dedekind Z_0-domain, G be a group, A be a a DG-module which is a D-torsion-free, $C_G(A) =< 1 >$. If A is a just infinite DG-module, then there exists a field $F \geq D$ and a simple FG-module $B \geq A$ such that $C_G(B) =< 1 >$ and $\mathbf{dim_F}(B)$ is finite.*

Corollary 1.3. *Let D be a Dedekind Z_0-domain, G be a locally radical group, A be a DG-module which is a D-torsion-free, $C_G(A) =< 1 >$. If A is a just infinite DG-module then G is an abelian-by-finite group.*

The statements 1.1 - 1.3 slightly generalize the main result of [15].

The following theorem describes the structure of just infinite DG-module A when A is a D-torsion-free.

Theorem 1.4. *Let D be a Dedekind Z_0-domain, G be an FC-hypercentral group, A be a just infinite DG-module which is D-torsion-free. If $C_G(A) =< 1 >$ then G includes a normal torsion-free abelian subgroup of finite index. Moreover, if $\mathbf{char}(D) = p > 0$ then $\mathbf{O_p}(G) =< 1 >$.*

In the case when a scalar ring D is a principal ideal domain, this theorem was proved in [8].

Corollary 1.5. *Let D be a Dedekind Z_0-domain, G be an FC-group, A be a just infinite DG-module which is D-torsion-free. If $C_G(A) =< 1 >$, then $G/\zeta(G)$ is finite. Moreover, a center $\zeta(G)$ includes a torsion-free subgroup of finite index. If* $\mathbf{char}(D) = p > 0$, *then* $\mathbf{O_p}(G) =< 1 >$.

Corollary 1.6. *Let D be a Dedekind Z_0-domain, G be a locally radical group, A be a just infinite DG-module which is D-torsion-free. If $C_G(A) =< 1 >$, then G includes a normal abelian torsion-free subgroup of finite index. Moreover, if* $\mathbf{char}(D) = p > 0$, *then* $\mathbf{O_p}(G) =< 1 >$.

The next result shows that in some special cases a reduction to a torsion-free case is possible.

Proposition 1.7. *Let D be a Dedekind Z_0-domain, G be a group, x be an element of infinite order of a center $\zeta(G)$, A be a just infinite DG-module, $C_G(A) =< 1 >$. If A is D-periodic, then* $\mathbf{Ann_D}(A) = P$ *is a maximal ideal of D. If $F = D/P$, then A is $F < x >$-torsion-free.*

Corollary 1.8. *Let D be a Dedekind Z_0-domain, G be an FC-hypercentral (respectively locally radical) group, the center of which contains elements of infinite order, A be a just infinite DG-module, $C_G(A) =< 1 >$. If A is D-periodic module, then* $\mathbf{Ann_D}(A) = P$ *is a maximal ideal and G includes a normal abelian torsion-free subgroup of finite index. Moreover,* $\mathbf{O_p}(G) =< 1 >$ *where* $p = \mathbf{char}(D/P)$.

Under the condition when the 0-rank of a group is finite, it is possible to obtain some additional specific information about both the group G and the structure of just infinite modules over G.

A group G is said to have **finite** $\mathbf{0-rank}$ $\mathbf{r_0(G) = r}$ if it has a finite subnormal series with exactly r infinite cyclic factors, being the others periodic. We note that every refinement of one of these series has only r factors which are infinite cyclic; any two finite subnormal series have isomorphic refinements. This allows us to convince ourselves that the 0-rank is independent of the series. This numerical invariant is also known as *the torsion-free rank of G* .

We shall need some assumptions on the underlying ring of coefficients. A Dedekind Z_0-domain D is called **a Dedekind $\mathbf{Z_1}$-domain** if the set of all maximal ideals of D is infinite.

The description of just infinite modules over groups of finite 0-rank falls into two parts depending on a characteristic of a scalars ring.

Theorem 1.9. *Let D be a Dedekind Z_1-domain of characteristic 0, G be an FC-hypercentral group of finite 0-rank and A be a just infinite DG-module which is D-torsion-free, $C_G(A) =< 1 >$. Then A has finite*

D-rank and G includes a torsion-free abelian normal subgroup of finite index. Moreover, if K is the field of fractions of D, then G is isomorphic to an irreducible subgroup of $\mathbf{GL_n}(K)$ where n is a D-rank of A.

A module A is called a minimax module if it possesses a finite series of submodules with either artinian or noetherian factors. An abelian group A is called minimax, if it is minimax as a $\mathbf{Z}$-module.

If R is a noetherian ring and A is a minimax R-module, then it is not hard to show that A includes a noetherian submodule B such that A/B is artinian.

Corollary 1.10. *Let G be an FC-hypercentral group of finite 0-rank and A be a torsion-free just infinite $\mathbf{Z}G$-module with $C_G(A) =< 1 >$. Then the following assertions hold:*

(i) G is a finitely generated abelian-by-finite group.

(ii) The additive group of A is minimax.

Corollary 1.11 [15]. *Let G be a locally radical group of finite 0-rank and A be a torsion-free just infinite $\mathbf{Z}G$-module with $C_G(A) =< 1 >$. Then the following assertions hold:*

(i) G is a finitely generated abelian-by-finite group.

(ii) The additive group of A is minimax.

