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RESEARCH ARTICLE

A Morita context related to finite groups acting partially on a ring

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Abstract. In this paper we consider partial actions of groups on rings, partial skew group rings and partial fixed rings. We study a Morita context associated to these rings, α-partial Galois extensions and related aspects. Finally, we establish conditions to obtain a Morita equivalence between R^α and $R \star_\alpha G$.

Dedicated to Professor Miguel Ferrero
on occasion of his 70-th anniversary

Introduction

Partial actions of groups have been introduced in the theory of operator algebras giving powerful tools of their study (see [4], [6], [5], [2] and the literature quoted therein). Also in [4] the authors introduced partial actions on algebras in a pure algebraic context. Let G be a group and R a unital k-algebra, where k is a commutative ring. A *partial action* α of G on R is a collection of ideals D_g, $g \in G$, of R and isomorphisms of (non-necessarily unital) k -algebras $\alpha_g : D_{g^{-1}} \to D_g$ such that:

(i) $D_1 = R$ and α_1 is the identity mapping of R;

(ii) $D_{(gh)^{-1}} \supseteq \alpha_h^{-1}(D_h \cap D_{g^{-1}})$, for any $g, h \in G$;

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(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for any $x \in \alpha_h^{-1}(D_h \cap D_{g^{-1}})$ and $g, h \in G$.

Using (iii) we can easily see that $\alpha_{g^{-1}} = \alpha_g^{-1}$, for every $g \in G$. Also the property (ii) can be written as $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$, for all $g, h \in G$.

Let α be a partial action of G on R. The partial skew group ring $S = R \star_\alpha G$ (see [4]) is defined as the set of all finite formal sums $\sum_{g \in G} a_g \delta_g$, $a_g \in D_g$ for every $g \in G$, where the addition is defined in the usual way and the multiplication is determined by $(a_g\delta_g)(b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}$.

Given a partial action α of a group G on R an enveloping action for α is an algebra T together with a global action $\beta = \{\beta_g \mid g \in G\}$ of G on T, where each β_g is an automorphism of T, such that the partial action is given by restriction of the global action (see [4] and [6] for more precise definition and properties). From Theorem 4.5 of [4] we know that a partial action α has an enveloping action if and only if all the ideals D_g are unital algebras, i.e., D_g is generated by a central idempotent of R, for any $g \in G$. In this case the partial skew group ring $R \star_\alpha G$ is an associative algebra, which is not true in general (see [4], Example 3.5).

Throughout this paper R is an associative k-algebra (which will be called frequently simply a ring) with an identity element 1_R, G is a finite group and $\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\}$ is a partial action of G on R. We assume, unless otherwise stated, that the partial action has an enveloping action denoted by (T, β). Then any of the ideals D_g is generated by a central idempotent of R which we denote by 1_g. Since that (T, β) is the enveloping action of (R, α) we have that $1_g = 1_R\beta_g(1_R)$ where $\beta_g(1_R)$ are central elements in T for every $g \in G$. These facts will be used freely in this paper.

In general, T does not need to have an identity element; but it has an identity when G is a finite group. In this case, the fixed ring of T will be denoted by T^G and the trace map by $tr_G = \sum_{g \in G} g$. The ring of the invariant elements of R under α (the partial fixed ring) is $R^\alpha = \left\{x \in R : \alpha_g(x1_{g^{-1}}) = x1_g, \text{for any } g \in G\right\}$ and the partial trace map is defined by $tr_\alpha\left(r\right) = \sum_{g \in G} \alpha_g\left(r1_{g^{-1}}\right)$, for any $r \in R$ (see [5] and [7] for details).

In [5], Dokuchaev, Ferrero and Paques introduced the notion of partial Galois extension and developed a Galois theory for partial actions. The existence of partial Galois coordinates introduced in [5] is necessary (but not sufficient) to establish a Morita equivalence between the partial fixed ring and the partial skew group ring. In this paper, among other results, we show some applications of these concepts. In the first Section, following the global case (see [3]), we establish a Morita context

$(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$. In Section 2 we study the non-degeneracy of Γ and Γ' and some consequences. In Section 3 we show, under the assumption that the partial trace map is onto, that R is an α-partial Galois extension of R^α if and only if the Morita Context given is strict, and in this case, R^α and S are Morita equivalent rings. Finally, Section 4 is devoted to establish some class of rings that are each one an α-partial Galois extension of its corresponding partial fixed subring.

