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Lie algebras associated with quadratic forms and their applications to Ringel-Hall algebras

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Abstract. We define and investigate Lie algebras associated with quadratic forms. We also present their connections with Lie algebras and Ringel-Hall algebras associated with representation directed algebras.

1. Introduction

Let q be a unit integral quadratic form

$$q(x) = q(x(1), \dots, x(n)) = \sum_{i=1}^{n} x(i)^2 + \sum_{i,j} a_{ij} x(i)x(j),$$

where $a_{ij} \in \{-1, 0, 1\}$. In [4], with q complex Lie algebras $G(q)$, $\widetilde{G}(q)$ are associated, where $\widetilde{G}(q)$ is the extension of $G(q)$ by the $\mathbb{C}$-dual of the radical of q and $\mathbb{C}$ is the complex number field.

The following facts were proved in [4].

- If q is positive definite and connected, then $G(q) = \widetilde{G}(q)$ is a finite dimensional simple Lie algebra.
- If q is connected and non-negative of corank one or two, then $\widetilde{G}(q)$ is isomorphic to an affine Kac-Moody algebra (if the corank of q equals one) or to elliptic (if the corank of q equals two and q is not of Dynkin type $\mathbb{A}_n$).

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In [4], the Lie algebra $G(q)$ was defined by generators and relations. Unfortunately, the set of relations defining $G(q)$ is infinite. In [5], the authors give a finite and small set of relations sufficient to define $G(q)$ for positive definite forms q.

In this paper, for any integral quadratic form q (2.1), we define by generators and relations, a Lie algebra $L(q, \mathfrak{r})$. For a positive definite form q, we describe a minimal set of relations defining $L(q, \mathfrak{r})$. Moreover we show that, for any representation directed $\mathbb{C}$-algebra A with Tits form q_A, there are isomorphisms of Lie algebras

$$L(q_A, \mathfrak{r}) \cong \mathcal{L}(A) \cong \mathcal{K}(A),$$

where $\mathcal{L}(A)$ is the Lie algebra associated with A in [11] by Ch. Riedtmann and $\mathcal{K}(A)$ is the Lie algebra associated with A in [13] by C. M. Ringel. The isomorphism $\mathcal{L}(A) \cong \mathcal{K}(A)$ is proved in [7]. Results of the present paper allow us to define Lie algebras $\mathcal{L}(A)$ and $\mathcal{K}(A)$ in a combinatorial way. Similar results are presented in [9] and [10] for Tits forms of posets of finite prinjective type.

The paper is organised as follows.

In Section 2 we give basic definitions and facts concerning weakly positive and positive definite quadratic forms and their roots.

In Sections 3, 4 we give a definition and prove basic properties of the Lie algebra $L(q, \mathfrak{r})$. Moreover (for some class of quadratic form q) we show that the Lie algebra $L(q, \mathfrak{r})$ is a Lie subalgebra of $G(q)$.

In Section 5 we prove the existence of isomorphisms of Lie algebras

$$L(q_A, \mathfrak{r}) \cong \mathcal{L}(A) \cong \mathcal{K}(A),$$

for any representation directed $\mathbb{C}$-algebra A. Moreover we present applications of these results to Ringel-Hall algebras of representation directed algebras.

In Section 6 we give a minimal set of relations sufficient to define the Lie algebra $L(q, \mathfrak{r})$, where q is a positive definite quadratic form. More precisely we prove the following theorem.

Theorem 1.1. *Let q be a positive definite quadratic form* (2.1)*. There is an isomorphism of Lie algebras*

$$L(q, \mathfrak{r}) \cong L(q)/(\mathfrak{j}),$$

where $L(q)$ is the free complex Lie algebra generated by the set $\{v_1, \ldots, v_n\}$ and $(\mathfrak{j})$ is the ideal of $L(q)$ generated by the set $\mathfrak{j}$, which is consisted of the following elements

- $[v_i, v_j]$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $a_{ij} \neq -1$,
- $[v_i, [v_i, v_j]]$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $a_{ij} = -1$,
- $[v_j, [v_i, v_j]]$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $a_{ij} = -1$,
- $[v_{i_1}, \ldots, v_{i_m}]$ *for all positive chordless cycles* $(i_1, \ldots, i_m)$ *(see Section* 6 *for definitions).*

Finally, in Section 7, we present some examples and remarks.

2. Preliminaries on weakly positive and positive definite quadratic forms

Let $e_1, \ldots, e_n$ be the standard basis of the free abelian group $\mathbb{Z}^n$. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected unit integral quadratic form defined by

$$q(x) = q(x(1), \ldots, x(n)) = \sum_{i=1}^{n} x(i)^2 + \sum_{i,j} a_{ij} x(i) x(j), \tag{2.1}$$

where $a_{ij} \in \{-1, 0, 1\}$. Let $B(q)$ be the bigraph associated with q (i.e. the set of vertices of $B(q)$ is $\{1, \ldots, n\}$; for $i \neq j$ there exists a solid edge i —— j if and only if $a_{ij} = -1$ and a broken edge $i - - - j$ if and only if $a_{ij} = 1$). An integral quadratic form q is said to be **weakly positive**, if $q(x) > 0$ for any $0 \neq x \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of non-negative integers.

A vector $x \in \mathbb{Z}^n$ is called **a root** of q, if $q(x) = 1$; if in addition $x(i) \geq 0$, for any $i = 1, \ldots, n$, then we call x **a positive root**. Denote by

$$\mathcal{R}_q = \{x \in \mathbb{Z}^n \; ; \; q(x) = 1\}, \;\; \mathcal{R}_q^+ = \{x \in \mathbb{N}^n \; ; \; q(x) = 1\} \tag{2.2}$$

the set of all roots and all positive roots of q, respectively.

We associate with q the symmetric $\mathbb{Z}$-bilinear form

$$\langle -, - \rangle_q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}, \tag{2.3}$$

where $\langle x, y \rangle_q = q(x+y) - q(x) - q(y)$, for all $x, y \in \mathbb{Z}$. It is straightforward to check that

$$\langle e_i, x \rangle_q = 2 \cdot x(i) + \sum_{i \neq j} a_{ij} x(j), \tag{2.4}$$

for any $i = 1, \ldots, n$. Let us recall the following useful facts concerning the $\mathbb{Z}$-linear form $\langle e_i, - \rangle_q$ (see [12]).

Lemma 2.5. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form* (2.1) *and let $i \in \{1, \dots, n\}$.*

(a) $\langle e_i, e_i \rangle_q = 2$.

(b) *Let x be a root of q and $0 \neq d \in \mathbb{Z}$. The vector $x - de_i$ is a root of q if and only if $d = \langle e_i, x \rangle_q$.*

(b′) *Let x, y be roots of q. The vector $x + y$ is a root of q if and only if $\langle x, y \rangle_q = -1$.*

Assume that q is positive definite.

(c) *If x is a root of q such that $x \neq e_i$, then*

$$-1 \leq \langle e_i, x \rangle_q \leq 1.$$

(d) *The set $\mathcal{R}_q$ of all roots of q is finite.*

Assume that q is weakly positive.

(e) *If x is a root of q, then*

$$-1 \leq \langle e_i, x \rangle_q.$$

(f) *The set $\mathcal{R}_q^+$ of all positive roots of q is finite.* □

Lemma 2.6. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive quadratic form and let $z \neq 0$ be a positive root of q. Then z is a* **Weyl root***, i.e. there exists a chain*

$$x^{(1)}, \dots, x^{(m)}$$

of positive roots of q such that

(a) $x^{(1)} = z$, $x^{(i)} = x^{(i-1)} - e_{j_i}$ *for $i = 1, \dots, m$ and for some $j_i \in \{1, \dots, n\}$,*

(b) $x^{(m)} = e_j$, *for some $j \in \{1, \dots, n\}$.* □

3. Lie algebras associated with quadratic forms

In a complex Lie algebra L, we use the following multibracket notation

$$[y_1, y_2, \dots, y_n] = [y_1, [y_2, [\dots [y_{n-1}, y_n]]]], \tag{3.1}$$

for all $y_1, \dots, y_n \in L$. Let L be a complex Lie algebra generated by a set $\{v_1, \dots, v_n\}$. Elements of the form $[v_{i_1}, \dots, v_{i_m}]$ we call **standard multibrackets**. All Lie algebras considered in this paper are assumed to be complex finitely generated Lie algebras and all quadratic forms are assumed to be integral quadratic forms (2.1).

Let S_n be the symmetric group of n-elements.

Lemma 3.2. (a) *Let L be a Lie algebra and let $y_1, \ldots, y_n, x \in L$. There exists a subset $\mathcal{S} \subseteq S_n$ and for any $\sigma \in \mathcal{S}$ there exists $\varepsilon_\sigma \in \{0,1\}$, such that*

$$[[y_1, \ldots, y_n], x] = \sum_{\sigma \in \mathcal{S}} (-1)^{\varepsilon_\sigma} [y_{\sigma(1)}, \ldots, y_{\sigma(n)}, x].$$

(b) *Let L be a Lie algebra generated by a set $\{y_1, \ldots, y_n\}$. Any element $y \in L$ is a linear combination of standard multibrackets $[y_{i_1}, \ldots, y_{i_m}]$, where $i_j \in \{1, \ldots, n\}$.*

Proof. (a) Apply recursively the Jacobi identity, see also [1, Lemma 1.1].
The statement (b) follows from (a). □

With a quadratic form q (2.1), we associate the complex free Lie algebra

$$L(q) = \mathrm{Lie}_{\mathbb{C}}\langle v_1, \ldots, v_n\rangle \tag{3.3}$$

generated by the set $\{v_1, \ldots, v_n\}$. Note that the Lie algebra $L(q)$ has a $\mathbb{N}^n$-gradation, if we define the degree of v_i to be e_i, for any $i = 1, \ldots, n$.

Let $\mathfrak{a} \subseteq L(q)$ be a subset consisting of some standard multibrackets and let $(\mathfrak{a})$ be the homogeneous ideal of the Lie algebra $L(q)$ generated by the set $\mathfrak{a}$. Let

$$L(q, \mathfrak{a}) = L(q)/(\mathfrak{a}) \tag{3.4}$$

be the quotient Lie algebra with induced $\mathbb{N}^n$-grading. Denote by $\pi : L(q) \to L(q, \mathfrak{a})$ the natural epimorphisms. Let $v = [v_{i_1}, \ldots, v_{i_m}] \in L(q)$. For the sake of simplicity, we will denote by $v = [v_{i_1}, \ldots, v_{i_m}]$ the element $\pi(v)$.

- We call an element $v = [v_{i_1}, \ldots, v_{i_m}] \in L(q)$ a **root**, if $m \geq 1$ and $\langle e_{i_k}, e_{i_{k+1}} + \ldots + e_{i_m}\rangle_v = -1$, for all $k = 1, \ldots, m-1$.
- Let $v = [v_{i_1}, \ldots, v_{i_m}] \in L(q)$, we set $\ell(v) = m$ and we call this number the **length** of the element v.
- We set $e_v = e_{i_1} + \ldots + e_{i_m} \in \mathbb{N}^n$.