Theorem 1.12. *Let D be a Dedekind Z_1-domain with $\mathbf{char}(D) = p > 0$, G be an abelian-by-finite group of finite 0-rank, A be an just infinite DG-module, $C_G(A) =< 1 >$. If A is D-torsion-free and $p \notin \mathbf{Sp}(G)$, then A has finite D-rank n. Moreover, G is isomorphic to an irreducible subgroup of $\mathbf{GL_n}(K)$ where K is the field of fractions for D.*

Theorem 1.13. *Let F be a finite field, $< x >$ be an infinite cyclic group, $D = F < x >$, G be an FC-hypercentral group of finite 0-rank and A be a just infinite DG-module which is D-torsion-free, $C_G(A) =< 1 >$. Then the following assertions hold:*

(i) G is a finitely generated abelian-by-finite group.

(ii) A is D-minimax.

Corollary 1.14 [15]. *Let F be a finite field, $< x >$ be an infinite cyclic group, $D = F < x >$, G be a locally radical group of finite 0-rank and A be a D-torsion-free just infinite DG-module with $C_G(A) =< 1 >$. Then the following assertions hold:*

(i) G is a finitely generated abelian-by-finite group.

(ii) A is D-minimax.

Corollary 1.15. *Let D be a Dedekind Z_0-domain, G be an FC-hypercentral group whose center contains an element x of infinite order and A be a D-periodic just infinite DG-module with $C_G(A) =< 1 >$. Then the following assertions hold:*

(i) G is a finitely generated abelian-by-finite group.

(ii) $\mathbf{Ann_D}(A) = P$ is a maximal ideal of D.

(iii) A is a torsion-free $F < x >$-minimax module, where $F = D/P$.

§2. Strong Noetherian modules

If A is a just infinite RG-module, then a scalar ring R must have finite homomorphic images. But not every ring has this property. Natural generalization of finite modules is noetherian and artinian modules. In [34] it was introduced a natural generalization of just infinite modules. Let R be a noetherian ring, G be a group, A be a noetherian RG-module. If A is noetherian as an R-module, then A is finitely generated over R. This situation is rather well investigated for many types of noetherian rings, especially, in representation theory. Suppose that A is non-noetherian R-module. Let $\mathcal{M} = \{B | B$ is an RG-submodule of A such that A/B is not noetherian as R-module $\}$.

Since $< 0 > \in \mathcal{M}$, then $\mathcal{M} \neq \emptyset$. Since A is a noetherian RG-module, the ordered by inclusion set $\mathcal{M}$ has a maximal element M. If $E \geq M$ and $E \neq M$, then the factor-module A/E is noetherian as an R-module.

Let R be a noetherian ring, G be a group, A be an RG-module. Following [34], we say that A is a strong noetherian RG-module if A is not noetherian R-module but every its proper factor-module (that is a factor-module by a non-zero submodule) is noetherian as an R-module.

Thus, every noetherian RG-module which is not noetherian as an R-module has a strong noetherian factor-module. This shows that the study of the strong noetherian RG-modules is one of the important stages in the process of investigation of noetherian RG-modules. The following two cases appeared for strong noetherian RG-module $\grave{A}$:

(NM) The intersection of all non-zero RG-submodules is zero.

(M) A includes the least non-zero RG-submodule B (i.e. A is a monolithic RG-module with the monolith B).

Consequently, the study of the strong noetherian RG-modules splits into two natural cases: the cases of non-monolithic and monolithic modules. The non-monolithic strong noetherian RG-modules have been investigated in the papers of L.A. Kurdachenko and I.Ya. Subbotin ([33], [34]). Consider the following useful result on existence of complements in modules.

Proposition 2.1 [34]. *Let H be a commutative algebra with identity over a field, F be a field and let A be an H-module. Suppose that B is an H-submodule of A satisfying the following conditions:*

(i) $\mathbf{dim_F}(A/B)$ is finite;

(ii) B has a finite composition H-series, every factor of which has an infinite dimension over F.

Then there exists a submodule C of A (necessary isomorphic to A/B) such that $A = B \oplus C$.

Let R be an integral domain, A be an R-module which is R-torsion-free. A submodule B is said to be $\mathbf{R-pure}$, if A/B is R-torsion-free.

If R is a noetherian integral domain, G is a group, A is a strong noetherian RG-module, then put $\mathcal{B}(A) = \{B|B$ *is the non-zero R-pure RG-submodule of A* $\}$.

Naturally, the following two situations arise:

$\cap\mathcal{B}(A) \neq <0>$ and $\cap\mathcal{B}(A) = <0>$.

In the first case we have

Proposition 2.2 [34]. *Let R be a noetherian integral domain, G be an FC-hypercentral group, A be a strong noetherian RG-module which is R-torsion-free. If A has an infinite R-rank and $\cap\mathcal{B}(A) = C \neq <0>$ then $A = C$. In particular, $A \otimes_R F$ is a simple FG-module where F is a field of fractions for a ring R.*

The following result is useful in the reduction to the case when a scalar ring is a field.

Proposition 2.3 [34]. *Let R be a noetherian integral domain, G be a group, A be a strong noetherian RG-module which is R-torsion-free, $C_G(A) = <1>$. Suppose that $\cap\mathcal{B}(A) = <0>$. If F is a field of fractions for a ring R then an FG-module $E = A \otimes_R F$ has the following properties:*

(i) every proper FG-factor-module of E has a finite F-dimension;

(ii) $C_G(E) = <1>$.

The next our step is a consideration of the case of strong noetherian FG-modules where F is a field. This case is a main one. In [33] the structure of a hypercentral group G for which a faithful non-monolithic strong noetherian FG-module exists was described. This description was extended to the case of FC-hypercentral group in [34].

Theorem 2.4. [34]. *Let D be a Dedekind domain, G be an FC-hypercentral locally soluble group, A be a non-monolithic strong noetherian DG-module, $C_G(A) = <1>$. Then either G is isomorphic to some subgroup of $\mathbf{GL_n}(F)$ for some field F or the following assertions hold:*

(i) G is abelian-by-finite;

(ii) the maximal normal periodic subgroup T of a group G is a p'-subgroup of finite special rank where $p = \mathbf{char} F$;

(iii) $T \cap \zeta(G)$ is a locally cyclic subgroup.