1. A Morita context for R^α and $R \star_\alpha G$

Following the global case ([3] and [1]), we will construct the partial version of a Morita context, that is, the six-tuple $(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$ where $V = {}_{R^\alpha}R_S$, $W = {}_S R_{R^\alpha}$ and $\Gamma : V \otimes_S W \to R^\alpha$ and $\Gamma' : W \otimes_{R^\alpha} V \to S$ are defined by

$$\Gamma(x \otimes y) = tr_\alpha(xy) = \sum_{g \in G} \alpha_g(xy1_{g^{-1}}), \tag{1}$$

$$\Gamma'(x \otimes y) = \sum_{g \in G} x\alpha_g(y1_{g^{-1}})\delta_g, \tag{2}$$

for all $x, y \in R$. For this we need some preparation.

First of all, it is clear that R has a structure of a (R^α, R^α)-bimodule via the multiplication of R and it is easy to check that R is a (S, R^α)-bimodule (resp. (R^α, S)-bimodule) with the left (resp. right) action of S on R given by $a\delta_g \cdot r = a\alpha_g(r1_{g^{-1}})$ (resp. $r \cdot a\delta_g = \alpha_{g^{-1}}(ra)$), for every $g \in G, a \in D_g$ and $r \in R$.

In order to prove that Γ and Γ' are well defined we need the following auxiliary result which is trivial in the global case.

1.1. Lemma. *$tr_\alpha(x) = tr_\alpha(\alpha_g(x))$, for any $g \in G$ and $x \in D_{g^{-1}}$.*

Proof. First, note that $\alpha_g(1_{g^{-1}}1_h) = 1_g 1_{gh}$ and $\alpha_h(\alpha_g(x1_{g^{-1}})1_{h^{-1}}) = \alpha_{hg}\left(x1_{g^{-1}h^{-1}}\right)1_h$, for any $g, h \in G$ and $x \in R$. Thus, for $x \in D_{g^{-1}}$ we have

$$\begin{aligned} tr_\alpha(\alpha_g(x)) &= \sum_{h \in G} \alpha_h\left(\alpha_g\left(x1_{g^{-1}}\right)1_{h^{-1}}\right) = \sum_{h \in G} \alpha_{hg}\left(x1_{g^{-1}h^{-1}}\right)1_h \\ &= \sum_{u \in G} \alpha_u\left(x1_{u^{-1}}\right)1_u 1_{ug^{-1}} = \sum_{u \in G} \alpha_u\left(x1_{u^{-1}}\right)\alpha_u\left(1_{u^{-1}}1_{g^{-1}}\right) \\ &= \sum_{u \in G} \alpha_u\left(x1_{g^{-1}}1_{u^{-1}}\right) = \sum_{u \in G} \alpha_u\left(x1_{u^{-1}}\right) = tr_\alpha(x). \end{aligned}$$

□

1.2. Proposition. *The applications* Γ *and* Γ'*, defined in (1) and (2) are well defined and are respectively* (R^α, R^α)*-bimodule and* (S,S)*-bimodule homomorphisms.*

Proof. Consider $\bar{\Gamma}: V \times W \to R^\alpha$, defined by $\bar{\Gamma}(x,y) = tr_\alpha(xy)$, for all $x, y \in R$. We will prove that $\bar{\Gamma}$ is S-balanced, hence Γ is well defined. Actually, let $r \in V$, $r' \in W$, $g \in G$ and $a \in D_g$. Since $r\,(a\delta_g \cdot r') = r a\alpha_g(r'1_{g^{-1}}) = \alpha_g\left(\alpha_{g^{-1}}(ra)\,r'\right)$ and $(r \cdot a\delta_g)\,r' = \alpha_{g^{-1}}(ra)\,r' \in D_{g^{-1}}$, then by Lemma 1.1, we have that $\bar{\Gamma}(r, a\delta_g \cdot r') = tr_\alpha(\alpha_g\left(\alpha_{g^{-1}}(ra)\,r'\right)) = tr_\alpha\left(\alpha_{g^{-1}}(ra)\,r'\right) = \bar{\Gamma}(r \cdot a\delta_g, r')$. The other properties of Γ are immediate. In a similar way we will check that Γ' is well defined considering $\bar{\Gamma}'$: $W \times V \to S$, defined by $\bar{\Gamma}'(x,y) = \sum\limits_{g\in G} x\alpha_g(y1_{g^{-1}})\delta_g$, for all $x, y \in R$. For $t \in R^\alpha$ and $r, r' \in R$ it easily follows that $\bar{\Gamma}'(rt, r') = \bar{\Gamma}'(r, tr')$. Further Γ' is an (S,S)-bimodule homomorphism. Actually, for all $h \in G$ and $y \in R$, we have $\sum\limits_{g\in G} \alpha_h\left(\alpha_g(y1_{g^{-1}})1_{h^{-1}}\right)\delta_{hg} = \sum\limits_{g\in G} \alpha_{hg}(y1_{(hg)^{-1}})\delta_{hg} = \sum\limits_{u\in G} \alpha_u(y1_{u^{-1}})\delta_u$. Therefore