For $e \in \mathbb{N}^n$, denote by $L(q, \mathfrak{a})_e$ the homogeneous space spanned by all standard multibrackets $v = [v_{i_1}, \ldots, v_{i_m}] \in L(q, \mathfrak{a})$, of degree $e_v = e$. Moreover, for any integer m, let

$$L(q, \mathfrak{a})_m = \bigoplus_{\substack{e \in \mathbb{N}^n \\ e(1)+\ldots+e(n) \leq m}} L(q, \mathfrak{a})_e \quad \text{and} \quad (\mathfrak{a})_m = (\mathfrak{a}) \cap L(q)_m. \tag{3.5}$$

The following lemma shows connections between roots in the complex Lie algebra $L(q)$ and positive roots of the quadratic form q.

Lemma 3.6. *Let $u, w, v = [v_{i_1}, \dots, v_{i_m}] \in L(q)$ be roots.*

(a) *At least one of the elements $[v_{i_1}, v_{i_2}]$, $[v_{i_2}, v_{i_1}, v_{i_3}, \dots, v_{i_m}]$ is not a root.*

(b) *If $\langle e_u, e_v + e_w\rangle_q = -1$, then $\langle e_u, e_v\rangle_q \geq 0$ or $\langle e_u, e_w\rangle_q \geq 0$.*

(c) *The vector $e_v = e_{i_1} + \ldots + e_{i_m}$ is a positive root of the quadratic form q.*

Let $v = [v_{i_1}, \dots, v_{i_m}] \in L(q)$ and let q be weakly positive.

(d) *If $e_v = e_{i_1} + \ldots + e_{i_m}$ is a positive root of q, then there exists a permutation $\sigma \in S_m$ such that the element $v_\sigma = [v_{i_{\sigma(1)}}, \dots, v_{i_{\sigma(m)}}]$ is a root in $L(q)$.*

Proof. (a) Let $v = [v_{i_1}, \dots, v_{i_m}] \in L(q)$ be a root and assume that $[v_{i_1}, v_{i_2}]$ is a root. It follows that $\langle e_{i_1}, e_{i_2}\rangle_q = -1$ and $\langle e_{i_1}, e_{i_3} + \ldots + e_{i_m}\rangle_q = \langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q - \langle e_{i_1}, e_{i_2}\rangle_q = -1 - (-1) = 0$. Therefore $[v_{i_2}, v_{i_1}, v_{i_3}, \dots, v_{i_m}]$ is not a root.

The statement (b) is obvious.

(c) Obviously e_{i_m} is a root of q. Let $2 \leq k \leq m$ and assume that $e_{i_k} + \ldots + e_{i_m}$ is a root of q. We have $\langle e_{i_{k-1}}, e_{i_k} + \ldots + e_{i_m}\rangle_q = -1$, because v is a root in $L(q)$. By Lemma 2.5(b), the vector $e_{i_{k-1}} + e_{i_k} + \ldots + e_{i_m}$ is a root of q. Inductively we finish the proof of the statement (c).

(d) Let v be a positive root of q. From Lemma 2.5(b) it follows that $v + e_i$ is a positive root of q if and only if $\langle e_i, v\rangle_q = -1$. Therefore the statement (d) follows easily from Lemma 2.6 and Lemma 2.5(b), because q is weakly positive. □

Definition 3.7. (a) *Let $\mathfrak{r}$ be the set of all standard multibrackets*

$$[v_{i_1}, \dots, v_{i_m}] \in L(q),$$

such that $[v_{i_1}, \dots, v_{i_m}]$ is not a root and $[v_{i_2}, \dots, v_{i_m}]$ is a root, where $i_1, \dots, i_m \in \{1, \dots, n\}$ and $m \in \mathbb{N}$.

(b) *Let*

$$L(q, \mathfrak{r}) = Lie_{\mathbb{C}}\langle v_1, \dots, v_n\rangle/(\mathfrak{r})$$

be a complex Lie algebra generated by the set $\{v_1, \dots, v_n\}$ modulo the ideal $(\mathfrak{r})$ generated by the set $\mathfrak{r}$. We consider $L(q, \mathfrak{r})$ as a Lie algebra with a $\mathbb{N}^n$-gradation, where we define the degree of v_i to be e_i, for any $i = 1, \dots, n$.

Lemma 3.8. *Let $\mathfrak{a} \subseteq L(q)$ be a subset consisting of some standard multibrackets. Let $v = [v_{i_1}, \dots, v_{i_m}]$, $y \in L(q, \mathfrak{a})$. Assume that for any $z \in L(q, \mathfrak{a})$, such that $\ell(z) \leq \ell(v) + \ell(y)$, the following condition is satisfied:*

$$\textit{if } \langle e_i, e_z\rangle_q \neq -1, \textit{ then } [v_i, z] = 0 \textit{ in } L(q, \mathfrak{a}). \tag{3.9}$$

Then $[v, y] = 0$ *in* $L(q, \mathfrak{a})$ *or there exists* $\sigma \in S_m$ *and* $\varepsilon \in \{0, 1\}$ *such that*

$$[v, y] = (-1)^{\varepsilon}[v_{i_{\sigma(1)}}, \ldots, v_{i_{\sigma(m)}}, y]$$

in $L(q, \mathfrak{a})$.

Proof. Let $v = [v_{i_1}, \ldots, v_{i_m}], y \in L(q, \mathfrak{a})$. We precede with induction on $m = \ell(v)$. For $m = 1$, the lemma is obvious.

Let $m > 1$. We apply the Jacobi identity and get

$$[v, y] = -[y, v] = [v_{i_1}, [[v_{i_2}, \ldots, v_{i_m}], y]] - [[v_{i_2}, \ldots, v_{i_m}], [v_{i_1}, y]].$$

Note that

$$\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m} + e_y\rangle_q = \langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q + \langle e_{i_1}, e_y\rangle_q.$$

Therefore $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q \neq -1$ or $\langle e_{i_1}, e_y\rangle_q \neq -1$ or $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m} + e_y\rangle_q \neq -1$.

If $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q \neq -1$, then, by (3.9), $v = [v_{i_1}, \ldots, v_{i_m}] = 0$ and $[v, y] = 0$. We are done. If $\langle e_{i_1}, e_y\rangle_q \neq -1$, then, by (3.9), we have $[v_{i_1}, y] = 0$ and $[v, y] = [v_{i_1}, [[v_{i_2}, \ldots, v_{i_m}], y]]$. We finish by induction on m applied to $[[v_{i_2}, \ldots, v_{i_m}], y]]$. In the case $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m} + e_y\rangle_q \neq -1$, we have $[v_{i_1}, [[v_{i_2}, \ldots, v_{i_m}], y]] = 0$. Finally

$$[v, y] = (-1)^{\varepsilon}[[v_{i_2}, \ldots, v_{i_m}], [v_{i_1}, y]]$$

and we finish by induction on m. □

4. Grading of $L(q, \mathfrak{r})$ and positive roots of q

Lemma 4.1. *Let q be a weakly positive quadratic form (2.1) and let $\mathfrak{a} \subseteq L(q)$ be a subset consisting of some standard multibrackets. Let m be a positive integer. Assume that the following conditions are satisfied.*

(a) *If $v = [v_{i_1}, \ldots v_{i_s}]$ is not a root in $L(q)$ and $\ell(v) = s \leq m$, then*

$$L(q, \mathfrak{a})_{e_v} = 0.$$

(b) *If $v = [v_{i_1}, \ldots v_{i_s}]$ is a root in $L(q)$ and $\ell(v) = s < m$, then*

$$\dim_{\mathbb{C}} L(q, \mathfrak{a})_{e_v} \leq 1.$$

Then $\dim_{\mathbb{C}} L(q, \mathfrak{a})_{e_v} \leq 1$, *if $v = [v_{i_1}, \ldots v_{i_s}]$ is a root in $L(q)$ and $\ell(v) = s = m$.*

Proof. Let $v = [v_{i_1}, \dots, v_{i_m}]$ be a root of $L(q)$. If $v = 0$ in $L(q, \mathfrak{a})$, then we are done. Assume that $0 \neq v \in L(q, \mathfrak{a})$. We have to prove that $\dim_{\mathbb{C}} L(q, \mathfrak{a})_{e_v} \leq 1$. If $m = 1$, the lemma is obvious. Let $m > 1$, and let $0 \neq w \in L(q, \mathfrak{a})_{e_v}$ be a standard multibracket. It follows that there exists a permutation $\sigma \in S_m$ such that $w = v_\sigma = [v_{i_{\sigma(1)}}, \dots, v_{i_{\sigma(m)}}] \neq 0$. Since $v_\sigma \neq 0$ in $L(q, \mathfrak{a})$, $\ell(v_\sigma) = m$, then the assumption (a) of our lemma yields that v_σ is a root. It is enough to prove that there exists $a \in \mathbb{C}$ such that $v_\sigma = av$. Since $v, v_\sigma \in L(q, \mathfrak{a})_{e_v}$, there exists $k = 1, \dots, m$ such that $i_k = i_{\sigma(1)}$. Note that we may assume, without loss of the generality, that $k < m$, because $[v_{i_{m-1}}, v_{i_m}] = -[v_{i_m}, v_{i_{m-1}}]$ and we may replace v by $-v$.

If $k = 1$, then $\overline{v} = [v_{i_{\sigma(2)}}, \dots, v_{i_{\sigma(m)}}]$, $[v_{i_2}, \dots, v_{i_m}] \in L(q, \mathfrak{a})_{e_{\overline{v}}}$, $\ell(\overline{v}) < m$, and the condition (b) yields that $\dim_{\mathbb{C}} L(q, \mathfrak{a})_{e_{\overline{v}}} \leq 1$. Then there exists $a \in \mathbb{C}$ such that $[v_{i_{\sigma(2)}}, \dots, v_{i_{\sigma(m)}}] = a[v_{i_2}, \dots, v_{i_m}]$. Therefore $v_\sigma = av$ and we are done.

Let $k > 1$. Consider the following set

$$\mathcal{Y} = \{v_\tau \,;\ \text{ for all } \tau \in S_m \text{ such that there exists } c \in \mathbb{C} \text{ such that } v_\tau = cv\}.$$

Note that for all $v_\tau \in \mathcal{Y}$ there exists l such that $i_{\tau(l)} = i_{\sigma(1)}$. We choose an element $v_\tau \in \mathcal{Y}$ such that l is minimal with this property. Without loss of the generality, we may assume that $\tau = \mathrm{id}$, $v_\tau = v$ and $k = l$. Let

$$v = [v_{i_1}, [v_{i_2}, [\dots, v_{i_{k-1}}, [v_{i_k}, y] \dots]]] = [v_{i_1}, v_{i_2}, \dots, v_{i_{k-1}}, v_{i_k}, y],$$

where we set $y = [v_{i_{k+1}}, \dots, v_{i_m}]$. By the choice of k it follows that $i_j \neq i_{\sigma(1)}$ for all $j < k$. Our assumptions yield: $\langle e_{i_k}, e_y \rangle_q = -1$, because v is a root and

$$\langle e_{i_k}, e_y + e_{i_1} + \dots + e_{i_{k-1}} \rangle_q = \langle e_{i_{\sigma(1)}}, e_{i_{\sigma(2)}} + \dots + e_{i_{\sigma(m)}} \rangle_q = -1,$$

because v_σ is a root. By the bilinearity of $\langle -, - \rangle_q$, it follows that $\langle e_{i_k}, e_{i_1} + \dots + e_{i_{k-1}} \rangle_q = 0$. We prove that

$$v = [[v_{i_1}, \dots, v_{i_k}], y]$$

in $L(q, \mathfrak{a})$.

