

n -ary comodules over n -ary (co)algebras

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ABSTRACT. In the paper we study connections between (co)modules over n -ary and binary (co)algebras.

Introduction

In this paper, the notations of right (left) (co)modules over (co)algebras are generalized from binary to n -ary case Definition 1.1 and 1.2 It is proved that a module M is a right n -ary C -comodule if and only if it is a left n -ary C^* -module Theorem 2.1. A dual statement is proved in Theorem 3.1. Notations of right (left) n -ary (co)module algebras are introduced and it is proved that an n -ary C -comodule algebra is an n -ary C^* -module algebra, Theorem 2.2. Moreover, n -ary C -module algebra is an n -ary C^* -comodule algebra Theorem 3.3.

All necessary notations and definitions can be founded in the papers listed in References.

1. Basic notions

Let k be a ground commutative associative ring with a unit, C and M modules over k . In what follows, \otimes is a tensor product over k . All homomorphisms are k -linear maps. In [Z1], the concept of n -ary algebras (C, m) is defined, where $m : C \otimes \cdots \otimes C \rightarrow C$ is n -ary multiplication,

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which is associative. It means that the following diagram is commutative:

$$\begin{array}{ccc}
 C^{\otimes(2n-1)} & \xrightarrow{m \otimes 1_C^{\otimes(n-1)}} & C^{\otimes n} \\
 1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-1)} \downarrow & & \downarrow m \\
 C^{\otimes n} & \xrightarrow{m} & C
 \end{array}$$

i.e. for any $i = 1, \dots, n$ we have

$$m \cdot (m \otimes 1_C^{\otimes(n-1)}) = m \cdot (1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-1)}).$$

The concept of n -ary coalgebra (C, Δ) is defined in [Z2], where $\Delta : C \rightarrow C \otimes \dots \otimes C$ is n -ary comultiplication, which is coassociative, that is the following diagram is commutative:

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C^{\otimes n} \\
 \Delta \downarrow & & \downarrow 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)} \\
 C^{\otimes n} & \xrightarrow{\Delta \otimes 1_C^{\otimes(n-1)}} & C^{\otimes(2n-1)}
 \end{array}$$

i.e. for any $i = 1, \dots, n$ we have

$$(\Delta \otimes 1_C^{\otimes(n-1)}) \cdot \Delta = (1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)}) \cdot \Delta.$$

Analogously, the concept of n -ary bialgebra (C, m, Δ) , where m is associative n -ary multiplication and Δ is coassociative n -ary comultiplication. It is shown that Δ is a homomorphism of n -ary algebras. We do not suppose an existence of the unit and counit. In [Z1] the notion of homomorphism $f : (C, m_C) \rightarrow (C', m_{C'})$ of n -ary algebras is defined as a morphism $f : C \rightarrow C'$, such that the following diagram is commutative:

$$\begin{array}{ccc}
 C^{\otimes n} & \xrightarrow{f^{\otimes n}} & C'^{\otimes n} \\
 m_C \downarrow & & \downarrow m_{C'} \\
 C & \xrightarrow{f} & C'
 \end{array}$$

i.e. $f \cdot m_C = m_{C'} \cdot f^{\otimes n}$.

Let C be an n -ary coalgebra and a finitely generated projective k -module. Denote by C^* the k -module $\text{Hom}(C, k)$. Then C^* is an n -ary algebra with multiplication $l_1 * \dots * l_n$, where for $c \in C$:

$$(l_1 * \dots * l_n)(c) = \sum_c l_1(c_{(1)}) \dots l_n(c_{(n)}) \tag{1}$$

if

$$\Delta(c) = \sum_c c_{(1)} \otimes \cdots \otimes c_{(n)} \in C^{\otimes n}.$$

Conversely, let C be an n -ary algebra and a finitely generated projective k -module. Define n -ary comultiplication Δ^* in $C^* = \text{Hom}(C, k)$, by the rule:

$$(\Delta^*)(x_1 \otimes \cdots \otimes x_n) = l(x_1 \cdots x_n) \quad (2)$$

where $x_i \in C$. Here we use the isomorphism of k -modules:

$$(C^{\otimes n})^* \simeq (C^*)^{\otimes n}$$

because C is a finitely-generated projective k -module. Then, C^* is a n -ary coalgebra. If C is a n -ary (co)algebra, then $(C^*)^* \simeq C$, [B].