$$\begin{aligned}
a\delta_h\Gamma'(x\otimes y) &= \sum_{g\in G} a\alpha_h\left(x\alpha_g(y1_{g^{-1}})1_{h^{-1}}\right)\delta_{hg} \\
&= \sum_{g\in G} a\alpha_h(x1_{h^{-1}})\alpha_h\left(\alpha_g(y1_{g^{-1}})1_{h^{-1}}\right)\delta_{hg} \\
&= \sum_{u\in G}(a\delta_h\cdot x)\alpha_u(y1_{u^{-1}})\delta_u = \Gamma'\left((a\delta_h\cdot x)\otimes y\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\Gamma'(x\otimes y)(a\delta_h) &= \sum_{g\in G}\left(x\alpha_g(y1_{g^{-1}})\delta_g\right)(a\delta_h) = \sum_{g\in G} x\alpha_g\left(ya1_{g^{-1}}\right)\delta_{gh} \\
&= \sum_{u\in G} x\alpha_{uh^{-1}}\left(ya1_{hu^{-1}}\right)\delta_u = \sum_{u\in G} x\alpha_u\left(\alpha_{h^{-1}}(ya)\,1_{u^{-1}}\right)\delta_u \\
&= \Gamma'\left(x\otimes\alpha_{h^{-1}}(ya)\right) = \Gamma'\left(x\otimes(y\cdot a\delta_h)\right).
\end{aligned}$$

Thus, the proposition is proved. □

1.3. Remark. From Proposition 1.2, for any $x, y \in R$, we get one sided ideals $\Gamma(V\otimes_S x) < {}_{R^\alpha}R^\alpha$, $\Gamma(y\otimes_S W) < R^\alpha_{R^\alpha}$, $\Gamma'(x\otimes_{R^\alpha} V) < S_S$ and $\Gamma'(W\otimes_{R^\alpha} y) < {}_SS$. In particular, $\Gamma(V\otimes_S W)$ is an ideal of R^α and $\Gamma'(W\otimes_{R^\alpha} V)$ is an ideal of S.

It remains to verify the associativity conditions.

1.4. Proposition. *Using the previous notations, we have* $x \cdot \Gamma'(y \otimes z) = \Gamma(x \otimes y) \cdot z$ *and* $\Gamma'(x \otimes y) \cdot z = x \cdot \Gamma(y \otimes z)$ *for all* $x, y, z \in R$.

Proof. Let $x, y, z \in R$. Then $x \cdot \Gamma'(y \otimes z) = x \cdot \sum_{g \in G} y\alpha_g(z1_{g^{-1}})\delta_g = \sum_{g \in G} \alpha_{g^{-1}}\left(xy\alpha_g(z1_{g^{-1}})\right) = \sum_{g \in G} \alpha_{g^{-1}}(xy1_g)\, z = tr_\alpha(xy)z = \Gamma(x \otimes y) \cdot z$. Moreover,

$$\begin{aligned}\Gamma'(x \otimes y) \cdot z &= \sum_{g \in G} x\alpha_g(y1_{g^{-1}})\delta_g \cdot z = \sum_{g \in G} x\alpha_g\left(yz1_{g^{-1}}\right) \\ &= xtr_\alpha(yz) = x \cdot \Gamma(y \otimes z).\end{aligned}$$

Thus, the assertions hold. □

As an immediate consequence of Propositions 1.2 and 1.4 we obtain

1.5. Theorem. *Using the previous notations, the six-tuple* $(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$ *is a Morita context.*

As simple application of Theorem 20 and Corollary 23 of [1] we have

1.6. Corollary. *Using the previous notations, the following assertions hold:*

1. $\Gamma(V \otimes_S rad(S)W) \subseteq rad(R^\alpha)$.
2. $\Gamma'(W \otimes_{R^\alpha} rad(R^\alpha)V) \subseteq rad(S)$.