Applying the Jacobi identity, we get

$$\begin{aligned}&[v_{i_1}, \dots, v_{i_{k-1}}, [v_{i_k}, y]] = \\ &\quad = -[v_{i_1}, \dots, v_{i_{k-2}}, [v_{i_k}, [y, v_{i_{k-1}}]]] - [v_{i_1}, \dots, v_{i_{k-2}}, [y, [v_{i_{k-1}}, v_{i_k}]]].\end{aligned}$$

Note that

$$\langle e_{i_{k-1}}, e_y \rangle_q = \langle e_{i_{k-1}}, e_y + e_{i_k} \rangle_q - \langle e_{i_{k-1}}, e_{i_k} \rangle_q = -1 - \langle e_{i_{k-1}}, e_{i_k} \rangle_q.$$

If $\langle e_{i_{k-1}}, e_y\rangle_q = -1$, then $\langle e_{i_{k-1}}, e_{i_k}\rangle_q = 0$. Therefore the condition (a) yields $[v_{i_{k-1}}, v_{i_k}] = 0$ and

$$v = [v_{i_1}, \ldots, v_{i_{k-2}}, [v_{i_k}, [v_{i_{k-1}}, y]]]$$

which is the contradiction with the choice of k. Therefore

$$v = [v_{i_1}, \ldots, v_{i_{k-2}}, [[v_{i_{k-1}}, v_{i_k}]\, y]].$$

Inductively, applying the Jacobi identity to

$$[v_{i_1}, \ldots, v_{i_s}, [[v_{i_{s+1}}, \ldots, v_{i_k}], y]]$$

we get

$$\begin{aligned}[v_{i_1}, \ldots, v_{i_s}, [[v_{i_{s+1}}, \ldots, v_{i_k}], y]] &= \\ &= -[v_{i_1}, \ldots, v_{i_{s-1}}, [[v_{i_{s+1}}, \ldots, v_{i_k}], [y, v_{i_s}]]] - \\ &\quad - [v_{i_1}, \ldots, v_{i_{s-1}}, [y, [v_{i_s}, [v_{i_{s+1}}, \ldots, v_{i_k}]]]].\end{aligned}$$

Consider

$$\begin{aligned}\langle e_{i_s}, e_y\rangle_q &= \langle e_{i_s}, e_y + e_{i_{s+1}} + \ldots + e_{i_k}\rangle_q - \langle e_{i_s}, e_{i_{s+1}} + \ldots + e_{i_k}\rangle_q = \\ &= -1 - \langle e_{i_s}, e_{i_{s+1}} + \ldots + e_{i_k}\rangle_q.\end{aligned}$$

If $\langle e_{i_s}, e_y\rangle_q = -1$, then $\langle e_{i_s}, e_{i_{s+1}} + \ldots + e_{i_k}\rangle_q = 0$. The condition (a) yields $[v_{i_s}, [v_{i_{s+1}}, \ldots, v_{i_k}]] = 0$ and

$$v = [v_{i_1}, \ldots, v_{i_{s-1}}, [[v_{i_{s+1}}, \ldots, v_{i_k}], [v_{i_s}, y]]].$$

Applying Lemma 3.8, we get the contradiction with the choice of k. Therefore

$$v = [v_{i_1}, \ldots, v_{i_{s-1}}, [[v_{i_s}, [v_{i_{s+1}}, \ldots, v_{i_k}]], y]].$$

Inductively we get $v = [[v_{i_1}, \ldots, v_{i_k}], y]$.

Since $v \neq 0$, $k < m$, then (by the assumption (a) of the lemma) the element $[v_{i_1}, \ldots, v_{i_k}]$ is a root. Therefore, by Lemma 3.6 (c), we have $q(e_{i_1} + \ldots + e_{i_k}) = 1$. Now consider

$$1 = q(e_{i_1} + \ldots + e_{i_k}) = q(e_{i_1} + \ldots + e_{i_{k-1}}) + q(e_{i_k}) + \langle e_{i_k}, e_{i_1} + \ldots + e_{i_{k-1}}\rangle_q.$$

Since we proved above that $\langle e_{i_k}, e_{i_1} + \ldots + e_{i_{k-1}}\rangle_q = 0$, we have

$$q(e_{i_1} + \ldots + e_{i_{k-1}}) = 1 - q(e_{i_k}) = 1 - 1 = 0.$$

We get a contradiction, because q is weakly positive. This shows that $k = 1$ and $v_\sigma \in \mathcal{Y}$. This finishes the proof of lemma. □

Let $L(q, \mathfrak{r})$ be the Lie algebra introduced in Definition 3.7

Proposition 4.2. *Let q be a weakly positive quadratic form* (2.1). *The following conditions hold.*

(a) *If $e = e_{i_1} + \ldots + e_{i_m}$ is not a root of q, then $L(q, \mathfrak{r})_e = 0$.*

(b) *If $e = e_{i_1} + \ldots + e_{i_m}$ is a root of q, then $\dim_{\mathbb{C}} L(q, \mathfrak{r})_e \leq 1$.*

Proof. (a) Assume that $e = e_{i_1} + \ldots + e_{i_m}$ is not a root of q. By Lemma 3.6, $v = [v_{i_1}, \ldots, v_{i_m}]$ is not a root in $L(q, \mathfrak{r})$. Let k be maximal with the property that $[v_{i_k}, \ldots, v_{i_m}]$ is a root. Since $v = [v_{i_1}, \ldots, v_{i_m}]$ is not a root, then $k > 1$. Therefore $[v_{i_{k-1}}, v_{i_k}, \ldots, v_{i_m}]$ is not a root and $[v_{i_{k-1}}, v_{i_k}, \ldots, v_{i_n}] \in \mathfrak{r}$. Finally $[v_{i_{k-1}}, v_{i_k}, \ldots, v_{i_m}] = 0$ and $v = 0$ in $L(q, \mathfrak{r})$.

The statement (b) follows easily by induction on the length $\ell(v)$ of v, if we apply Lemma 4.1. □

As a consequence we get the following corollary.

Corollary 4.3. *Let q be a weakly positive quadratic form and let $\mathcal{R}_q^+$ be the set of all positive roots of q. Then $L(q, \mathfrak{r})$ is a nilpotent Lie algebra and*

$$L(q, \mathfrak{r}) = \bigoplus_{e \in \mathcal{R}_q^+} L(q, \mathfrak{r})_e \quad \textit{and} \quad \dim_{\mathbb{C}} L(q, \mathfrak{r}) \leq |\mathcal{R}_q^+|.$$

□

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ quadratic form (2.1). With q we associate Cartan matrix $C = (c_{ij}) \in \mathbb{M}_n(\mathbb{Z})$ defined by $c_{ij} = q(e_i + e_j) - q(e_i) - q(e_j)$. Following [4], to q we attach a $\mathbb{Z}^n$-graded complex Lie algebra $G(q)$ with generators $x_i, x_{-i}, h_i,\ i = 1, \ldots, n$, which are homogeneous of degree $e_i, -e_i, 0$, respectively, and subject to the following relations:

1. $[h_i, h_j] = 0$, for all $i, j = 1, \ldots, n$,
2. $[h_i, x_{\varepsilon j}] = \varepsilon c_{ij} x_{\varepsilon j}$, for all $i, j = 1, \ldots, n$ and $\varepsilon \in \{-1, 1\}$,
3. $[x_{\varepsilon i}, x_{-\varepsilon i}] = \varepsilon h_i$, for all $i = 1, \ldots, n$ and $\varepsilon \in \{-1, 1\}$,
4. $[x_{\varepsilon_1 i_1}, \ldots, x_{\varepsilon_n i_n}] = 0$, if $q(\varepsilon_1 e_{i_1} + \ldots + \varepsilon_n e_{i_n}) > 1$ for $\varepsilon_j \in \{-1, 1\}$.

Denote by $G^+(q)$ a Lie subalgebra of $G(q)$ generated by the elements $x_1, \ldots, x_n$.

Proposition 4.4. *If q is weakly positive and positive semi-definite, then*

$$L(q, \mathfrak{r}) \simeq G^+(q).$$

Proof. By [4, Proposition 2.2] and Corollary 4.3, we have

$$\dim_{\mathbb{C}} G^+(q) \geq |\mathcal{R}_q^+| \geq \dim_{\mathbb{C}} L(q, \mathfrak{r}).$$

On the other hand, it is easy to see that all relations $\mathfrak{r}$ are satisfied in $G^+(q)$. Therefore we may define a homomorphism of Lie algebras

$$\Psi : L(q, \mathfrak{r}) \to G^+(q)$$

by $\Psi(u_i) = x_i$ for all $i = 1, \ldots, n$. Since $G^+(q)$ is generated by the elements $x_1, \ldots, x_n$, the homomorphism Ψ is surjective. Therefore Ψ is an isomorphism, because $\dim_{\mathbb{C}} G^+(q) \geq \dim_{\mathbb{C}} L(q, \mathfrak{r})$. □

5. Connections with Ringel-Hall algebras

We present applications of Lie algebras $L(q, \mathfrak{r})$ to Lie algebras and Ringel-Hall algebras associated with representation directed algebras. We get a description of these Ringel-Hall algebras by generators and relations. For the basic concepts of representation theory the reader is referred to [2], [3] and for the basic concepts of Ringel-Hall algebras to [13], [14].

Let $Q = (Q_0, Q_1)$ be a finite quiver without oriented cycles. Let $\mathbb{C}Q$ be the complex path algebra of Q. Assume that I is an admissible ideal of $\mathbb{C}Q$ such that $A = \mathbb{C}Q/I$ is a representation directed algebra. By mod (A) we denote the category of all right finite dimensional A-modules and by ind(A) we denote the set of all representatives of isomorphism classes of indecomposable A-modules. For any A-module M denote by $\mathbf{dim}\, M \in \mathbb{N}^{Q_0}$ the dimension vector of M (i.e. $(\mathbf{dim}\, M)(i)$ equals the number of composition factors of M which are isomorphic to the simple A-module S_i corresponding to the vertex $i \in Q_0$). Let $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be the Tits form of A (see [6]). By [6, Theorem 3.3], q_A is weakly positive and there is a bijection (given by **dim**) between the set ind(A) and the set $\mathcal{R}_{q_A}^+$. Let $\mathcal{K}(A)$ be the corresponding complex Lie algebra defined in [13]. Recall that, for a representation directed algebra A, the $\mathbb{C}$-Lie algebra $\mathcal{K}(A)$ is the free $\mathbb{C}$-linear space with basis $\{u_X \ ; \ X \in \text{ind}(A)\}$. If X, Y are non-isomorphic indecomposable A-modules such that $\text{Ext}_A^1(X, Y) = 0$, then the Lie bracket in $\mathcal{K}(A)$ is defined by

$$[u_Y, u_X] = \begin{cases} \varphi_{YX}^Z(1) \cdot u_Z & \text{if there is an indecomposable } A-\text{module } Z \\ & \text{and a short exact sequence} \\ & 0 \to X \to Z \to Y \to 0, \\ 0 & \text{otherwise,} \end{cases},$$

where φ^Z_{YX} are Hall polynomials (see [13]). In [7] it is proved that the Lie algebra $\mathcal{K}(A)$ is isomorphic to $\mathcal{L}(A)$, where $\mathcal{L}(A)$ is the Lie algebra associated with A in [11]. Let $\mathcal{H}(A)$ be the universal enveloping algebra of the Lie algebra $\mathcal{K}(A)$. Recall that $\mathcal{H}(A) = \mathcal{H}_1(A)$, where $\mathcal{H}_q(A)$ is the generic Ringel-Hall algebra associated in [13] with the algebra A. In fact, in [13], generic Ringel-Hall algebras were associated with directed Auslander-Reiten quivers. However, it is possible to associate generic Ringel-Hall algebras with representation directed $\mathbb{C}$-algebras. The reader is referred to [7] for details.