§3. Artinian modules over group rings

The theory of artinian modules over the group rings is a large and important part of the theory of modules over group rings. This theory is rather well developed. It has its methodology and is rich in many interesting results. Many important aspects of this theory are represented in the book [26]. Therefore we consider here only some selected results. One of the first step is the study of artinian modules over periodic groups. B. Hartley and D. McDougall [14] considered artinian modules over a ring having the form $\mathbf{Z}G$, where G is an abelian Chernikov group. Their results could be extended to the case of a ring of the form DG where G is a periodic abelian group of finite special rank and D is a Dedekind domain. Therefore the question of the structure of artinian modules over DG where G is an non-periodic abelian group of finite special rank seems very logical. The following principal obstacles arised here. If G is a periodic abelian group of finite special rank, then the reduction to the case when G has rank 1 is possible. If G is not periodic, then a similar reduction is not possible. S. A. Kruglyak [17] argued that by the theory of representations the problem of description of finite modules (and hence artinian modules) over a free abelian group of 0-rank at most two is wild. The following analogy could be useful at this point. The problem of the description of torsion-free abelian groups of finite 0-rank is also "wild" [47], though the $\mathbf{Z}$-injective envelope of these groups is a direct sum of finitely many copies of the full rational group $\mathbf{Q}$. Thus we came to injective envelopes of artinian modules over group rings, a partial case of which is the description of simple modules over group rings. In [22] L.A. Kurdachenko has described the injective envelope of an artinian module over JG where J is a principal ideal domain and G is an abelian group of finite special rank. These results can be extended (almost without changes) to the case of a ring DG where D is some Dedekind domain (see, [26], Chapter 14). We adduce this description here.

Before further consideration, we need to make the following remark. Let R be a ring, A be an R-module, $L = \mathbf{Ann_R}(A)$. We can consider A as K-module where $K = R/L$. Then for an injective envelope the following two cases appear: we can consider a K-injective envelope E of A and then consider E as an R-module putting $L = Ann_R(E)$. This seems promisingr since L has no influence on the structure of A. On the other hand, if we will consider a full injective envelope I, then $\mathbf{Ann_R}(I)$ may does not include L and therefore the structure of I becomes dependent from the elements of an ideal L, which we excluded from the consideration at the beginning. Since that we shall consider the first case for the injective envelope. In fact, such way has been chosen by B. Hartley and

D. McDougall [14]. They reduced the general situation of an artinian module A over a group ring $\mathbf{Z}G$ where G is a Chernikov group to the case in which the additive group of A is a p-group. Then the general case is reduced to the case when $G/C_G(A)$ is a p'-group. After this they came to the injective envelope.

A Dedekind domain D is said to be **a Dedekind $\mathbf{Z}$ – domain**, if D satisfies the two following conditions:

(1) The set of all maximal ideals of D is infinite;

(2) If P is a maximal ideal of D, then the field D/P is locally finite.

Theorem 3.1 [22]. *Let D be a Dedekind Z-domain, G be an abelian group of finite section rank, A be an artinian DG-module, I =$\mathbf{Ann_{DG}}(A)$, $Q = DG/I$. If E is an Q-injective envelope of A, then E is an artinian DG-module.*

Let R be a ring, A be an artinian R-module. Then $\mathbf{Soc_R}(A) = \oplus_{\mathbf{j=1,...,n}} M_j$, where M_j is a minimal R-submodule of A for each $j = 1, ..., n$. Let U_j be an R-submodule of A which is the maximal submodule with the respect to the properties $M_j \cap U_j =< 1 >$ and $\oplus_{\mathbf{k \neq j}} M_k \leq U_j$.

Then A/U_j is a monolithic R-module with R-monolith $(M_j + U_j)/U_j$. Let $U = \cap_{j=1,...,n} U_j$, then $U \cap \mathbf{Soc}_R(A) =< 0 >$. Since A is an artinian R-module, it follows that $U =< 0 >$. By Remak's Theorem we obtain the imbedding

$$A \leq \oplus_{j=1,...,n} A/U_j.$$

This show that the case of a monolithic artinian R-module is basic for this consideration.

Proposition 3.2 [22]. *Let D be a Dedekind Z-domain, G be an abelian group of finite section rank, A be an artinian monolithic DG-module with the DG-monolith M. Suppose that $H = C_G(M)$ includes a finitely generated torsion-free subgroup L such that H/L does not include the subgroup of index p where p =$\mathbf{char}(D/\mathbf{Ann}_D(M))$. Then G includes a subgroup $K \geq L$ such that K/L is a periodic p'-group of special rank 1 and every DK-submodule of A is likewise DG-submodule. In particular, a DK-module A is artinian.*

This result reduces the general situation to the case when G includes a finitely generated torsion-free subgroup $L \leq C_G(M)$ such that G/L is a periodic p'-group of finite special rank.