In both cases, rad denotes one of the following radicals: Prime, Jacobson, Levitzki or the Nil upper if R satisfies Köthe's Conjecture.

We will keep throughout all the next sections the same notations introduced in this one.

2. Non-degeneracy of Γ and Γ'

Recall that, if A, B and C are additive groups, a bilinear form $F : A \times B \to C$ is nondegenerate if, for all $0 \neq a \in A$ and $0 \neq b \in B$, we have $F(a, B) \neq 0$ and $F(A, b) \neq 0$. The non-degeneracy of Γ and Γ' provides some consequences that we will list in this section. Firstly, we also recall that an ideal I of R is said to be α-invariant if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I \cap D_g$, for any $g \in G$. Note that this notion is equivalent to $\alpha_g(I \cap D_{g^{-1}}) = I \cap D_g$, for any $g \in G$, (see [6], Definition 2.1).

2.1. Lemma. *If $x \in W$, then $x^{\perp} = \{y \in V : \Gamma'(x \otimes y) = 0\}$ is a right α-invariant ideal of R contained in $ran_R(x)$ (the right annihilator of x in R). Analogously, if $y \in V$, then $y^{\perp} = \{x \in W : \Gamma'(x \otimes y) = 0\}$ is a left α-invariant ideal of R contained in $lan_R(y)$ (the left annihilator of y in R).*

Proof. Consider $x \in W, r \in R$ and $y \in x^{\perp}$, we have $\Gamma'(x \otimes yr) = \Gamma'(x \otimes y \cdot r\delta_1) = \Gamma'(x \otimes y)r\delta_1 = 0r\delta_1 = 0$, thus $yr \in x^{\perp}$. Now $0 = \Gamma'(x \otimes y) = x\sum_{g\in G}\alpha_g(y1_{g^{-1}})\delta_g$ and hence $0 = x\alpha_1(y1_R)\delta_1 = xy$ implies $y \in ran_R(x)$. It follows that $x^{\perp} \subseteq ran_R(x)$. Moreover, since $\Gamma'(x \otimes \alpha_g(y1_{g^{-1}})) = \Gamma'(x \otimes y \cdot 1_{g^{-1}}\delta_{g^{-1}}) = \Gamma'(x \otimes y) \cdot 1_{g^{-1}}\delta_{g^{-1}} = 0$, it follows that $\alpha_g(y1_{g^{-1}}) \in x^{\perp}$. The second assertion follows by similar arguments. □

2.2. Lemma. Γ *is nondegenerate if and only if* Γ' *is nondegenerate.*

Proof. Let $r \in R$. By Proposition 1.4 we have $R\Gamma(r \otimes_S W) = \Gamma'(W \otimes_{R^\alpha} r)R$ and $R\Gamma'(r \otimes_{R^\alpha} V) = \Gamma(V \otimes_S r)R$. Since R is unital, the result follows. □

In the following proposition we will see that Γ and Γ' are nondegenerate and, as a consequence, that some radical properties are transferable from the partial skew group ring to the partial fixed ring.

2.3. Proposition. *The following statements hold:*

1. Γ *and* Γ' *are nondegenerate.*

2. $rad\,(S) = 0$ *if and only if* $rad\,(R^\alpha) = 0$, *where rad denotes someone of the following radicals: Prime, Jacobson, Levitzki or the Nil upper radical if R satisfies the Köthe's conjecture.*

3. *If* $I <\ _S S$ *is minimal, then* $V \cdot I = (0)$ *or* $V \cdot I$ *is a simple* R^α*-module*

Proof. 1. Take $x \in R$, $x \neq 0$. Since R is unital, we have that $ran_R\,(x) \neq R$. Now using the Lemma 2.1, $x^{\perp} \subseteq ran_R\,(x) \neq R$ implies that there exists $y \in V$ such that $\Gamma'(x \otimes y) \neq 0$. Hence $\Gamma'(x \otimes_{R^\alpha} V) \neq 0$. In an analogous way, we get that $lan_R\,(y) \neq R$ and $\Gamma'(W \otimes_{R^\alpha} y) \neq 0$ for any $0 \neq y \in V$.
2. The result follows from item 1 and Corollary 1.6.
3. Assume that $V \cdot I \neq 0$ and consider $0 \neq J \subseteq V \cdot I$, where J is a left R^α-submodule of R. By item 1, $\Gamma'(W \otimes_{R^\alpha} J) \neq 0$. Then $0 \neq \Gamma'(W \otimes_{R^\alpha} J) \subseteq \Gamma'(W \otimes_{R^\alpha} V \cdot I) = \Gamma'(W \otimes_{R^\alpha} V)I \subseteq I$. Since I is minimal in S, it follows that $\Gamma'(W \otimes_{R^\alpha} J) = I$, hence $V \cdot \Gamma'(W \otimes_{R^\alpha} J) = V \cdot I$.