Definition 1.1. Let C be a n -ary coalgebra. We say that k -module M is a right n -ary C -comodule, if there exists a map $\rho : M \rightarrow M \otimes C^{\otimes(n-1)}$, such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C^{\otimes(n-1)} \\ \rho \downarrow & & \downarrow 1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)} \\ M \otimes C^{\otimes(n-1)} & \xrightarrow{\rho \otimes 1_C^{\otimes(n-1)}} & M \otimes C^{\otimes(2n-1)} \end{array}$$

i.e.

$$\left(1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)}\right) \cdot \rho = \left(\rho \otimes 1_C^{\otimes(n-1)}\right) \cdot \rho$$

for any i .

Definition 1.2. Let C be a n -ary algebra. k -module M is called a left n -ary C -module, if there exists a map $\gamma : C^{\otimes(n-1)} \otimes M \rightarrow M$, such that the following diagram is commutative:

$$\begin{array}{ccc} C^{\otimes(n-1)} \otimes M & \xrightarrow{\gamma} & M \\ \uparrow 1_C^{\otimes(n-1)} \otimes \gamma & & \uparrow \gamma \\ C^{\otimes(2n-1)} \otimes M & \xrightarrow{1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M} & C^{\otimes(n-1)} \otimes M \end{array}$$

i.e.

$$\gamma \cdot \left(1_C^{\otimes(n-1)} \otimes \gamma\right) = \gamma \cdot \left(1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M\right)$$

for any $i = 1, \dots, n$.

Definition 1.3. Let C be a n -ary bialgebra. n -ary algebra M is called a left n -ary C -module algebra, if:

- 1) M is a left n -ary C -module;
- 2) for any $c_1, \dots, c_{n-1} \in C$, $m_1, \dots, m_n \in M$

$$\begin{aligned} & (c_1 \otimes \dots \otimes c_{n-1})(m_1 \dots m_n) \\ &= \sum_{c_1, \dots, c_{n-1}} (c_{1(1)} \dots c_{n-1(1)} m_1) \dots (c_{1(n)} \dots c_{n-1(n)} m_n) \end{aligned}$$

Definition 1.4. Let C be an n -ary bialgebra. n -ary algebra M is called a right n -ary C -comodule algebra, if:

- 1) M is a right n -ary C -comodule with the structure morphism $\rho : M \rightarrow M \otimes C^{\otimes(n-1)}$;
- 2) ρ is a homomorphism of n -ary algebras.

2. The relations between n -ary comodules and modules

Theorem 2.1. Let C be a n -ary coalgebra. Then M is a right n -ary C -comodule if and only if M is a left n -ary C^* -module.

Proof. Suppose that M is a right n -ary C -comodule and

$$\rho(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)} \otimes \dots \otimes m_{(n-1)}, \tag{3}$$

where $m \in M$ and $m_{(0)} \in M$, $m_{(i)} \in C$ for $i = 1, \dots, n-1$. If $l_1, \dots, l_{n-1} \in C^*$, $m \in M$, then we put

$$\rho^*(l_1 \otimes \dots \otimes l_{n-1} \otimes m) = \sum_{(m)} m_{(0)} l_1(m_{(1)}) \dots l_{n-1}(m_{(n-1)}) \in M. \tag{4}$$

Further, if $l_1, \dots, l_{2n-2} \in C^*$ and $m \in M$, then by Definition 1.2 we have:

$$\begin{aligned} & \left(\rho^* \cdot \left(1_{C^*}^{\otimes(n-1)} \otimes \rho^* \right) \right) (l_1 \otimes \dots \otimes l_{2n-2} \otimes m) \\ &= \rho^* \left[l_1 \otimes \dots \otimes l_{n-1} \otimes \sum_{(m)} m_{(0)} l_n(m_{(1)}) \dots l_{2n-2}(m_{(n-1)}) \right] \\ &= \sum_{m_{(0)}} m_{(0)(0)} l_1(m_{(0)(1)}) \dots l_{n-1}(m_{(0)(n-1)}) l_n(m_{(1)}) \dots l_{2n-2}(m_{(n-1)}) . \end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}
 & \left(\rho^* \cdot \left(1_{C^*}^{\otimes(i-1)} \otimes m^* \otimes 1_{C^*}^{\otimes(n-1-i)} \otimes 1_M \right) \right) (l_1 \otimes \cdots \otimes l_{2n-2} \otimes m) \\
 &= \rho^* (l_1 \otimes \cdots \otimes l_{i-1} \otimes (l_i * \cdots * l_{i+n-1}) \otimes l_{n+i} \otimes \cdots \otimes l_{2n-2} \otimes m) \\
 &= \sum_m m_{(0)} l_1 (m_{(1)}) \cdots l_{i-1} (m_{(i-1)}) (l_1 * \cdots * l_{i+n-1}) \times \\
 & (m_{(i)}) l_{i+n} (m_{(i+1)}) \cdots l_{2n-2} (m_{(n-1)}) \\
 &= \sum_m m_{(0)} l_1 (m_{(1)}) \cdots l_{i-1} (m_{(i-1)}) \times \\
 & \left[\sum_{m^{(i)}} l_1 (m_{(i)(1)}) \cdots l_{i+n-1} (m_{(i)(n)}) \right] l_{i+n} (m_{(i+1)}) \cdots l_{2n-2} (m_{(n-1)}) .
 \end{aligned} \tag{6}$$

By Definition 1.2 we have:

$$\begin{aligned}
 & \sum_m m_{(0)(0)} \otimes m_{(0)(1)} \otimes \cdots \otimes m_{(0)(n-1)} \otimes m_{(1)} \otimes \cdots \otimes m_{(n-1)} \\
 &= \sum_m m_{(0)} \otimes \cdots \otimes m_{(i-1)} \otimes m_{(i)(0)} \otimes \cdots \\
 & \cdots \otimes m_{(i)(n-1)} \otimes m_{(i+1)} \otimes \cdots \otimes m_{(n-1)} .
 \end{aligned}$$

It proves the equalities (5) and (6), i.e. M is a left n -ary C^* -module.

Since, the module C is a finite-generated projective k -module, then $(C^*)^* \simeq C$ and therefore the converse statement follows. \square

Theorem 2.2. *Let C be an n -ary bialgebra. If M is a right n -ary C -comodule algebra, then M is a left n -ary C^* -module algebra.*

Proof. Suppose that M is a right n -ary C -comodule algebra and C is an n -ary bialgebra. We shall show that M is left n -ary C^* -module algebra with respect to the action (4). It is necessary to prove the following equality:

$$\begin{aligned}
 & \rho^* [l_1 \otimes \cdots \otimes l_{n-1} \otimes (m_1 \cdots m_n)] \\
 &= \sum_{l_1, \dots, l_{n-1}} \rho^* (l_{(1)(1)} \otimes \cdots \otimes l_{(n-1)(1)} \otimes m_1) \times \\
 & \cdots \times \rho^* (l_{(1)(n)} \otimes l_{(n-1)(n)} \otimes m_n) .
 \end{aligned}$$

By (3) and (4), we have:

$$\begin{aligned}
 & \rho^* [l_1 \otimes \cdots \otimes l_{n-1} \otimes (m_1 \cdots m_n)] \\
 &= \sum (m_1 \cdots m_n)_{(0)} l_1 \left((m_1 \cdots m_n)_{(1)} \right) \cdots l_{n-1} \left((m_1 \cdots m_n)_{(n-1)} \right) .
 \end{aligned} \tag{7}$$

where $f_1, \dots, f_t \in [C^{\otimes(n-1)}]^* = (C^*)^{\otimes(n-1)}$, and so C is finitely generated. If the system of vectors $\{a_1, \dots, a_t\}$ can be enlarged by a vector a_{t+1} and then we can put $f_{t+1} = 0$.

Lemma 3.2. *Let $b_1, \dots, b_d \in \langle a_1, \dots, a_t \rangle$ be given and $g_1, \dots, g_d \in (C^*)^{\otimes(n-1)}$, such that for all $c_1, \dots, c_{n-1} \in C$:*

$$m(c_1 \otimes \cdots \otimes c_{n-1}) = \sum_{i=1}^d g_i(c_1 \otimes \cdots \otimes c_{n-1}) b_i.$$

Then $\sum_{i=1}^d g_i \otimes b_i = \sum_{j=1}^t f_j \otimes a_j$ holds in $(C^*)^{\otimes(n-1)} \otimes M$.