Let D be a Dedekind Z-domain, G be an abelian group of finite 0-rank, A be an artinian monolithic DG-module with the DG-monolith M. Then $\mathbf{Ann}_D(M) = P$ is a maximal ideal of D (see, for example, [25], Theorem 1.15) and D/P is a locally finite field. Let $\mathbf{char}(D/P) = p$ and $H = C_G(M)$. Then G/H is a periodic locally cyclic p'-group (see, for example, ([25], Theorem 2.3). Suppose that G has finite p-rank and choose in H a finitely generated torsion-free subgroup L such that G/L is periodic. Let S/L be a Sylow p-subgroup of G/L, then S/L is a Chernikov subgroup. Moreover, $S \leq H$. Let U/L be a Sylow p'-subgroup of G/L, then $H \leq U$ and

$$G/L = U/L \times V/L \times W/V,$$

where V/L is a divisible Chernikov p-subgroup, W/L is a finite p-subgroup. Let $S = UV$. Then a DS-module A is artinian. In other words, we can further suppose that a subgroup $H = C_G(M)$ includes a finitely generated torsion-free subgroup L such that H/L does not include the subgroup of index p.

Now we may give a description of an injective envelope.

Theorem 3.3 [22]. *Let F be a locally finite field, $p =$* **char**F*, G be an abelian group of finite special rank, A be an artinian monolithic FG-module with monolith M, $C_G(A) =< 1 >$, $H = C_G(M)$. Suppose that H includes a torsion-free finitely generated subgroup $L =\mathbf{Dr}_{j=1,\dots,n} < g_j >$ such that G/L is a periodic p'-group. If E is an FG-injective envelope of A, then E has a series of submodules*

$$E = E_0 \geq E_1 \geq \dots \geq E_{n-1} \geq E_n = M$$

satisfying the following properties:

(i) $E_1 = C_E(g_1), E_2 = C_E(x_1, x_2), \dots, E_{n-1} = C_E(x_1, \dots, x_{n-1})$;

(ii) E_j possesses an ascending series of submodules

$$E_{j+1} = M_{j+1,0} \leq M_{j+1,1} \leq \dots \leq M_{j+1,k} \leq \dots \cup_{k\in \mathbf{N}} M_{j+1,k} = E_j$$

such that

$$M_{j+1,k+1}/M_{j+1,k} \simeq E_{j+1}$$

and

$$M_{j+1,k+1}(g_{j+1} - 1) = M_{j+1,k}$$

for all $k \in \mathbf{N}$.

Theorem 3.4 [22]. *Let D be a Dedekind Z-domain, G be an abelian group of finite section rank, A be an artinian monolithic DG-module with a monolith M, $P =$ $\mathbf{Ann}_D(M)$, E be a DG-injective envelope of A. Then E satisfies the following conditions:*

(i) E is artinian P-module over DG;

(ii) $\mathbf{\Omega}_{P,1}(E)$ is an FG-injective envelope of $\mathbf{\Omega}_{P,1}(A)$ where $F = D/P$;

(iii) E is a D-divisible envelope of $\mathbf{\Omega}_{P,1}(E)$.

Corollary 3.5 [22]. *Let D be a Dedekind Z-domain, G be an abelian group of finite section rank, A be an artinian monolithic DG-module with* *a monolith M, $P = \mathbf{Ann}_D(M)$. Then A is DG-injective if and only if A is D-divisible and $\mathbf{\Omega}_{P,1}(A)$ is FG-injective where $F = D/P$.*

Let R be a ring and A be an R-module. The submodule $\mathbf{Soc}_R(A)$ generated by all minimal R-submodules of A is said to be **the R-socle of** A . If A has no such minimal submodules, we define $\mathbf{Soc}_R(A) =< 0 >$.

Starting from the socle, we define the upper socular series or ascending Loewy series of A as

$$< 0 >= S_0 \leq S_1 \leq ...S_\alpha \leq S_{\alpha+1} \leq ...S_\rho$$

where $S_1 = \mathbf{Soc}_R(A)$ and $S_{\alpha+1}/S_\alpha = \mathbf{Soc}_R(A/S_\alpha)$ for a given ordinal α. Note that $S_\mu = \cup_{\beta<\mu} S_\beta$ for any limit ordinal μ.

The least ordinal ρ such that $S_\rho = S_{\rho+1}$ is called the socular height of A.

One of the important problems of artinian modules theory is the evaluation of their socular height. For the artinian modules over a group ring this problem can be reformulated in the following form.

Let G be a group, D be a Dedekind domain. What can be said about a group G if *the socular* height of *an arbitrary* artinian DG-module is at most ω ?

More general: What can be said about a countable group G if an arbitrary artinian DG-module has a countable set of generators as a D-module?

Now we are in a position to deduce a result due to P. Neumann (see B. Hartley [13]).

Theorem 3.6. *An artinian module A over a commutative ring R has socular height at most ω , the first infinite ordinal. In particular, if R is countable so is A.*

B. Hartley in [13] constructed the series of impressive examples of uncountable artinian monolithic $\mathbf{Z}G$-modules for distinct types of groups G.

A module A is called *uniserial, if the set of all submodules of A is well ordered by inclusion. The ordinal corresponding to the order type of the set of proper submodules of uniserial module is called the length of this module.* Clearly every uniserial module is artinian.

Proposition 3.7 [13]. *Let F be an arbitrary field and G be the free group freely generated by a countable infinite subset X. Then there exists an uniserial module A over a group ring FG of length Ω (Ω denotes the first uncountable ordinal).*

B. Hartley [13] has constructed important examples of uncountable artinian modules over group rings. Let p, q be the distinct primes and Q be a quasicyclic q-group. There exists a simple $\mathbf{F_p}Q$-module A (see, for example, [25], Corollary 2.4]) such that $C_Q(A) =< 0 >$. Consider a natural semidirect product $G = AQ$, where A is a normal subgroup of G and $A \cap Q = E$. This group is called *Charin group* (it was constructed by V.S. Charin [2]).