Now, by Proposition 1.4, we have $V \cdot \Gamma'(W \otimes_{R^\alpha} J) = \Gamma(V \otimes_S W)J \subseteq J$. Therefore, $J \subseteq V \cdot I = V \cdot \Gamma'(W \otimes_{R^\alpha} J) \subseteq J$, that is, $J = V \cdot I$, thus $V \cdot I$ is a simple R^α-module. □

Recall that N is an essential submodule of a module M if, for all nonzero submodules X of M, one has $N \cap X \neq 0$. If an ideal (resp.a left ideal, a right ideal) I is an essential submodule of ${}_RR_R$ (resp. ${}_RR$, R_R) it is called an essential (resp. left, right) ideal.

2.4. Proposition. *The following statements hold:*

1. *If $x \in R$ is such that $\Gamma'(W \otimes_{R^\alpha} V) \cdot x = 0$, then $x = 0$; analogously, if $y \in R$, is such that $y \cdot \Gamma'(W \otimes_{R^\alpha} V) = 0$, then $y = 0$.*
2. *$lan_R(\Gamma(V \otimes_S W)) = ran_R(\Gamma(V \otimes_S W)) = 0$. In particular, $\Gamma(V \otimes_S W)$ is an essential ideal of R^α.*
3. *If A is a subset of R^α and $lan_{R^\alpha}(A) = 0$, then $lan_R(A) = 0$. The same holds for right annihilators.*
4. *If E is an essential submodule of ${}_RR$ or ${}_{R^\alpha}R$, then $\Gamma(V \otimes_S E)$ is an essential submodule of ${}_{R^\alpha}R^\alpha$.*

Proof. 1. It is an immediate consequence of Propositions 1.4 and 2.3.

2. By Proposition 2.3, we have that Γ' is nondegenerate, then we can prove that $lan_R(\Gamma(V \otimes_S W)) = ran_R(\Gamma(V \otimes_S W)) = 0$ using similar arguments as in 1. So, it follows that $\Gamma(V \otimes_S W)$ is an essential ideal of R^α.

3. For $A \subseteq R^\alpha$, $\Gamma(1_R \otimes_S lan_R(A)) \subseteq lan_{R^\alpha}(A)$. Actually $\Gamma(1_R \otimes_S lan_R(A)) \subseteq tr_\alpha(R) \subseteq R^\alpha$ and $\Gamma(1_R \otimes_S lan_R(A))A = tr_\alpha(lan_R(A)A) = 0$. Again, since $\Gamma(ran_R(A) \otimes_S W) \subseteq tr_\alpha(R) \subseteq R^\alpha$ and $A\Gamma(ran_R(A) \otimes_S W) = \Gamma(Aran_R(A) \otimes_S W) = 0$, it follows that $\Gamma(ran_R(A) \otimes_S W) \subseteq ran_{R^\alpha}(A)$. By the non-degeneracy of Γ, we obtain the result.

4. Let E be a essential left ideal of R and $0 \neq J < {}_{R^\alpha}R^\alpha$. Hence $0 \neq J \subseteq RJ < {}_RR$ implies $RJ \cap E \neq 0$. Thus there exist $n > 0$, $r_1, \cdots, r_n \in R$ and $j_1, \cdots, j_n \in J$, such that $0 \neq \sum_{i=1}^n r_i j_i \in E$. By assumption, we have $0 \neq \Gamma(V \otimes_S \sum_{i=1}^n r_i j_i) = \sum_{i=1}^n \Gamma(V \otimes_S r_i) j_i \subseteq J$. Hence $\Gamma(V \otimes_S E) \cap J \neq 0$. The remaining part follows similarly. □