Proposition 5.1. *Let A be a representation directed $\mathbb{C}$-algebra.*

(a) *The Lie algebra $\mathcal{K}(A)$ is generated by the set $\{u_i\ ;\ i \in Q_0\}$, where $u_i = u_{S_i}$ and S_i is a simple A-module corresponding to the vertex $i \in Q_0$.*

(b) *In the Lie algebra $\mathcal{K}(A)$ the relations from the set $\mathfrak{r}$ hold, if we interchange u_i's by v_i's.*

Proof. The statement (a) is proved in [14, Proposition 6]. Let $[v_{i_1}, \ldots, v_{i_n}]$ be an element from the set $\mathfrak{r}$. It follows that $[v_{i_2}, \ldots, v_{i_n}]$ is a root and the element $[v_{i_1}, v_{i_2}, \ldots, v_{i_n}]$ is not a root. By Lemma 3.6, the vector $m = e_{i_2} + \ldots + e_{i_m}$ is a positive root of the Tits form q_A of A. If $[v_{i_2}, \ldots, v_{i_n}] = 0$ in $\mathcal{K}(A)$, then we are done. Otherwise $[v_{i_2}, \ldots, v_{i_n}] = a \cdot u_M$ for some $0 \neq a \in \mathbb{C}$ and the unique indecomposable A-module $M \in \operatorname{ind}(A)$ with $\mathbf{dim}\, M = m = e_{i_2} + \ldots + e_{i_n}$, because A is representation directed $\mathbb{C}$-algebra. Since q_A is weakly positive, then, by Lemma 3.6, the vector $e_{i_1} + m$ is not a root of q_A. Therefore there exists no indecomposable A-module with dimension vector $e_{i_1} + m$. Then, by [13, Theorem 2], $[v_{i_1}, \ldots, v_{i_n}] = 0$ in $\mathcal{K}(A)$ and we are done. □

Corollary 5.2. *If A is a representation directed $\mathbb{C}$-algebra, then there is an isomorphism of $\mathbb{C}$-algebras*

$$F : L(q_A, \mathfrak{r}) \to \mathcal{K}(A) \cong \mathcal{L}(A)$$

given by $F(v_i) = u_i$, in particular $\dim_{\mathbb{C}} L(q_A, \mathfrak{r}) = |\mathcal{R}^+_q|$.

If, in addition, q_A is positive semi-definite, then $L(q_A, \mathfrak{r}) \cong G^+(q_A)$.

Proof. By Proposition 5.1, F is a well-defined homomorphism of graded Lie algebras. Since the Lie algebra $\mathcal{L}(q_A, \mathfrak{r})$ is generated by the set $\{v_i\ ;\ i \in Q_0\}$ and the Lie algebra $\mathcal{K}(A)$ is generated by the set $\{u_i\ ;\ i \in Q_0\}$, the homomorphism F is an epimorphism. By Corollary 4.3, F is a monomorphism, because $\dim_{\mathbb{C}} \mathcal{K}(A) = |\mathcal{R}^+_{q_A}|$. Finally F is an isomorphism of Lie algebras.

The final assertion follows from Proposition 4.4. □

6. A minimal set of relations defining $L(q,\mathfrak{r})$ for a positive definite form q

The set $\mathfrak{r}$ usually is not a minimal set generating the ideal $(\mathfrak{r})$ of the Lie algebra $L(q)$. In this section we describe a minimal set of elements defining the ideal $(\mathfrak{r})$ of $L(q)$ for a positive definite form q (2.1). In this section all quadratic forms are assumed to be positive definite.

Remark 6.1. The following easily verified facts are essentially used in this section.

1. Let $i, j \in \{1, \dots, n\}$ be such that $\langle e_i, e_j\rangle_q \neq -1$, then
$$[\dots,[v_i,[v_j,\dots]]] = [\dots,[v_j,[v_i,\dots]]]$$
in $L(q,\mathfrak{r})$. Indeed, apply the Jacobi identity and note that in this case $[v_i, v_j] \in (\mathfrak{r})$.

2. If $a \in L(q)$ is a standard multibracket such that e_a is not a root of q, then $a \in (\mathfrak{r})$. Indeed, apply Lemma 3.2 (b) and Proposition 4.2 (a).

3. Let $a, b \in L(q)$ be standard multibrackets such that $\langle e_a, e_b\rangle_q \geq 0$, then $[a, b] \in (\mathfrak{r})$. Indeed,
$$q(e_a + e_b) = q(e_a) + q(e_b) + \langle e_a, e_b\rangle_q \geq 1 + 1 = 2,$$
then $e_a + e_b$ is not a root of q. Therefore $[a, b] \in (\mathfrak{r})$.

4. Let $a, b \in L(q)$ be standard multibrackets such that $\langle e_a, e_b\rangle_q \geq 0$, then
$$[\dots,[a,[b,\dots]]] = [\dots,[b,[a,\dots]]]$$
in $L(q,\mathfrak{r})$. Indeed, apply the Jacobi identity and the fact that in this case $[a, b] \in (\mathfrak{r})$.

5. If $a \in L(q)$ and $\langle e_i, e_a\rangle_q \leq -2$, for some $i = 1, \dots, n$, then $a \in (\mathfrak{r})$. Indeed, by Lemma 2.5, e_a is not a root of q and therefore $a \in (\mathfrak{r})$.

6.1. The first step of reduction

Let $\mathfrak{r}_1 \subseteq \mathfrak{r} \subseteq L(q)$ be the set consisting of the following elements:

- $[v_i, v_j]$ for all $i, j \in \{1, \dots, n\}$ such that $i < j$ and $\langle e_i, e_j\rangle_q \neq -1$,
- $[v_i,[v_i, v_j]]$ for all $i, j \in \{1, \dots, n\}$ such that $i < j$ and $\langle e_i, e_j\rangle_q = -1$,

- $[v_j,[v_i,v_j]]$ for all $i,j\in\{1,\dots,n\}$ such that $i<j$ and $\langle e_i,e_j\rangle_q=-1$.

Let $\mathfrak{r}_0\subseteq\mathfrak{r}\subseteq L(q)$ be the set consisting of all elements $[v_{i_1},\dots,v_{i_m}]$ of $\mathfrak{r}$ such that $\langle e_{i_1},e_{i_2}+\ldots+e_{i_m}\rangle_q=0$. Define $\mathfrak{p}\subseteq\mathfrak{r}$ to be

$$\mathfrak{p}=\mathfrak{r}_1\cup\mathfrak{r}_0. \tag{6.2}$$

Proposition 6.3. *If q is a positive definite quadratic form* (2.1), *then the ideals* $(\mathfrak{p})$ *and* $(\mathfrak{r})$ *of the Lie algebra $L(q)$ are equal.*

Proof. The inclusion $(\mathfrak{p})\subseteq(\mathfrak{r})$ is obvious. It is enough to prove that $(\mathfrak{r})\subseteq(\mathfrak{p})$. Let $v=[v_{i_2},\dots,v_{i_m}]$ be a root in $L(q)$ and let $i_1\in\{1,\dots,n\}$ be such that $[v_{i_1},v_{i_2},\dots,v_{i_m}]\in\mathfrak{r}$ and $\langle e_{i_1},e_{i_2}+\ldots+e_{i_m}\rangle_q\neq 0$. Since $[v_{i_1},v_{i_2},\dots,v_{i_m}]$ is not a root, we have $\langle e_{i_1},e_{i_2}+\ldots+e_{i_m}\rangle_q\geq 1$. We claim that $[v_{i_1},v]\in\mathfrak{p}$.

We precede with induction on $\ell(v)=m-1$. For $\ell(v)<3$ our statement easily follows by a case by case inspection on all possible cases.

Let $\ell(v)\geq 3$, then $m\geq 4$, $v=[v_{i_2},[v_{i_3},\overline{v}]]$, $\langle e_{i_2},e_{i_3}+e_{\overline{v}}\rangle_q=-1$, $\langle e_{i_3},e_{\overline{v}}\rangle_q=-1$ and $\ell(\overline{v})\geq 1$. Note that $\langle e_{i_1},e_{i_2}\rangle_q=a_{ij}\in\{-1,0,1\}$, if $i_1\neq i_2$. Therefore it is enough to consider the following three cases.

1) If $\langle e_{i_1},e_{i_2}\rangle_q\in\{0,1\}$, then $\langle e_{i_1},e_{i_3}+\ldots+e_{i_m}\rangle_q\geq 0$. Therefore $[v_{i_1},v_{i_2}]$, $[v_{i_1},[v_{i_3}\dots,v_{i_m}]]\in\mathfrak{r}$ and by the induction hypothesis we have $[v_{i_1},[v_{i_3}\dots,v_{i_m}]]$, $[v_{i_1},v_{i_2}]\in(\mathfrak{p})$. Finally

$$[v_{i_1},\dots,v_{i_m}]=-[v_{i_2},[[v_{i_3}\dots,v_{i_m}],v_{i_1}]]-[[v_{i_3},\dots,v_{i_m}],[v_{i_1},v_{i_2}]]\in(\mathfrak{p}).$$

2) Let $i_1=i_2$. Applying the Jacobi identity to $[v_{i_1},v]$ we get

$$\begin{aligned}&[v_{i_1},[v_{i_1},[v_{i_3},\overline{v}]]]=-[v_{i_1},[v_{i_3},[\overline{v},v_{i_1}]]]-[v_{i_1},[\overline{v},[v_{i_1},v_{i_3}]]]=\\&=[v_{i_3},[[\overline{v},v_{i_1}],v_{i_1}]]+[[\overline{v},v_{i_1}],[v_{i_1},v_{i_3}]]+[\overline{v},[[v_{i_1},v_{i_3}],v_{i_1}]]+[[\overline{v},v_{i_1}],[v_{i_1},v_{i_3}]].\end{aligned}$$

Note that, we have $\langle e_{i_1},e_{i_1}+e_{\overline{v}}\rangle_q=2+\langle e_{i_1},e_{\overline{v}}\rangle_q\geq 2+(-1)=1$, and therefore by the induction hypothesis $[v_{i_1},[v_{i_1},\overline{v}]]\in(\mathfrak{p})$. Moreover, $[v_{i_1},[v_{i_1},v_{i_3}]]\in(\mathfrak{r}_1)\subseteq(\mathfrak{p})$. Finally $[v_{i_1},v]=2[[v_{i_1},v_{i_3}][v_{i_1},\overline{v}]]$. If $[v_{i_1},v_{i_3}]\in(\mathfrak{p})$, then $[v_{i_1},v]\in(\mathfrak{p})$. Assume that $[v_{i_1},v_{i_3}]\notin(\mathfrak{p})$. In this case $\langle e_{i_1},e_{i_3}\rangle_q=-1$ and

$$\langle e_{i_1},e_{\overline{v}}\rangle_q=\langle e_{i_1},e_{i_2}+e_{i_3}+e_{\overline{v}}\rangle_q-\langle e_{i_1},e_{i_2}\rangle_q-\langle e_{i_1},e_{i_3}\rangle_q\geq 1-2-(-1)=0.$$

Then $[v_{i_1},v]=2[[v_{i_1},v_{i_3}][v_{i_1},\overline{v}]]\in(\mathfrak{r}_0)\subseteq(\mathfrak{p})$ and we are done.