Theorem 3.8 [13].*Let G be a Charin group and let r be a prime such that $r \notin \Pi(G)$. Then there exists an uniserial $\mathbf{F_r}G$-module of length Ω.*

Let $D =< d >< c >$ be a natural semidirect product, where $< d >$ is a normal subgroup of D, $< d > \cap < c >= E$. D is a dihedral group, that is $|d| = 4|, |c| = 2$, and $d^c = d^{-1}$. For every $n \in \mathbf{N}$ denote by $D_n =< d_n >< c_n >$. D_n is a natural semidirect product. It is an isomorphic copy of D. Put $R = \mathbf{Dr}_{n \in \mathbf{N}} D_n$ and $E =< d_n^2 d_{n+1}^{-2} | n \in \mathbf{N} >$, $U = R/E$.

Theorem 3.9 [13]. *Let F be a field of characteristic not equal 2. Then there exists an uniserial FU-module of length Ω.*

All these examples are examples of uniserial modules, in particular, they are monolithic. We can see, that in all these examples the factor-group $G/C_G(\mu_{\mathbf{Z}G}(A))$ is not abelian-by-finite. Therefore the question about the case of a (generalized) nilpotent group G when $G/C_G(\mu_{\mathbf{Z}G}(A))$ is abelian-by-finite arises naturally. We can observe that abelian-by-finite groups play a special role in the theory of artinian modules over group rings. The problem of countability will be considered for fairly wide generalization of nilpotent groups (for FC-hypercentral groups). These results have been obtained by L.A. Kurdachenko, N.N. Semko and I.Ya. Subbotin in [27]. We adduce the basic results of this work.

Theorem 3.10 [27]. *Let G be a countable FC-hypercentral group, F be a field and A be an artinian FG-module. If a factor-group $G/C_G(\mathbf{Soc}_{FG}(A))$ is abelian-by-finite, then $\mathbf{dim}_F A$ is countable. In particular, if a field F is countable, then A is likewise countable.*

Theorem 3.11 [27]. *Let G be a countable FC-hypercentral group, D be a Dedekind domain and A be an artinian DG-module. If a factor-group $G/C_G(\mathbf{Soc}_{DG}(A))$ is abelian-by-finite, then $|A| \leq |D|\aleph_{\mathbf{0}}$. In particular, if a ring D is countable, then A is likewise countable.*

Theorem 3.12 [27]. *Let G be an FC-hypercentral group, D be a Dedekind domain with infinite set of maximal ideals, A be an artinian monolithic DG-module and $\mathbf{char}(D/Ann_D(\mu_{DG}(A))) = p$. Suppose also that G satisfies the following conditions:*

if $p > 0$, then G has a finite section p-rank;

if $\eth = 0$, then G has finite 0-rank.

If $G/C_G(\mu_{DG}(A))$ is abelian-by-finite, then each finitely generated DG-submodule of A has finite DG-composition series. In particular, the

socular *height of A is at most* ω*.*

Corollary 3.13 [27]. *Let* G *be an* FC*-hypercentral group of finite section rank,* D *be a Dedekind domain with infinite set of maximal ideals and* A *be an artinian* DG*-module. If a factor-group* $G/C_G(\mathbf{Soc}_{DG}(A))$ *is abelian-by-finite, then each finitely generated* DG*-submodule of* A *has finite* DG*-composition series. In particular, the* **socular** *height of* A *is at most* ω*.*

As we have seen from the adduced results there are not too many cases in which artinian modules can be satisfactory described, although many problems require the investigation of some specific artinian modules. We consider one of the important types of artinian modules, namely, the quasifinite modules. These modules appear in the following way. Let A be an artinian DG-module and $\mathcal{U}$ be a family of all infinite submodules of A. Then $\mathcal{U}$ has a minimal element M. Then either M is a minimal (infinite) and, hence, a simple submodule of A, or M is infinite and not simple, but every proper submodule of M is finite. D.I. Zaitcev introduced this type of modules in connection with the study of complementability of normal subgroups [48]. These modules also appeared in some other investigations, for instance, in the study of groups with the weak maximal or minimal conditions for normal subgroups ([18], [19], [20], [21], [22], [50]). In [49], D.I. Zaitcev initiated investigation of modules over integral group rings in which all proper submodules are finite. Besides this, there are many types of Dedekind domains for which finite modules can be only zero modules. Therefore the condition of being a finite submodule is reasonable to change to the condition of being a finitely generated submodule. In other words, we come to a module A over a group ring DG with the following property:

every proper DG**-submodule of** A **is finitely generated as** D**-submodule.**

The following two cases we meet here:

(i) A *includes a proper* DG*-submodule* B *such that* A/B *is a simple* DG*-module.*

(ii) every proper DG*-submodule of* A *is finitely generated as a* D*-submodule and* A *is an union of its proper* DG*-submodules.*

The first case one can reduces to the cases of finitely generated over D modules and simple DG-modules. Therefore the second case is more interesting. Thus we come to the following definition.

Let R be a ring, G be a group. An RG-module A is said to be a **quasifinite RG – module**, if A satisfies the following conditions:

(**QF1**) A is not finitely generated as an R-module;

(**QF2**) if B is a proper RG-submodule of A, then B is finitely generated as an R-submodule:

(**QF3**) A is an union of its proper RG-submodules.

As we have already noted, the case when $R = F$ is a finite field has been considered by D.I. Zaitsev [49], the case of arbitrary field has been considered in the paper L.A. Kurdachenko and I.Ya. Subbotin [32] and a case when $R = D$ is a Dedekind domain has been considered in the paper L.A. Kurdachenko [23].

Let D be a Dedekind domain. Put $\mathbf{Spec}(D) = \{P|P$ is a maximal ideal of $D\}$.