Following [10], Chapter 1, we say that α has a *nondegenerate partial trace* if R^α is semiprime and for any non-zero left α-invariant ideal H of R we have $tr_\alpha(H) \neq 0$. It is easy to see that if R^α is semiprime and Γ is nondegenerate then α has a nondegenerate partial trace. We will use this in the next result. Before, recall that a nonzero left module U is *uniform* if each nonzero left submodule of U is essential in U. We also recall that a left module M is said to have *finite uniform dimension* if it contains no infinite direct sum of nonzero left submodules. In this case, any direct sum of uniform left submodules of M which is essential in M has precisely the same quantity of summands. Such quantity is called the *left uniform dimension* of M, and is written $udimM$. In particular, if R is a ring, $udimR$ will denote the left uniform dimension of ${}_RR$. Finally, a ring R is a left Goldie ring if it has finite left uniform dimension and satisfies the ascending chain condition on the left annihilators (see [9], Sections 2.2 and 2.3, for details). Theorems 5.5 and 5.6 of [7] assert that if R is a semiprime $|G|$-torsion free ring, then $udim\, R^\alpha \leq udim\, R \leq |G|\, udim\, R^\alpha$. This same result also holds under another different hypotheses.

2.5. Corollary. *Assume that R and R^α are semiprime. If R is a left Goldie ring, then R^α is a left Goldie ring. Furthermore $udim\, R^\alpha \leq udim\, R \leq |G|\, udim\, R^\alpha$.*

Proof. By Proposition 2.3, the first part is immediate from Corollary 5.2 of [7]. Now, by Corollary 1.15 of [6] and Theorem 1.4 of [7] we have that the enveloping T and its subring T^G are semiprime. Then α and its enveloping action have a nondegenerate partial trace on R and T respectively. By the first part of this corollary applied to T, Proposition 1.18 of [6], Theorem 1.4 of [7] and Theorem 5.3 of [10], the result follows. □

We finish this section with an example showing that the hypotheses of Corollary 2.5 are, in fact, not equivalent to the claimed in Theorems 5.5 and 5.6 of [7].

2.6. Example. Take $R = Ke_1 \oplus Ke_2 \oplus Ke_3$, where K is a ring and e_1, e_2, e_3 are orthogonal central idempotents of R. Let G be the cyclic group of order 5 with generator g and define a partial action of G on R by: $\alpha_1 = id_R$, $\alpha_g : Ke_1 \oplus Ke_2 \to Ke_2 \oplus Ke_3$, $\alpha_g(e_1) = e_2$ and $\alpha_g(e_2) = e_3$; $\alpha_{g^2} : Ke_1 \to Ke_3$, $\alpha_{g^2}(e_1) = e_3$; $\alpha_{g^3} : Ke_3 \to Ke_1$, $\alpha_{g^3}(e_3) = e_1$; $\alpha_{g^4} : Ke_2 \oplus Ke_3 \to Ke_1 \oplus Ke_2$, $\alpha_{g^4}(e_2) = e_1$ and $\alpha_{g^4}(e_3) = e_2$. If $K = \mathbb{Z}/15\mathbb{Z}$ we have that $R^\alpha = K1_R$. Then R and R^α are semiprime rings, but R is not a $|G|$-torsion free.

3. Morita equivalence

The main purpose of this section is to show that the existence of partial Galois coordinates of R over R^α is a necessary and sufficient condition for the map Γ' to be surjective, and if in addition the trace map tr_α from R to R^α is onto then the Morita context $(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$ is strict.

Recall from [5] Section 3, that R is *an α-partial Galois extension of* R^α if there exist elements $x_i, y_i \in R$, $i = 1, ..., n$, such that

$$\sum_{i=1}^{n} x_i \alpha_g \left(y_i 1_{g^{-1}}\right) = \delta_{1,g} 1_R,$$

for any $g \in G$. Such elements are called *partial Galois coordinates* of R over R^α.

3.1. Theorem. *The following statements are equivalent:*

1. *R is an α-partial Galois extension of R^α.*
2. *R is a finitely generated projective right R^α-module and $\varphi : S \longrightarrow End(R_{R^\alpha})$ defined by $\varphi(a\delta_g)(x) = a\alpha_g(x1_{g^{-1}})$ is an isomorphism of rings.*
3. *$RtR = S$, where $t = \sum_{h \in G} 1_h \delta_h$.*
4. *The map Γ' is surjective.*
5. *R is a generator for the category of the left S-modules.*

Moreover, if at least one of the above statements holds, then the following additional statements also are equivalent:

6. *$R^\alpha = tr_\alpha(R)$.*
7. *R is a generator for the category of the right R^α-modules.*
8. *The Morita context $(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$ is strict.*

Proof. 1. $\Leftrightarrow$ 2. It follows by the same arguments used in the proof of Theorem 4.1 of [5].