3) Let $\langle i_1,i_2\rangle_q=-1$. By Lemma 2.5 (c), $\langle e_{i_1},e_{i_3}+\ldots+e_{i_m}\rangle\in\{-1,0,1\}$, because q is positive definite. On the other hand

$$1\leq\langle e_{i_1},e_{i_2}+\ldots+e_{i_m}\rangle_q=-1+\langle e_{i_1},e_{i_3}+\ldots+e_{i_m}\rangle_q\leq 0.$$

This contradiction shows that the case **3)** does not hold.

This finishes the proof. □

Corollary 6.4. *If q is a positive definite quadratic form (2.1), then*

$$L(q, \mathfrak{r}) \cong L(q, \mathfrak{p}).$$

□

6.2. The second step of reduction

Let $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Following [5], we call the sequence $(i_1, \ldots, i_m)$ a **chordless cycle** of the form q (2.1), if the following conditions are satisfied:

- the elements $i_1, \ldots, i_m$ are pairwise different,
- $a_{ij} = \langle e_{i_j}, e_{i_k} \rangle_q \neq 0$ if and only if $|k - j| = 1 \mod m$.

Chordless cycles are playing an important role in [5], where Lie algebras associated with positive definite quadratic forms are investigated.

A chordless cycle $(i_1, \ldots, i_m)$ is called **positive**, if $\langle e_{i_1}, e_{i_m} \rangle_q = 1$ and $\langle e_{i_j}, e_{i_k} \rangle_q = -1$ for all j, k such that $\{j, k\} \neq \{1, m\}$ and $|j - k| = 1 \mod m$.

Remark 6.5. We note that if $(i_1, \ldots, i_m)$ is a chordless cycle, then $(i_1, \ldots, i_m)$ is a simple cycle in the bigraph $B(q)$. Moreover, if the chordless cycle $(i_1, \ldots, i_m)$ is positive, then the cycle $(i_1, \ldots, i_m)$ in $B(q)$ has exactly one broken edge $i_1 - - - i_m$.

Let $\mathfrak{r}_2 \subseteq L(q)$ be the set consisting of all elements $[v_{i_1}, \ldots, v_{i_m}]$ such that $(i_1, \ldots, i_m)$ is a positive chordless cycle.

Lemma 6.6. $\mathfrak{r}_2 \subseteq \mathfrak{p}$.

Proof. Let $v = [v_{i_1}, \ldots, v_{i_m}] \in \mathfrak{r}_2$. From the definition of a positive chordless cycle, it follows easily that $[v_{i_k}, \ldots, v_{i_m}]$ is a root for any $k > 1$. Moreover $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m} \rangle_q = 0$, and therefore $v \in \mathfrak{p}$ □

Set

$$\mathfrak{j} = \mathfrak{r}_1 \cup \mathfrak{r}_2. \tag{6.7}$$

For all elements $x, y \in L(q)$ we write $x \equiv y$ if $x - y \in (\mathfrak{j})$. Obviously $\equiv$ is an equivalence relation.

Before we prove that the ideals $(\mathfrak{p})$ and $(\mathfrak{j})$ of $L(q)$ are equal, we need to prove two technical lemmata.

Lemma 6.8. *Let q be a positive definite quadratic form (2.1). Let $m \geq 3$ be an integer, let $v = [v_{i_2}, \dots, v_{i_m}]$ be a root and let $(\mathfrak{j})_{m-1} = (\mathfrak{p})_{m-1}$. Let $i_1 \in \{1, \dots, n\}$ be such that $[v_{i_1}, v_{i_2}, \dots, v_{i_m}] \in \mathfrak{p}$, $\langle e_{i_1}, e_{i_2}\rangle_q = -1$ and $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q = 0$. Then $[v_{i_1}, v] \in (\mathfrak{j})$ or there exists $\varepsilon \in \{0,1\}$ such that $[v_{i_1}, v] \equiv (-1)^{\varepsilon}[v_{i_1}, [a, x]]$, where*

(a) $a = [v_{i_k}, \dots, v_{i_2}]$, $x = [v_{i_{k+1}}, \dots, v_{i_m}]$, *for some* $2 \leq k < m$,
(b) $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, *for all* $j = 3, \dots, k$, *and*
(c) $\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1$.

Proof. Let $m \geq 3$ and let $v = [v_{i_2}, \dots, v_{i_m}]$ be a root. Let $i_1 \in \{1, \dots, n\}$ be such that $[v_{i_1}, v_{i_2}, \dots, v_{i_m}] \in \mathfrak{p}$, $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q = 0$ and $\langle e_{i_1}, e_{i_2}\rangle_q = -1$. Note that $[v_{i_1}, v] \equiv [v_{i_1}, [\overline{a}, x]]$, where $\overline{a} = [v_{i_l}, \dots, v_{i_2}]$, $x = [v_{i_{l+1}}, \dots, v_{i_m}]$ and $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, for all $j = 3, \dots, l$ (i.e. the conditions (a), (b) are satisfied). Indeed, it is enough to set $l = 2$, $\overline{a} = x_{i_2}$ and $x = [v_{i_3}, \dots, v_{i_m}]$.

Fix $\overline{a}$ and $\overline{x}$ such that $[v_{i_1}, v] \equiv [v_{i_1}, [\overline{a}, \overline{x}]]$, where $\overline{a} = [v_{i_l}, \dots, v_{i_2}]$, $\overline{x} = [v_{i_{l+1}}, \dots, v_{i_m}]$ and $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, for all $j = 3, \dots, l$. As we noted above there exists at least one such a presentation of $[v_{i_1}, v]$. Consider the element i_{l+1}. By Lemma 2.5, $\langle e_{i_1}, e_{i_{l+1}}\rangle_q \in \{-1, 0, 1, 2\}$.

- If $\langle e_{i_1}, e_{i_{l+1}}\rangle_q = 1$, then we set $k = l$ and note that $[v_{i_1}, v]$ has the required form, i.e. $[v_{i_1}, v] \equiv [v_{i_1}, [a, x]]$, where $a = [v_{i_k}, \dots, v_{i_2}]$, $x = [v_{i_{k+1}}, \dots v_{i_m}]$, $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, for all $j = 3, \dots, k$, and $\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1$.

- If $\langle e_{i_1}, e_{i_{l+1}}\rangle_q = -1$, then

$$\begin{aligned}\langle e_{i_1}, e_{i_{l+2}} + \ldots + e_{i_m}\rangle_q &= \langle e_{i_1}, e_v\rangle_q - \langle e_{i_1}, e_{\overline{a}}\rangle_q - \langle e_{i_1}, e_{i_{l+1}}\rangle_q \\ &= 0 - (-1) - (-1) = 2.\end{aligned}$$

  Therefore, by Lemma 2.5, $m = l + 2$ and $i_1 = i_{l+2}$. Note that

$$\begin{aligned}&[v_{i_1}, [v_{i_1}, v_{i_{l+1}}]] \in (\mathfrak{r}_1) \subseteq (\mathfrak{j}),\\ &\langle e_{i_1}, e_{i_1} + e_{\overline{a}}\rangle_q = 1, \text{ then } [v_{i_1}, [v_{i_1}, \overline{a}]] \in (\mathfrak{p})_{(m-1)} = (\mathfrak{j})_{(m-1)} \subseteq (\mathfrak{j}),\\ &\langle e_{i_1}, e_{i_{l+1}} + e_{\overline{a}}\rangle_q = -2, \text{ therefore } [v_{i_{l+1}}, \overline{a}] \in (\mathfrak{p})_{(m-1)} = (\mathfrak{j})_{(m-1)} \subseteq (\mathfrak{j}).\end{aligned}$$

  Then we have

$$\begin{aligned}[v_{i_1}, v] &\equiv [v_{i_1}, [\overline{a}, [v_{i_{l+1}}, v_{i_1}]]]\\ &= -[\overline{a}, [[v_{i_{l+1}}, v_{i_1}], v_{i_1}]] - [[v_{i_{l+1}}, v_{i_1}], [v_{i_1}, \overline{a}]]\\ &\equiv [[v_{i_1}, \overline{a}], [v_{i_{l+1}}, v_{i_1}]]\\ &= -[v_{i_{l+1}}, [v_{i_1}, [v_{i_1}, \overline{a}]]] - [v_{i_1}, [[v_{i_1}, \overline{a}], v_{i_{l+1}}]]\\ &\equiv [v_{i_1}, [v_{i_{l+1}}, [v_{i_1}, \overline{a}]]]\\ &= -[v_{i_1}, [v_{i_1}, [\overline{a}, v_{i_{l+1}}]]] - [v_{i_1}, [\overline{a}, [v_{i_{l+1}}, v_{i_1}]]]\\ &\equiv -[v_{i_1}, [\overline{a}, [v_{i_{l+1}}, v_{i_1}]]]\\ &\equiv -[v_{i_1}, v].\end{aligned}$$

Therefore $2 \cdot [v_{i_1}, v] \in (\mathfrak{j})$ and $[v_{i_1}, v] \in (\mathfrak{j})$.

- If $\langle e_{i_1}, e_{i_{l+1}}\rangle_q = 2$, then $i_1 = i_{l+1}$ and $[v_{i_1}, v] \equiv [v_{i_1}, [\overline{a}, [v_{i_1}, y]]]$, where $y = [v_{i_{l+2}}, \ldots, v_{i_m}]$. By the bilinearity of $\langle -, -\rangle_q$ and assumptions, we have $\langle e_{i_1}, e_y\rangle_q = -1$. Moreover

  $$\langle e_{i_1}, e_{i_1} + e_y\rangle_q = 1, \text{ then } [v_{i_1}, [v_{i_1}, y]] \in (\mathfrak{p})_{(m-1)} = (\mathfrak{j})_{(m-1)} \subseteq (\mathfrak{j}),$$
  $$\langle e_{i_1}, e_{i_1} + e_{\overline{a}}\rangle_q = 1, \text{ then } [v_{i_1}, [v_{i_1}, \overline{a}]] \in (\mathfrak{p})_{(m-1)} = (\mathfrak{j})_{(m-1)} \subseteq (\mathfrak{j}),$$
  $$\langle e_{i_1}, e_y + e_{\overline{a}}\rangle_q = -2, \text{ therefore } [v_{i_{l+1}}, \overline{a}] \in (\mathfrak{p})_{(m-1)} = (\mathfrak{j})_{(m-1)} \subseteq (\mathfrak{j}).$$

  Similarly as above we can prove that $[v_{i_1}, v] \in (\mathfrak{j})$.