Proof. By the assumption $b_t = \sum_{j=1}^t \alpha_{ij} a_j$ where $\alpha_{ij} \in k$. Then

$$\begin{aligned} m(c_1 \otimes \cdots \otimes c_{n-1}) &= \sum_{i=1}^d g_i(c_1 \otimes \cdots \otimes c_{n-1}) \sum_{j=1}^t \alpha_{ij} a_j \\ &= \sum_{j=1}^t \left[\sum_{i=1}^d g_i(c_1 \otimes \cdots \otimes c_{n-1}) \alpha_{ij} \right] a_j. \end{aligned}$$

Since a_1, \dots, a_t are linearly independent, for all j by (8) we get:

$$f_j(c_1 \otimes \cdots \otimes c_{n-1}) = \sum_{i=1}^d g_i(c_1 \otimes \cdots \otimes c_{n-1}) \alpha_{ij}.$$

Thus, $f_j = \sum_{i=1}^d g_i \alpha_{ij}$ in $(C^*)^{\otimes(n-1)}$. Then

$$\sum_{j=1}^t f_j \otimes a_j = \sum_{j=1}^t \sum_{i=1}^d g_i \alpha_{ij} \otimes a_j = \sum_{i=1}^d \left(g_i \otimes \sum_{j=1}^t \alpha_{ij} a_j \right) = \sum_{i=1}^d g_i \otimes b_i.$$

□

Define the map $\rho : M \rightarrow (C^*)^{\otimes(n-1)} \otimes M$ by the following rule: if (8) holds, then we put

$$\rho(m) = f_1 \otimes a_1 + \cdots + f_t \otimes a_t. \quad (9)$$

By Lemma 3.2 this definition is correct. Let us show now that if M is a right n -ary C -module, then M is a left n -ary C^* -comodule with respect to (9). In fact,

$$\left(1_{C^*}^{\otimes(n-1)} \otimes \rho \right) \rho(m) = f_1 \otimes \rho(m_1) + \cdots + f_t \otimes \rho(m_t).$$

But $\rho(m_i) = \sum_j f_{ij} \otimes m_j$ where $f_{ij} \in (C^*)^{\otimes(n-1)}$. So

$$\left(1_{C^*}^{\otimes(n-1)} \otimes \rho\right) \rho(m) = \sum_{i,j} f_i \otimes f_{ij} \otimes m_j.$$

On the other hand,

$$\begin{aligned} & \left(1_{C^*}^{\otimes i} \otimes \Delta \otimes 1_{C^*}^{\otimes(n-i-2)} \otimes 1_M\right) \rho(m) \\ &= \sum_j \left(1_{C^*}^{\otimes i} \otimes \Delta \otimes 1_{C^*}^{\otimes(n-i-2)}\right) f_j \otimes m_j. \end{aligned}$$

Further, by associativity Definition 1.2, for all $c_1, \dots, c_{2n-2} \in C$ and all $m \in M$, for any i we have

$$\begin{aligned} & [m(c_1 \otimes \dots \otimes c_{n-1})](c_n \otimes \dots \otimes c_{2n-2}) \\ &= m[c_1 \otimes \dots \otimes c_i \otimes (c_{i+1} \otimes \dots \otimes c_{i+n}) \otimes c_{i+n+1} \otimes \dots \otimes c_{2n-2}] \end{aligned} \quad (10)$$

Suppose that m_1, \dots, m_t be as above. Then, as in (8)

$$\begin{aligned} & m_j(c_n \otimes \dots \otimes c_{2n-2}) \\ &= g_{j1}(c_n \otimes \dots \otimes c_{2n-2})m_1 + \dots + g_{jt}(c_n \otimes \dots \otimes c_{2n-2})m_t \end{aligned}$$

where $g_{j1}, \dots, g_{jt} \in (C^*)^{\otimes(n-1)}$. So, by (8) and (9), we have

$$\left(1_{C^*}^{\otimes(n-1)} \otimes \rho\right) \rho(m) = \sum_{i,j} f_i \otimes g_{ij} \otimes m_j.$$