1. $\Leftrightarrow$ 3. It suffices to observe that $RtR = S$ if and only if there exists elements $a_1, ..., a_n, b_1, ..., b_n \in R$ such that $\sum_{i=1}^{n} a_i t b_i = 1_R$, if only if $\{a_i, b_i\}_{i=1}^{n}$ are partial Galois coordinates.

1. $\Leftrightarrow$ 4. Γ' is onto if and only if there exist elements $x_i, y_i \in R$, $1 \leq i \leq n$ such that $\sum\limits_{i=1}^{n} \sum_{g \in G} x_i \alpha_g \left(y_i 1_{g^{-1}}\right) \delta_g = 1$, if and only if there exist elements $x_i, y_i \in R$, $1 \leq i \leq n$ such that $\sum\limits_{i=1}^{n} x_i \alpha_g \left(y_i 1_{g^{-1}}\right) = \delta_{1,g} 1_R$, for any $g \in G$.

2. $\Leftrightarrow$ 5. We have that $(R^\alpha)^{op} \backsimeq End_S(R)$. Actually, For all $a \in R^\alpha$ we define $\phi : R^\alpha \to End_S(R)$ by $\phi(a) = \phi_a$ where $\phi_a(x) = xa$, for all $x \in R$. Since $\phi_a(u\delta_g \cdot r) = u\alpha_g(r1_{g^{-1}})a = u\alpha_g(ra1_{g^{-1}}) = u\delta_g \cdot (r\phi_a)$, for any $u \in D_g$ and $r \in R$, it follows that $\phi_a \in End_S(R)$. It is easy to see that ϕ is a monomorphism of rings. Let $f \in End_S(R)$ and $r \in R$. Since $f(r) = f(r\delta_1 \cdot 1_R) = r\delta_1 \cdot f(1_R) = rf(1_R)$ and for any $g \in G$, $\alpha_g(f(1_R)1_{g^{-1}}) = 1_g\delta_g \cdot f(1_R) = f(1_g\delta_g \cdot 1_R) = f(1_g) = 1_g f(1_R)$, we have that ϕ is an isomorphism. Finally, from Theorem 0.4 of [10] we have the equivalence.

Now by assumption that one of the above statements holds,

6. $\Leftrightarrow$ 7. Assuming that $R^\alpha = tr_\alpha(R)$ it follows that the map tr_α is surjective and so R is a right R^α-generator. Conversely, first observe that R is a right R^α-generator if and only if the trace ideal of R^α, defined by $T(R_{R^\alpha}) := \sum_{f\in Hom(R_{R^\alpha},R^\alpha)} f(R)$, equals R^α (see, for instance Theorem 18.8 of [8]). Now, take $f \in Hom(R_{R^\alpha}, R^\alpha)$. By the assertion 3. above, there exists $y \in S$, $y = \sum_{g\in G} a_g\delta_g$, such $\varphi(y) = f$. Then for any $r \in R$ we have $\varphi(y)(r) = \sum_{g\in G} a_g\alpha_g(r1_{g^{-1}}) \in R^\alpha$. Thus, for any $h \in G$,

$$\begin{aligned}\sum_{g\in G} a_g\alpha_g(r1_{g^{-1}})1_h &= \alpha_h(\sum_{g\in G} a_g\alpha_g(r1_{g^{-1}})1_{h^{-1}}) \\ &= \sum_{g\in G} \alpha_h(a_g1_{h^{-1}})\alpha_h\left(\alpha_g(r1_{g^{-1}})1_{h^{-1}}\right) \\ &= \sum_{g\in G} \alpha_h\left(a_g1_{h^{-1}}\right)\alpha_{hg}\left(r1_{(hg)^{-1}}\right) \\ &= \sum_{\tau\in G} \alpha_h\left(a_{h^{-1}\tau}1_{h^{-1}}\right)\alpha_\tau\left(r1_{\tau^{-1}}\right).\end{aligned}$$

and so

$$\varphi(\sum_{g\in G} a_g1_h\delta_g)(r) = \varphi(\sum_{g\in G} \alpha_h\left(a_{h^{-1}g}1_{h^{-1}}\right)\delta_g)(r).$$

Hence

$$\sum_{g\in G} a_g1_h\delta_g = \sum_{g\in G} \alpha_h\left(a_{h^{-1}g}1_{h^{-1}}\right)\delta_g,$$

which implies

$$a_g1_h = \alpha_h\left(a_{h^{-1}g}1_{h^{-1}}\right).$$

for any $g, h \in G$. In particular, for $h = g$, we have $a_g = \alpha_g(a_11_{g^{-1}})$. Therefore, $y = \sum_{g\in G} a_g\delta_g = \sum_{g\in G} \alpha_g(a_11_{g^{-1}})\delta_g = ta_1$, $f = tr_\alpha(a_1\_)$ and, consequently, $R^\alpha \subseteq tr_\alpha(R)$.