- Let $\langle e_{i_1}, e_{i_{l+1}}\rangle_q = 0$ and $y = [v_{i_{l+2}}, \ldots, v_{i_m}]$. Consider

  $$\begin{aligned}[v_{i_1}, [\overline{a}, [v_{i_{l+1}}, y]]] &= \\ &= -[v_{i_1}, [v_{i_{l+1}}, [y, \overline{a}]]] - [v_{i_1}, [y, [\overline{a}, v_{i_{l+1}}]]] = \\ &= [v_{i_{l+1}}, [[y, \overline{a}], v_{i_1}]] + [[y, \overline{a}], [v_{i_1}, v_{i_{l+1}}]] - [v_{i_1}, [[v_{i_{l+1}}, \overline{a}], y]].\end{aligned}$$

  Note that $[v_{i_1}, v_{i_{l+1}}] \in (\mathfrak{r}_1) \subseteq (\mathfrak{j})$, $\langle e_{i_1}, e_{\overline{a}} + e_y\rangle_q = 0$ and therefore $[[y, \overline{a}], v_{i_1}] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1} \subseteq (\mathfrak{j})$. Then

  $$[v_{i_1}, v] \equiv [v_{i_1}, [\overline{a}, [v_{i_{l+1}}, y]]] \equiv -[v_{i_1}, [[v_{i_{l+1}}, \overline{a}], y]].$$

  We may set $\overline{a} := [v_{i_{l+1}}, \overline{a}]$, $\overline{x} := y$ and continue this procedure inductively.

Note that there exists k such that $\langle e_{i_1}, e_{i_{k+1}}\rangle_q \geq 1$, because $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q = 0$ and $\langle e_{i_1}, e_{i_2}\rangle_q = -1$. Therefore continuing this procedure inductively, we prove that $[v_{i_1}, v] \in (\mathfrak{j})$ or $[v_{i_1}, v] \equiv (-1)^{\varepsilon}[v_{i_1}, [a, x]]$, where $a = [v_{i_k}, \ldots, v_{i_2}]$, $x = [v_{i_{k+1}}, \ldots v_{i_m}]$, $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, for all $j = 3, \ldots, k$, and $\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1$. □

Lemma 6.9. *Assume that q is a positive definite quadratic form* (2.1). *Let $m \geq 3$ be an integer, $v = [v_{i_2}, \ldots, v_{i_m}]$ be a root and $(\mathfrak{j})_{m-1} = (\mathfrak{p})_{m-1}$. Let $i_1 \in \{1, \ldots, n\}$ be such that $[v_{i_1}, v_{i_2}, \ldots, v_{i_m}] \in \mathfrak{p}$, $\langle e_{i_1}, e_{i_2}\rangle_q = -1$ and $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q = 0$. Moreover assume that $[v_{i_1}, v] \equiv (-1)^{\varepsilon}[v_{i_1}, [a, x]]$ and the conditions* (a)-(c) *of Lemma 6.8 are satisfied. Then $[v_{i_1}, v] \in (\mathfrak{j})$ or $[v_{i_1}, v] \equiv [v_{i_1}, [a, x]] \equiv [v_{i_1}, [a, [b, y]]]$, where*

(i) $a = [v_{i_k}, \ldots, v_{i_2}]$, $b = [v_{i_s}, v_{i_{s-1}} \ldots, v_{i_{k+1}}]$, $y = [v_{i_{s+1}}, \ldots, v_{i_m}]$,
(ii) $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, *for all* $j = 3, \ldots, k$ *and* $j = k+2, \ldots, s$,
(iii) $\langle e_{i_1}, e_{i_2}\rangle_q = -1$, $\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1$,
(iv) $\langle e_b, e_a\rangle_q = 0$,
(v) *if* $s < m$ *then* $\langle e_{i_{s+1}}, e_a\rangle_q = -1$ *and* $\langle e_{i_{s+1}}, e_b\rangle_q = -1$.

Proof. Note that $[v_{i_1}, v] \equiv [v_{i_1}, [a, x]] \equiv [v_{i_1}, [a, [b, y]]]$, where $b = v_{i_{k+1}}$, $y = [v_{i_{k+2}}, \dots, v_{i_m}]$ and the conditions (i), (ii), (iii) are satisfied, if we put $s = k+1$. We may assume that the condition (iv) is also satisfied. Indeed, it is enough to show that $\langle e_{i_{k+1}}, e_a\rangle_q = 0$.

- If $\langle e_{i_{k+1}}, e_a\rangle_q = -1$, then $\langle e_{i_{k+1}}, e_a + e_y\rangle_q = -2$. Therefore $[a, y] \in (\mathfrak{r})_{m-1} = (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1} \subseteq (\mathfrak{j})$. Then

$$\begin{aligned}[v_{i_1}, [a, [v_{i_{k+1}}, y]]] &= -[v_{i_1}, [v_{i_{k+1}}, [y, a]]] - [v_{i_1}, [y, [a, v_{i_{k+1}}]]] \\ &\equiv [v_{i_1}, [[a, v_{i_{k+1}}], y]] \\ &= -[[a, v_{i_{k+1}}], [y, v_{i_1}]] - [y, [v_{i_1}, [a, v_{i_{k+1}}]]].\end{aligned}$$

  Since $\langle e_{i_1}, e_y\rangle_q = 0 = \langle e_{i_1}, e_{i_{k+1}} + e_a\rangle_q$, then $[y, v_{i_1}] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}$ and $[v_{i_1}, [a, v_{i_{k+1}}]] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}$.Therefore we have $[v_{i_1}, [a, [v_{i_{k+1}}, y]]] \in (\mathfrak{j})$.

- If $\langle e_{i_{k+1}}, e_a\rangle_q = 2$, then $a = v_{i_{k+1}}$. It is a contradiction, because

$$\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1 \neq -1 = \langle e_{i_1}, e_a\rangle_q.$$

- If $\langle e_{i_{k+1}}, e_a\rangle_q = 1$, then $\langle e_{i_{k+1}}, e_a + e_y\rangle_q = 0$. Therefore $[v_{i_{k+1}}, [y, a]]$, $[a, v_{i_{k+1}}] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}$ and

$$[v_{i_1}, [a, [v_{i_{k+1}}, y]]] = -[v_{i_1}, [v_{i_{k+1}}, [y, a]]] - [v_{i_1}, [y, [a, v_{i_{k+1}}]]] \in (\mathfrak{j}).$$

Finally, we can assume that $\langle e_{i_{k+1}}, e_a\rangle_q = \langle e_b, e_a\rangle_q = 0$ and the condition (iv) is satisfied. Therefore $[v_{i_1}, [a, [b, y]]] \equiv [v_{i_1}, [b, [a, y]]]$, because $\langle e_b, e_a\rangle_q = 0$.

If $k+1 = m$, then we are done. Assume that $k+1 < m$ and consider the element i_{k+2}.

1. Let $\langle e_{i_{k+2}}, e_b\rangle_q = -1$.

   (a) If $\langle e_{i_{k+2}}, e_a\rangle_q = -1$, then we put $s = k+1$ and we are done.

   (b) If $\langle e_{i_{k+2}}, e_a\rangle_q \geq 0$, then $[v_{i_{k+2}}, a] \in (\mathfrak{j})_{m-1}$.
   If $m = k+2$, then

$$[v_{i_1}, v] \equiv -[v_{i_1}, [b, [v_{i_{k+2}}, a]]] - [v_{i_1}, [v_{i_{k+2}}, [a, b]]] \in (\mathfrak{j})$$

   and we are done.
   Assume that $m > k+2$. We can assume that $\langle e_{i_{k+1}}, e_a\rangle_q = 0$. Indeed, since $[v_{i_{k+2}}, a] \in (\mathfrak{j})_{m-1}$, we have

$$\begin{aligned}[v_{i_1}, v] &\equiv [v_{i_1}, [b, [a, [v_{i_{k+2}}, v_{i_{k+3}}, \dots, v_{i_m}]]]] \\ &\equiv [v_{i_1}, [b, [v_{i_{k+2}}, [a, [v_{i_{k+3}}, \dots, v_{i_m}]]]]].\end{aligned}$$

If $\langle e_{i_{k+2}}, e_a\rangle_q \geq 1$, then $\langle e_{i_{k+2}}, e_a + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q \geq 1 - 1 = 0$. It follows that

$$[v_{i_{k+2}}, [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}.$$

Therefore we can assume that $\langle e_{i_{k+2}}, e_a\rangle_q = 0$. Moreover

$$\begin{aligned} [v_{i_1}, v] &\equiv [v_{i_1}, [b, [v_{i_{k+2}}, [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \\ &= -[v_{i_1}, [v_{i_{k+1}}, [[a, [v_{i_{k+3}}, \ldots, v_{i_m}]], b]]] \\ &\quad -[v_{i_1}, [[a, [v_{i_{k+3}}, \ldots, v_{i_m}]], [b, v_{i_{k+2}}]]] \\ &\equiv -[v_{i_1}, [[v_{i_{k+2}}, b], [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]], \end{aligned}$$

because $\langle e_{i_{k+2}}, e_b\rangle_q = -1$ and $\langle e_{i_{k+2}}, e_a + e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = -2$. We can assume that $\langle e_{i_1}, e_{i_{k+2}}\rangle_q = 0$. Indeed, if $\langle e_{i_1}, e_{i_{k+2}}\rangle_q = -1$, then

$$\begin{aligned} 0 = \langle e_{i_1}, e_v\rangle_q &= \langle e_{i_1}, e_a + e_b + e_{i_{k+2}}\rangle_q + \langle e_{i_1}, e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q \\ &= -1 + \langle e_{i_1}, e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q. \end{aligned}$$

It follows that $\langle e_{i_1}, e_{i_{k+3}} + \ldots + v_{i_m}\rangle_q = 1$, $\langle e_{i_1}, e_a + e_{i_{k+3}} + \ldots + v_{i_m}\rangle_q = 0$ and $[v_{i_1}, [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}$. Therefore

$$\begin{aligned} [v_{i_1}, v] &\equiv -[v_{i_1}, [[v_{i_{k+2}}, b], [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \\ &\equiv -[[v_{i_{k+2}}, b], [v_{i_1}, [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \in (\mathfrak{j}), \end{aligned}$$

because $\langle e_{i_1}, e_{i_{k+2}} + e_b\rangle_q = 0$. If $\langle e_{i_1}, e_{i_{k+2}}\rangle_q \geq 1$, then

$$\begin{aligned} 0 = \langle e_{i_1}, e_v\rangle_q &= \langle e_{i_1}, e_a + e_b + e_{i_{k+2}}\rangle_q + \langle e_{i_1}, e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q \\ &\geq 1 + \langle e_{i_1}, e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q. \end{aligned}$$

It follows that $\langle e_{i_1}, e_{i_{k+3}} + \ldots + v_{i_m}\rangle_q \leq -1$, $\langle e_{i_1}, e_a + e_{i_{k+3}} + \ldots + v_{i_m}\rangle_q \leq -2$ and $[a, [v_{i_{k+3}}, \ldots, v_{i_m}]] \in (\mathfrak{p})_{m-1} = (\mathfrak{j})_{m-1}$. Therefore, $[v_{i_1}, v] \in (\mathfrak{j})$ and we are done. Finally, $\langle e_{i_{k+2}}, e_a\rangle_q = 0$, $\langle e_{i_1}, e_{i_{k+2}}\rangle_q = 0$, $\langle e_{i_{k+2}} + e_b, e_a\rangle_q = 0$ and

$$\begin{aligned} [v_{i_1}, v] &\equiv -[v_{i_1}, [[v_{i_{k+2}}, b], [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \\ &\equiv -[v_{i_1}, [a, [[v_{i_{k+2}}, b], [v_{i_{k+3}}, \ldots, v_{i_m}]]]]. \end{aligned}$$

We set $\overline{a} = a$, $\overline{b} = [v_{i_{k+2}}, b]$, $\overline{y} = [v_{i_{k+3}}, \ldots, v_{i_m}]$ and continue this procedure inductively using $[v_{i_1}, [\overline{a}, [\overline{b}, \overline{y}]]]$ instead of $[v_{i_1}, [a, [b, y]]]$.