If I is an ideal of D, then put

$$A_I = \ a \in A|aI^n =< 0 > \quad for \ \ some \ \ n \in \mathbf{N}.$$

It is not hard to see that A_I is a D-submodule of A. This submodule A_I is called the *I-component* of A. If $\grave{A}$ coincides with its I-component, then we will say that A is *an I-module over a ring D*. If $k \in \mathbf{N}$, we define

$$\Omega_{I,k}(A) = \{a \in A|aI^k =< 0 >\}.$$

It is easy to see that $\Omega_{I,k}(A)$ is an R-submodule and

$$\Omega_{I,1}(A) \leq \Omega_{I,2}(A) \leq ... \leq \Omega_{I,k}(A) \leq ... \ .$$

The R-submodule $A_I = \cap_{k \in \mathbf{N}} \Omega_{I,k}(A)$ is said to be *the I-component of A. If $A = A_I$, then A is said to be an I – module. If $I = P$ is a prime ideal, a P-module is generally called primary module.*

Put $\mathbf{Ass}_D(A) = \{P \in \mathbf{Spec}(D)|A_P \neq < 0 >\}$.

Then $\mathbf{Tor}_R(A) = \oplus_{P \in \pi} A_P$ where $\pi = \mathbf{Ass}_D(A)$ (see, for example, [26], Corollary 6.25).

Let D be a Dedekind domain, A be a simple D-module. Then $A \simeq D/P$ for some maximal ideal P. We observe that D/P^k and P/P^{k+1} are isomorphic as D-modules for any $k \in \mathbf{N}$. In particular, the D-module D/P^k is embedded in the D-module $D/P^{k+1}, k \in \mathbf{N}$. Therefore we can consider the injective limit of the family of D-modules $\{D/P^k|k \in \mathbf{N}\}$.

Put $\mathbf{C_{P^\infty}} = \mathbf{limlnj}\{D/P^k|k \in \mathbf{N}\}$.

The D-module $\mathbf{C_{P^\infty}}$ is called the Prüfer P-module.

By the construction $\mathbf{C_{P^\infty}}$ is a P-module, moreover $\mathbf{\Omega_{P,k}}(\mathbf{C_{P^\infty}}) \simeq_D D/P^k, k \in \mathbf{N}$. Furthermore,

$\mathbf{\Omega_{P,k+1}}(\mathbf{C_{P^\infty}})/\mathbf{\Omega_{P,k}}(\mathbf{C_{P^\infty}}) \simeq (D/P^{k+1})/(P/P^{k+1}) \simeq D/P.$

Hence, if C is a D-submodule of $\mathbf{C_{P^\infty}}$ and $C \neq \mathbf{C_{P^\infty}}$, then $C = \mathbf{\Omega_{P,k}}(\mathbf{C_{P^\infty}})$ for some $k \in \mathbf{N}$. Similarly, if $b \notin \mathbf{\Omega_{P,k}}(\mathbf{C_{P^\infty}})$, then $C = bD$.

Observe also that a Prüfer P-module is monolithic and its monolith coincides with $\mathbf{\Omega_{P,1}}(\mathbf{C_{P^\infty}})$.

Theorem 3.14 [23]. *Let D be a Dedekind domain, which is not a field, G be a locally soluble group, A be a quasifinite DG-module, $C_G(A) =< 1 >$. Suppose that A is D-periodic, then the following assertions hold:*

(i) $\mathbf{Ass}_D(A) = P$, $P \in \mathbf{Spec}(D)$;

(ii) $A = C_1 \oplus ... \oplus C_n$, *where* C_j *is a Prüfer* P*-module, j=1,...,n;*

(iii) G *includes a normal abelian subgroup* U *of finite index;*

(iv) the periodic part of U *has finite special rank;*

(v) if $\mathbf{char}D = p > 0$, *then* $\mathbf{O}_p(G) =< 1 >$.

Let A be a quasifinite DG-module. It is not hard to see that either A is D-periodic and D-divisible, or $\mathbf{Ann}_D(A) = P$ is a maximal ideal of D. The first case has been considered in **Theorem 5.16**. Therefore we need to consider the case, when $\mathbf{Ann}_D(A) = P$ is a maximal ideal. In other words, we will consider the case of an FG-module where $F = D/P$ is a field.

Theorem 3.15 [23]. *Let F be a field, G be a locally soluble group, A be a quasifinite FG-module, $C_G(A) =< 1 >$, $S = \mathbf{Soc}_{FG}(A)$. If $C_G(S) =< 1 >$, the the following assertion holds:*

(i) G *includes a normal abelian subgroup* U *of finite index;*

(ii) the periodic part of subgroup U *has a finite special rank;*

(iii) if $\mathbf{char}F = p > 0$, *then* $\mathbf{O}_p(G) =< 1 >$;

(iv) U *contains an element* x *of infinite order and* A *includes a quasifinite* FU*-submodule* B *such that* $\mathbf{Ass}_{F<x>}(B) = P$ *for some* $P \in \mathbf{Spec}(F < x >)$ *and*

$$B = C_1 \oplus ... \oplus C_n$$

where C_j *is a Prüfer* P*-module,* $j = 1, ..., n$;

(v) $A = B \oplus Bg_1 \oplus ... \oplus Bg_t$ *where* $\{1, g_1, ..., g_t\}$ *is a transversal to* U *in* G.