7. $\Leftrightarrow$ 8. Immediate. $\square$

3.2. Corollary. *Suppose that at least one of the elements $tr_\alpha(1_R)$ and $|G|1_R$ is invertible in R. Then, R is an α-partial Galois extension of R^α if and only if the Morita context $(R^\alpha, S = R \star_\alpha G, V, W, \Gamma, \Gamma')$ is strict.*

Proof. It suffices to show that $tr_\alpha(R) = R^\alpha$. One easily sees that $tr_\alpha(1_R)$ is invertible in R if and only if there exists $c \in R^\alpha$ such that $tr_\alpha(c) = 1_R$. And if $|G|1_R$ is invertible in R then the result follows from Lema 2.1 of [5] and Proposition 2.5 of [2]. □

4. Applications

The main purpose of this section is to establish some sufficient conditions for a ring R to be an α-partial Galois extension of R^α. Recall that a right (resp. left) S-module M is *faithful* if $ann\, M_S = 0$ (resp. $ann\, _SM = 0$). In general, V and W are not faithful S-modules (see Example 2.1 of [3]). In fact, it easily follows from Proposition 1.2 and the non-degeneracy of Γ' that $ann\, V_S = ran_S\Gamma'(W \otimes_{R^\alpha} V)$ and $ann\, _SW = lan_S\Gamma'(W \otimes_{R^\alpha} V))$.

4.1. Proposition. *The following statements hold:*

1. *If V_S is faithful, then $\Gamma'(W \otimes_{R^\alpha} V)$ is a essential left ideal of S.*
2. *If $_SW$ is faithful, then $\Gamma'(W \otimes_{R^\alpha} V)$ is a essential right ideal of S.*

Moreover, if R is semiprime and $|G|$-torsion free, then the converse of 1. and 2. also holds.

Proof. 1. If V_S is faithful, then we have $ran_S\Gamma'(W \otimes_{R^\alpha} V) = 0$. Now, consider a left ideal J of S such that $J \cap \Gamma'(W \otimes_{R^\alpha} V) = 0$. Then we have $\Gamma'(W \otimes_{R^\alpha} V)\, J \subseteq J \cap \Gamma'(W \otimes_{R^\alpha} V) = 0$ and so

$$J \subseteq ran_S\Gamma'(W \otimes_{R^\alpha} V) = 0.$$

2. It is analogous to item 1.

Finally assume that R is semiprime and $|G|$-torsion free. Then by Proposition 5.3 of [6], we have that S is a semiprime ring. Therefore, if $\Gamma'(W \otimes_{R^\alpha} V)$ is a essential right ideal of S, we have $ran_S\Gamma'(W \otimes_{R^\alpha} V) = 0$ and so V_S is faithful. The converse of the item 2. follows similarly. □

4.2. Remark. We observe that if R is semisimple and $tr_\alpha(1_R)$ is invertible in R, then the assertions 1. and 2. in Proposition 4.1 are in fact equivalences, by Corollary 6.8 of [6].

We end with the following proposition which gives some sufficient conditions on R in order to obtain a Morita equivalence between R^α and S.

4.3. Proposition. *Assume that R is a ring such that R_S (or ${}_SR$) is faithful. If R is semisimple and at least one of the elements $tr_\alpha(1_R)$ or $|G|1_R$ is invertible in R, then R is an α-partial Galois extension of R^α. In particular, R^α and S are Morita equivalents.*

Proof. By Maschke Theorem (see Theorem 3.1 or Corollary 3.3 of [6] for the partial case) we have that S is semisimple. Now, by Proposition 4.1, $\Gamma'(V \otimes_{R^\alpha} W)$ is a essential left (or right) ideal of S. Thus $\Gamma'(V \otimes_{R^\alpha} W) = S$, that is, Γ' is onto. Then, the result follows by Corollary 3.2. □

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