2. Let $\langle e_{i_{k+2}}, e_b\rangle_q \neq -1$, then $[v_{i_{k+2}}, b] \in (\mathfrak{j})_{m-1}$. If $m = k + 2$, then

$$[v_{i_1}, v] \equiv [v_{i_1}, [a, [b, v_{i_{k+2}}]]] \in (\mathfrak{j})$$

and we are done. Assume that $m > k+2$. Since $[v_{i_{k+2}}, b] \in (\mathfrak{j})_{m-1}$, we have

$$[v_{i_1}, [a, [b, [v_{i_{k+2}}, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \equiv [v_{i_1}, [a, [v_{i_{k+2}}, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]].$$

For the sake of simplicity we present partial results in tables. In the first column of the following table we consider all possible values of $\langle e_{i_{k+2}}, e_b\rangle_q$. In the second column we give the corresponding value of $\langle e_{i_{k+2}}, e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q$. The third column contains the sign "$+$", if we can deduce that $X = [v_{i_{k+2}}, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]] \in (\mathfrak{j})_{m-1}$, and the sign "$-$", otherwise.

$\langle e_{i_{k+2}}, e_b\rangle_q$	$\langle e_{i_{k+2}}, e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q$	$X \in (\mathfrak{j})_{m-1}$
0	-1	$-$
1	0	$+$
2	1	$+$

Therefore we may assume that $\langle e_{i_{k+2}}, e_b\rangle_q = 0$, because otherwise

$$[v_{i_1}, v] \equiv [v_{i_1}, [a, [v_{i_{k+2}}, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \in (\mathfrak{j}).$$

(a) Assume that $\langle e_{i_{k+2}}, e_a\rangle_q \neq -1$. Then $[v_{i_{k+2}}, a] \in (\mathfrak{j})_{m-1}$ and

$$\begin{aligned}[v_{i_1}, v] &\equiv [v_{i_1}, [a, [v_{i_{k+2}}, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \\ &\equiv [v_{i_1}, [v_{i_{k+2}}, [a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]].\end{aligned}$$

In the first column of the following table we consider all possible values of $\langle e_{i_{k+2}}, e_a\rangle_q$. In the second column we give the corresponding value of $x = \langle e_{i_{k+2}}, e_a + e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q$. The third column contains the sign "$+$", if we can deduce that

$$X = [v_{i_{k+2}}, [a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \in (\mathfrak{j})_{m-1},$$

and the sign "$-$", otherwise.

$\langle e_{i_{k+2}}, e_a\rangle_q$	x	$X \in (\mathfrak{j})_{m-1}$
0	-1	$-$
1	0	$+$
2	1	$+$

Therefore we may assume that $\langle e_{i_{k+2}}, e_a\rangle_q = 0$, because otherwise $[v_{i_1}, v] \in (\mathfrak{j})$. Moreover,

$$\begin{aligned}&[v_{i_1}, [v_{i_{k+2}}, [a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \equiv \\ &\qquad \equiv [v_{i_1}, [v_{i_{k+2}}, [b, [a, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]],\end{aligned}$$

because $\langle e_a, e_b\rangle_q = 0$.

In the first column of the following table we consider all possible values of $\langle e_{i_1}, e_{i_{k+2}}\rangle_q$. In the second column we present a consequences of the information contained in the first column. Finally, in the second table we present conclusions of the results presented in the first table.

$\langle e_{i_1}, e_{i_{k+2}}\rangle_q$	consequences
-1	$\langle e_{i_1}, e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = 2$
0	$\langle e_{i_1}, e_a + e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = 0$
1	$\langle e_{i_1}, e_a + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = -2$
2	$\langle e_{i_1}, e_a + e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = -2$

$\langle e_{i_1}, e_{i_{k+2}}\rangle_q$	conclusions
-1	$[b, [v_{i_{k+3}}, \ldots, v_{i_m}]] \in (\mathfrak{j})_{m-1}$
0	$[v_{i_1}, [a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \in (\mathfrak{j})_{m-1}$
1	$[a, [v_{i_{k+3}}, \ldots, v_{i_m}]] \in (\mathfrak{j})_{m-1}$
2	$[a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]] \in (\mathfrak{j})_{m-1}$

All these cases imply that $[v_{i_1}, v] \in (\mathfrak{j})$.

(b) Assume that $\langle e_{i_{k+2}}, e_a\rangle_q = -1$. In this case

$$\langle e_{i_{k+2}}, e_a + e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = -2.$$

It follows that

$$[a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]] \in (\mathfrak{j})_{m-1}$$

and

$$\begin{aligned}[v_{i_1}, v] &\equiv [v_{i_1}, [a, [v_{i_{k+2}}, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \\ &= [v_{i_1}, [v_{i_{k+2}}, [a, [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]]] \\ &\quad -[v_{i_1}, [[b, [v_{i_{k+3}}, \ldots, v_{i_m}]], [a, v_{i_{k+2}}]]] \\ &\equiv -[v_{i_1}, [[v_{i_{k+2}}, a], [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]].\end{aligned}$$

Consider $\langle e_{i_1}, e_{i_{k+2}}\rangle_q$ and

$$\begin{aligned}&-[v_{i_1}, [[v_{i_{k+2}}, a], [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] = \\ &\qquad = [[v_{i_{k+2}}, a], [[b, [v_{i_{k+3}}, \ldots, v_{i_m}]], v_{i_1}]] + \\ &\qquad\qquad + [[b, [v_{i_{k+3}}, \ldots, v_{i_m}]], [v_{i_1}, [v_{i_{k+2}}, a]]].\end{aligned}$$

We present again partial results in tables.

$\langle e_{i_1}, e_{i_{k+2}}\rangle_q$	consequences
-1	$\langle e_{i_1}, e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = 2$
1	$\langle e_{i_1}, e_b + e_{i_{k+3}} + \ldots + e_{i_m}\rangle_q = 0$ and $\langle e_{i_1}, e_{i_{k+2}} + e_a\rangle_q = 0$
2	$i_1 = i_{k+2}$

$\langle e_{i_1}, e_{i_{k+2}}\rangle_q$	conclusions
-1	$[b, [v_{i_{k+3}}, \ldots, v_{i_m}]] \in (\mathfrak{j})_{m-1}$
1	$[[b, [v_{i_{k+3}}, \ldots, v_{i_m}]], v_{i_1}] \in (\mathfrak{j})_{m-1}$ and $[v_{i_1}, [v_{i_{k+2}}, a]] \in (\mathfrak{j})_{m-1}$
2	contradiction, because $\langle e_{i_1}, e_b\rangle_q = 1 \neq 0 = \langle e_{i_{k+2}}, e_b\rangle_q$

Therefore, we can assume that $\langle e_{i_1}, e_{i_{k+2}}\rangle_q = 0$, because otherwise $[v_{i_1}, v] \in (\mathfrak{j})$. Moreover

$$\begin{aligned}[v_{i_1}, v] &\equiv -[v_{i_1}, [[v_{i_{k+2}}, a], [b, [v_{i_{k+3}}, \ldots, v_{i_m}]]]] \\ &\equiv -[v_{i_1}, [b, [[v_{i_{k+2}}, a], [v_{i_{k+3}}, \ldots, v_{i_m}]]]],\end{aligned}$$

because $\langle e_{i_{k+2}} + e_a, e_b\rangle_q = 0$. Therefore

$$[v_{i_1}, v] \equiv -[v_{i_1}, [\overline{a}, [\overline{b}, \overline{y}]]],$$

where $\overline{a} = [v_{i_{k+2}}, a]$, $\overline{b} = b$, $\overline{y} = [v_{i_{k+3}}, \ldots, v_{i_m}]$ and the conditions (i)-(iv) are satisfied.

Continuing this procedure inductively we show that $[v_{i_1}, v] \in (\mathfrak{j})$ or $[v_{i_1}, v] \equiv [v_{i_1}, [a, x]] \equiv [v_{i_1}, [a, [b, y]]]$ and the conditions (i)-(v) are satisfied. □

Proposition 6.10. *Let q be a positive definite quadratic form. The ideals $(\mathfrak{j})$ and $(\mathfrak{p})$ of the Lie algebra $L(q)$ are equal.*

Proof. The inclusion $(\mathfrak{j}) \subseteq (\mathfrak{p})$ is obvious. It is enough to prove that $(\mathfrak{p}) \subseteq (\mathfrak{j})$. Let $v = [v_{i_2}, \ldots, v_{i_m}]$ be a root in $L(q)$ and let $i_1 \in \{1, \ldots, n\}$ be such that $[v_{i_1}, v_{i_2}, \ldots, v_{i_m}] \in \mathfrak{p}$ and $\langle e_{i_1}, e_{i_2} + \ldots + e_{i_m}\rangle_q = 0$. We claim that $[v_{i_1}, v] \in \mathfrak{j}$.

We prove our claim by induction on $\ell(v) = m - 1$. For $\ell(v) < 3$ our statement easily follows by a case by case inspection on all possible cases.

Let $\ell(v) \geq 3$, then $m \geq 4$, $v = [v_{i_2}, [v_{i_3}, \overline{v}]]$, $\langle e_{i_2}, e_{i_3} + e_{\overline{v}}\rangle_q = -1$, $\langle e_{i_3}, e_{\overline{v}}\rangle_q = -1$ and $\ell(\overline{v}) \geq 1$. By Lemma 2.5, it is enough to consider the following three cases.

1) If $\langle e_{i_1}, e_{i_2}\rangle_q = 0$, then by the bilinearity of $\langle -,-\rangle_q$, we have $\langle e_{i_1}, e_{i_3}+\ldots+e_{i_m}\rangle_q = 0$. Moreover

$$[v_{i_1}, v] = -[v_{i_2}, [[v_{i_3}, \ldots, v_{i_m}], v_{i_1}]] - [[v_{i_3}, \ldots, v_{i_m}], [v_{i_1}, v_{i_2}]].$$

By definitions, $[v_{i_1}, v_{i_2}] \in \mathfrak{j}$ and $[v_{i_1}, [v_{i_3}, \ldots, v_{i_m}]] \in \mathfrak{r}$. Then, by Proposition 6.3, $[v_{i_1}, [v_{i_3}, \ldots, v_{i_m}]] \in (\mathfrak{p})$ and by the induction hypothesis we have $[v_{i_1}, [v_{i_3}, \ldots, v_{i_m}]] \in (\mathfrak{j})$. Finally, $[v_{i_1}, v] \in (\mathfrak{j})$.