Theorem 3.16 [23]. *Let F be a field, G be a hypercentral group, A be a quasifinite FG-module, $C_G(A) =< 1 >$, $S = \mathbf{Soc}_{FG}(A)$. If $C_G(S) \neq < 1 >$, then*

(i) G *is abelian-by-finite;*

(ii) the periodic part T *of a group* G *is a* p'*-group of finite special rank where* $p = \mathbf{char}F$;

(iii) $T \cap \zeta(G)$ *is locally cyclic;*

(iv) $C_G(S) \cap \zeta(G)$ *contains an element* x *of infinite order such that* A *is* $F < x >$*-periodic and* $\mathbf{Ass}_{F<x>}(A) = P$ *where* $P = (x-1)F < x >$;

(v) $A = C_1 \oplus ... \oplus C_n$ *where* C_j *is a Prüfer* P*-module, j=1,...,n.*

D.I. Zaitsev considered the quasifinite FG*-modules where* F *is a finite field. For this case it is possible to obtain more detailed information on periodic subgroups of* G.

Theorem 3.17 [49]. *Let F be a finite field, G be a group, A be a quasifinite FG-module, $C_G(A) =< 1 >$. Suppose that $\zeta(G)$ is infinite. Then*

(i) if S is a periodic normal subgroup of G then S is finite;

*(ii) if S is a periodic subgroup of G then S includes a nilpotent bounded p-subgroup of finite index, where $p =$***char***F.*

§4. Finitary modules

Let F be a field, G be a group and A be an FG-module. A group G is called **finitary** if for each element $g \in G$ the quotient space $A/C_A(g)$ has finite dimension over F. In this case we say that A is a finitary module. The theory of finitary linear groups is well established now and many interesting results have been proved there (see [41], for example). Some generalization of finitary groups have been considered by B.A.F. Wehrfritz (see ([44], [45], [46]). Taking into account that finite, artinian and noetherian modules over rings are natural extensions of finite dimensional vector spaces, B.A.F. Wehrfritz introduced the following classes of groups and modules.

Let R be a ring, G be a group and A be an RG-module. A group G is called **artinian – finitary** (respectively **noetherian – finitary**) if for every element $g \in G$, the factor-module $A/C_A(g)$ is artinian (respectively noetherian) **as an R – module**. In this case we will say that A is artinian-finitary (respectively noetherian-finitary) RG-module.

A finitary linear group G is said to be the **bounded finitary linear group**, if there is a positive integer b such that $dim_F(A/C_A(g)) \leq b$ for each element $g \in G$.

A finitary linear group can be considered as a linear analog of an FC-group (that is, a group with finite conjugacy classes), a concept introduced by R. Baer [1]. One of the first important results in the theory of FC-groups was a theorem due to B. H. Neumann describing the structure of FC-groups with boundedly finite conjugacy classes (BFC-groups). More precisely, B.H. Neumann proved that if there exists a positive integer b such that if $|g^G| \leq b|$ for each element $g \in G$, then the derived subgroup $[G, G]$ is finite ([37], Theorem 3.1). Thus we can consider bounded finitary group as a linear analogies of BFC-groups. Let ωRG be the **augmentation ideal** of the group ring RG, the two-sided ideal of RG generated by the all elements $g - 1$ where $g \in G$. In our analogy between groups with finite orbits and FC-groups, the **derived submodule** $A(\omega FG)$ of A is an analogy of the derived subgroup of a group G and, in view of Neumann's results, a natural conjecture would be that if G is a bounded finitary group then the dimension of $A(\omega FG)$ is finite. At once, we note that this analogy is not right. The corresponding counterexample was constructed in [16]. However, under some natural restrictions this analogy is possible not only for finitary groups but in some general situations.

Let D be a Dedekind domain. If A is an artinian D-module, then A is D-periodic and the set $\mathbf{Ass}_D(A)$ is finite. Furthermore, $A = K_1 \oplus \ldots \oplus K_d \oplus B$ where K_j is a Prüfer submodule, $j = 1, \ldots, d$, B is a finitely generated submodule. Observe that this decomposition is unique up to isomorphism. It follows that a number $\mathbf{d}$ is an invariant of the module A. Put $\mathbf{d} = \mathbf{l}_D(A)$. The submodule B has a finite series of submodules with D-simple factors. The Jordan-Holder Theorem implies that the length of this composition series is also an invariant of B, and hence of A. Denote this number by $\mathbf{l}_F(A)$.

Let D be a Dedekind domain and G be a group. The D-module A is said to be a **bounded artinian finitary**, if A is artinian finitary and there are the positive integers $\mathbf{b}_F(A) = \mathbf{b}$, $\mathbf{b}_D(A) = \mathbf{d}$ and a finite subset $\mathbf{b}_\sigma(A) = \tau \subseteq \mathbf{Spec}(D)$ such that $\mathbf{l}_F(A/C_A(g)) \leq \mathbf{b}$, $\mathbf{l}_D(A/C_A(g)) \leq \mathbf{d}$ and $\mathbf{Ass}_D(A/C_A(g)) \subseteq \mathbf{b}_\sigma(A)$. We will use the following notation: $\pi(A) = \{p | p = \mathbf{char}\ D/P \text{ for all } P \in \mathbf{b}_\sigma(A)\}$.

The structure of bounded artinian finitary modules has been described in [35].

Theorem 4.1 [35].*Let D be a Dedekind domain, G be a locally generalized radical group, and A be a DG-module. Suppose that A is a bounded artinian finitary module. Assume also that there exists a positive integer r such that the section p-rank of G is at most r for all $p \in \pi(A)$. Then*

(i) the submodule $A(\omega DG)$ is artinian as D-module,

(ii) the factor-group $G/C_G(A)$ has a finite special rank.

Corollary 4.2 [16]. *Let F be a field, G be a locally generalized radical group and A be an FG-module. Suppose that there exists a positive integer r such that the section p-rank of G is at most r where $p =$ **char** F. If A is a bounded finitary modules, then*

(i) the submodule $A(\omega FG)$ is finite dimemsional,

(ii) the factor-group $G/C_G(A)$ has a finite special rank.