By (10) for all j :

$$\begin{aligned} & \sum_i f_i(c_1 \otimes \dots \otimes c_{n-1})g_{ij}(c_n \otimes \dots \otimes c_{2n-2}) \\ &= f_j[c_1 \otimes \dots \otimes c_i \otimes (c_{i+1} \otimes \dots \otimes c_{i+n}) \otimes c_{i+n+1} \otimes \dots \otimes c_{2n-2}]. \end{aligned}$$

In other words we have that in $(C^*)^{\otimes(2n-2)}$ that

$$\sum_i f_i \otimes g_{ij} = \left(1_{C^*}^{\otimes i} \otimes \Delta_{C^*} \otimes 1_{C^*}^{\otimes(n-i-2)}\right) f_j.$$

Tensor-multiplying by m_j and summing on j , we obtain

$$\sum_{i,j} f_i \otimes g_{ij} \otimes m_j = \left(1_{C^*}^{\otimes i} \otimes \Delta_{C^*} \otimes 1_{C^*}^{\otimes(n-i-2)}\right) \sum_j f_j \otimes m_j.$$

But, the left side is equal to $\left(1_{C^*}^{\otimes(n-1)} \otimes \rho\right) \rho(m)$ and right side is equal to $\left(1_{C^*}^{\otimes i} \otimes \Delta_{C^*} \otimes 1_{C^*}^{\otimes(n-i-2)}\right) \rho(m)$. Consequently, M is a left n -ary C^* -comodule. \square

Theorem 3.3. *Let C be an n -ary bialgebra, now. If M is a right n -ary C -module algebra, then M is a left n -ary C^* -comodule algebra.*

Proof. Assume that M is a right n -ary C -module algebra. It means that for all $c_1, \dots, c_{n-1} \in C$ and all $m_1, \dots, m_n \in M$ we have:

$$\begin{aligned} & (m_1 \cdots m_n) (c_1 \otimes \cdots \otimes c_{n-1}) \\ &= \sum_{c_1, \dots, c_{n-1}} m_1 (c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}) \cdots m_n (c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}) \quad (11) \end{aligned}$$

Let us show now that M is a left n -ary C^* -comodule algebra, i.e. the map (9) is a homomorphism of n -ary algebras. Suppose that $m_1, \dots, m_r \in M$ and a_1, \dots, a_t is a base in $\sum_{j=1}^r m_j C^{\otimes(n-1)}$. Assume that $\rho(m_i) = \sum_{j=1}^t f_{ij} \otimes a_j$ where $f_{ij} \in (C^*)^{\otimes(n-1)}$. But,

$$\begin{aligned} & \sum_{c_1, \dots, c_{n-1}} m_1 (c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}) \cdots m_n (c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}) \\ &= \sum_{c_1, \dots, c_{n-1}} \left[\sum_{j_1} f_{1,j_1} (c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}) a_{j_1} \right] \text{ times} \\ & \cdots \times \left[\sum_{j_n} f_{1,j_n} (c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}) a_{j_n} \right] \\ & \sum_{j_1, \dots, j_n} a_{j_1} \cdots a_{j_n} \left[\sum_{j_1} f_{1,j_1} (c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}) \times \right. \\ & \left. \cdots \times f_{n,j_n} (c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}) \right] \end{aligned}$$

For all j_1, \dots, j_n we have:

$$\begin{aligned} & \sum_{c_1, \dots, c_{n-1}} f_{1,j_1} (c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}) \cdots f_{n,j_n} (c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}) \\ &= (f_{1,j_1} * \cdots * f_{n,j_n}) (c_1 \otimes \cdots \otimes c_{n-1}). \end{aligned}$$

In that way, the right side of (11) is equal to:

$$\sum_{j_1, \dots, j_n} (f_{1,j_1} * \cdots * f_{n,j_n}) (c_1 \otimes \cdots \otimes c_{n-1}) a_{j_1} \cdots a_{j_n}.$$

By Lemma 3.2 $b_* = a_{j_1} \cdots a_{j_n}$ for all j_1, \dots, j_n and by (11), we obtain that

$$\begin{aligned} \rho(m_1 \cdots m_n) &= \sum_{j_1, \dots, j_n} (f_{1,j_1} * \cdots * f_{n,j_n}) \otimes a_{j_1} \cdots a_{j_n} \\ &= \prod_{s=1}^n \left[\sum_{j_s} (f_{s,j_s} \otimes a_{j_s}) \right] = \rho(m_1) \cdots \rho(m_n). \end{aligned}$$

□

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