§5. Modules over grouprings with the conditions *min-nad* and *max-nad*

The conditions of minimality and maximality on subgroups can be refered to very classical finiteness conditions in group theory. We shall apply these conditions to the theory of modules over group rings.

Let A be DG-module where D is Dedekind domain, G be a group. If $H \leq G$, then the quotient module $A/C_A(H)$ is called the cocentralizer of H in module A. If A is a DG-module such that the cocentralizer of group G in module A is not Artinian D-module then let $L_{nad}(G)$ be a system of all subgroups of group G for which the cocentralizers in the module A are not Artinian D-modules. Introduce on $L_{nad}(G)$ the ordering with respect to the usual inclusion of subgroups. If $L_{nad}(G)$ satisfies the maximal condition on subgroups, then we shall say that the group G satisfies the maximal condition on subgroups for which the cocentralizers in the module A are not Artinian D-modules or simply that the group G satisfies the condition $max - nad$. If the system $L_{nad}(G)$ satisfies the minimal condition on subgroups then we shall say that the group G satisfies the condition $min - nad$.

The subject of the investigation is a DG-module A where D is Dedekind domain, $C_G(A) = 1$, $A/C_A(G)$ is not Artinian D-module, G is a locally soluble group. It appears that the structure of a solube group G with $max-nad$ depends on the fact that the system of generators of G whether finite or infinite. Let $AD(G)$ be a set of all elements $x \in G$, such that the cocentralizer of group $< x >$ in the module A is Artinian D-module. Then $AD(G)$ is a normal subgroup of G [3].

At first, we consider a DG-module A such that the quotient group $G/[G,G]$ is not finitely generated.

Theorem 5.1 [3]. *Let A be a DG-module and suppose that G is a soluble group satisfying the condition $max - nad$. If the quotient group $G/[G,G]$ is not finitely generated then G satisfies the following conditions:*

(1) A has the finite series of DG-submodules

$$< 0 >= S_0 \leq S_1 \leq S_2 \leq S_3 \leq ... \leq S_m = A,$$

such that S_2/S_1 is a divisible D-module, which is a direct sum of finite number of Prüfer D-modules, every factor $S_{i+1}/S_i, i = 2, ..., m-1$, is a simple DG-module and the quotient group $Q = G/C_G(C_1)$ is a Prüfer p-group for some prime p;

(2) $H = C_G(S_1) \cap C_G(S_2/S_1) \cap ... \cap C_G(S_m/S_{m-1})$ is a nilpotent normal subgroup such that the cocentralizer of it in a module A is an Artinian **D***-module;*

(3) group G has the series of normal subgroups $H \leq L \leq N \leq M \leq G$ such that the quotient group G/M is finite, the quotient group M/N is a Prüfer p-group for some prime p, the quotient group N/L is finitely generated, L/H and H are nilpotent.

The next natutal step is the consideration of the case when group G is finitely generated.

Theorem 5.2 [3]. *Let A be a DG-module and suppose that G is a soluble group satisfying the condition $max - nad$. If the cocentralizer of a* *subgroup $AD(G)$ in a module A is an Artinian D-module then G has the series of normal subgroups $H \leq L \leq G$ such that the quotient group G/L is polycyclic, L/H and H are nilpotent.*

Theorem 5.3 [3]. *Let A be a DG-module and suppose that G is a soluble group satisfying the condition $max - nad$. If the cocentralizer of a subgroup $AD(G)$ in a module A is not Artinian D-module then G contains the normal subgroup L satisfying the following conditions:*

(1) The quotient group G/L is polycyclic.

(2) $L \leq AD(G)$ and the cocentralizer of a subgroup L in a module A is not Artinian D-module.

(3) The quotient group $L/[L, L]$ is not finitely generated.

Now we shall consider a structure of a locally soluble group G with $min - nad$.

Theorem 5.4 [6]. *Let A be a DG-module and suppose that G is a locally soluble group satisfying the condition $min-nad$. Then either G is soluble or G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq ... \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$, such that the cocentralizer of subgroup W_n in a module A is an Artinian D-module, and the factors W_{n+1}/W_n are abelian for $n \geq 0$. Moreover, G/W_ω is a soluble Chernikov group.*

In the case where $\mathbf{D} = \mathbf{Z_{p^\infty}}$ is a ring of p-adic integers the structure of a locally soluble group with $min - nad$ is sufficiently simple.

Theorem 5.5 [4]. *Let A be a $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group which satisfies the condition $min - nad$. If the cocentralizer of a group G in a module A is not Artinian $\mathbf{Z_{p^\infty}}$-module then a group G is soluble.*

Theorem 5.6 [4]. *Let A be a $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group which satisfies the condition $min - nad$. If the cocentralizer of a group G in a module A is not Artinian $\mathbf{Z_{p^\infty}}$-module then a group G contains the normal nilpotent subgroup H such that the quotient group G/H is Chernikov group.*

In the case where $\mathbf{D} = \mathbf{Z}$ is a ring of integers the structure of a locally soluble group with $min - nad$ is the same as in the case $\mathbf{D} = \mathbf{Z_{p^\infty}}$ [5].

In conclusion it should be noted that the following theorem has some connection with Shmidt's problem.

Theorem 5.7 [4]. *Let A be a $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group. Suppose that the cocentralizer of a group G in a module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. If the cocentralizer of each proper subgroup of a group G in a module A is an Artinian $\mathbf{Z_{p^\infty}}$-module then $G \simeq C_{q^\infty}$ for some prime q.*

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