2) If $i_1 = i_2$, then $\langle e_{i_1}, e_{i_2}\rangle_q = 2$ and $\langle e_{i_1}, e_{i_3}+\ldots+e_{i_m}\rangle_q = -2$. This is a contradiction with Lemma 2.5 and therefore the case 2) does not hold.

3) Let $\langle e_{i_1}, e_{i_2}\rangle_q \in \{1, -1\}$. In this case we apply the Jacobi identity and develop Lemmata 6.8, 6.9 to find an element $w \in L(q)$ such that $[v_{i_1}, v] - w \in (\mathfrak{j})$ (i.e. $[v_{i_1}, v] \equiv w$). Finally we show that $w \in (\mathfrak{j})$, which implies that $[v_{i_1}, v] \in (\mathfrak{j})$.

3.1) Let $\langle e_{i_1}, e_{i_2}\rangle_q = 1$. We reduce this case to the case 3.2) presented below. Since $\langle e_{i_1}, e_v\rangle_q = 0$ and $\langle e_{i_1}, e_{i_2}\rangle_q = 1$, then there exists $k \in \{3, \ldots, m\}$ such that $\langle e_{i_1}, e_{i_k}\rangle_q = -1$. Choose k minimal with this property. We may assume that $k < m$, because $[v_{i_{m-1}}, v_{i_m}] = -[v_{i_m}, v_{i_{m-1}}]$ and we can work with $-v$ instead of v. Note that for all $s = 3, \ldots, k-1$, we have $\langle e_{i_1}, e_{i_s}\rangle_q = 0$. Indeed, if there exists $s = 3, \ldots, k-1$ such that $\langle e_{i_1}, e_{i_s}\rangle_q \neq 0$, then by the choice of k, we have $\langle e_{i_1}, e_{i_s}\rangle_q \geq 1$. Then $\langle e_{i_1}, e_{i_k}+\ldots+e_{i_m}\rangle_q \leq \langle e_{i_1}, e_v\rangle_q - \langle e_{i_1}, e_{i_2}\rangle_q - \langle e_{i_1}, e_{i_s}\rangle_q = -2$ and we get a contradiction, because $[v_{i_k}, \ldots, v_{i_m}]$ is a root.

Now applying the Jacobi identity we get

$$[v_{i_2}, \ldots, v_{i_{k-1}}, [v_{i_k}, y]] = \\ = [v_{i_2}, \ldots, v_{i_k}, [v_{i_{k-1}}, y]] + [v_{i_2}, \ldots, v_{i_{k-2}}, [[v_{i_{k-1}}, v_{i_k}], y]].$$

By Lemma 3.6 (a), the element $[v_{i_k}, [v_{i_{k-1}}, y]]$ or the element $[v_{i_{k-1}}, v_{i_k}]$ is not a root, then $[v_{i_{k-1}}, v_{i_k}] \in (\mathfrak{p})$ or $[v_{i_k}, [v_{i_{k-1}}, y]] \in (\mathfrak{p})$. By the induction hypothesis $[v_{i_k}, [v_{i_{k-1}}, y]] \in (\mathfrak{j})$ or $[v_{i_{k-1}}, v_{i_k}] \in (\mathfrak{j})$. Therefore $v \equiv [v_{i_2}, \ldots, v_{i_{k-2}}, [x, z]]$, where $x = v_{i_k}$ and $z = [v_{i_{k-1}}, y]$ or $x = [v_{i_{k-1}}, v_{i_k}]$, $z = y$. In both cases $\langle e_{i_1}, e_x\rangle_q = -1$. Continuing this procedure (i.e. x plays a role of v_{i_k} and z plays a role of y), we get

$$[v_{i_1}, v] \equiv [v_{i_1}, [v_{i_2}, [x, z]]],$$

where $\langle e_{i_1}, e_x\rangle_q = -1$. Applying the Jacobi identity, we get

$$[v_{i_2}, [x, z]] = [x, [v_{i_2}, z]] + [[v_{i_2}, x], z].$$

By Lemma 3.6 (b), $[v_{i_2}, z]$ or $[v_{i_2}, x]$ is not a root, then $[x, [v_{i_2}, z]] \in (\mathfrak{p})$ or $[v_{i_2}, x] \in (\mathfrak{p})$. By the induction hypothesis $[x, [v_{i_2}, z]] \in (\mathfrak{j})$ or $[v_{i_2}, x] \in (\mathfrak{j})$. Therefore $[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_2}, z]]]$ or $[v_{i_1}, v] \equiv [v_{i_1}, [[v_{i_2}, x], z]]$. If $[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_2}, z]]]$, then applying Lemma 3.8 we get a reduction to the case 1) or to the case 3.2) below. If $[v_{i_1}, v] \equiv [v_{i_1}, [[v_{i_2}, x], z]]$, then

$$[v_{i_1}, v] \equiv [v_{i_1}, [[v_{i_2}, x], z]] = -[[v_{i_2}, x], [z, v_{i_1}]] - [z, [v_{i_1}, [v_{i_2}, x]]].$$

Note that $\ell(z) \geq 1$, because we choose k with the property $k < m$. Then $\langle e_{i_1}, e_z\rangle_q = \langle e_{i_1}, e_v\rangle_q - \langle e_{i_1}, e_x\rangle_q - \langle e_{i_1}, e_{i_2}\rangle_q = 0 - (-1) - 1 = 0$ and $\langle e_{i_1}, e_{i_2} + e_x\rangle_q = 1 - 1 = 0$, and therefore by the induction hypothesis

$$[v_{i_1}, v] = -[[v_{i_2}, x], [z, v_{i_1}]] - [z, [v_{i_1}, [v_{i_2}, x]]] \in (\mathfrak{j}).$$

3.2) Let $\langle e_{i_1}, e_{i_2}\rangle_q = -1$. Applying Lemma 6.8 and 6.9 we get

$$[v_{i_1}, v] \in (\mathfrak{j})$$

or

$$[v_{i_1}, v] \equiv [v_{i_1}, [a, x]] \equiv [v_{i_1}, [a, [b, y]]],$$

where

(i) $a = [v_{i_k}, \dots, v_{i_2}]$, $b = [v_{i_s}, v_{i_{s-1}} \dots, v_{i_{k+1}}]$, $y = [v_{i_{s+1}}, \dots v_{i_m}]$,
(ii) $\langle e_{i_1}, e_{i_j}\rangle_q = 0$, for all $j = 3, \dots, k$ and $j = k+2, \dots, s$,
(iii) $\langle e_{i_1}, e_{i_2}\rangle_q = -1$, $\langle e_{i_1}, e_{i_{k+1}}\rangle_q = 1$,
(iv) $\langle e_b, e_a\rangle_q = 0$,
(v) if $s < m$ then $\langle e_{i_{s+1}}, e_a\rangle_q = -1$ and $\langle e_{i_{s+1}}, e_b\rangle_q = -1$.

Consider the following cases.

(a) If $s = m$, then $[v_{i_1}, v] \equiv [v_{i_1}, [a, b]]$ and

$$q(e_a + e_b) = q(e_a) + q(e_b) + \langle e_a, e_b\rangle_q = 1 + 1 + 0 = 2.$$

Therefore $e_a + e_b$ is not a root of q, by the induction hypothesis $[a, b] \in (\mathfrak{j})$ and $[v_{i_1}, v] \equiv [v_{i_1}, [a, b]] \in (\mathfrak{j})$.

We may assume that $s < m$ and consider $\langle e_{i_1}, e_{i_{s+1}}\rangle_q$. Partial results are presented in the following tables.

$\langle e_{i_1}, e_{i_{s+1}}\rangle_q$	consequence
-1	$q(e_{i_1} + (e_{i_{s+1}} + e_a)) =$ $q(e_{i_1}) + q(e_{i_{s+1}} + e_a) + \langle e_{i_1}, e_{i_{s+1}} + e_a\rangle_q =$ $1 + 1 - 1 - 1 = 0$
1	$q(-e_{i_1} + (e_{i_{s+1}} + e_b)) =$ $q(e_{i_1}) + q(e_{i_{s+1}} + e_b) - \langle e_{i_1}, e_{i_{s+1}} + e_b\rangle_q =$ $1 + 1 - 1 - 1 = 0$
2	$i_1 = i_{s+1}$

$\langle e_{i_1}, e_{i_{s+1}}\rangle_q$	conclusions
-1	contradiction, because q is positive definite
1	contradiction, because q is positive definite
2	contradiction, because $\langle e_{i_{s+1}}, e_b\rangle_q = -1 \neq 1 = \langle e_{i_1}, e_b\rangle_q$

Therefore, we may assume that $\langle e_{i_1}, e_{i_{s+1}}\rangle_q = 0$.

(b) Let $m \geq s + 2$. Then

$$[v_{i_1}, v] \equiv [v_{i_1}, [a, [b, [v_{i_{s+1}}, z]]]],$$

where $\ell(z) \geq 1$. Moreover $\langle e_{i_{s+1}}, e_b + e_z\rangle_q = -1 - 1 = -2$, then $[z, b]$ is not a root and by the induction hypothesis $[z, b] \in (j)$. Therefore

$$\begin{aligned}[v_{i_1}, [a, [b, [v_{i_{s+1}}, z]]]] &= -[v_{i_1}, [a, [v_{i_{s+1}}, [z, b]]]] - [v_{i_1}, [a, [z, [b, v_{i_{s+1}}]]]] \\ &\equiv [v_{i_1}, [a, [[b, v_{i_{s+1}}], z]]].\end{aligned}$$

Applying the Jacobi identity, we get

$$[v_{i_1}, [a, [[b, v_{i_{s+1}}], z]]] = -[v_{i_1}, [[b, v_{i_{s+1}}], [z, a]]] - [v_{i_1}, [z, [a, [b, v_{i_{s+1}}]]]].$$

Note that $\langle e_{i_{s+1}}, e_a + e_z\rangle_q = -1 - 1 = -2$, then $[z, a]$ is not a root and by the induction hypothesis $[z, a] \in (\mathfrak{j})$. Therefore

$$\begin{aligned}[v_{i_1}, [a, [[b, v_{i_{s+1}}], z]]] &\equiv [v_{i_1}, [[a, [b, v_{i_{s+1}}]], z]] \\ &= -[[a, [b, v_{i_{s+1}}]], [z, v_{i_1}]] - [z, [v_{i_1}, [a, [b, v_{i_{s+1}}]]]].\end{aligned}$$

Note that $\langle e_{i_1}, e_a + e_b + e_{i_{s+1}}\rangle_q = -1 + 1 + 0 = 0$ and $\langle e_{i_1}, e_z\rangle_q = \langle e_{i_1}, e_v\rangle_q - \langle e_{i_1}, e_a + e_b + e_{i_{s+1}}\rangle_q = 0$. By the induction hypothesis $[v_{i_1}, [a, [b, v_{i_{s+1}}]]] \in (\mathfrak{j})$ and $[z, v_{i_1}] \in (\mathfrak{j})$. Therefore $[v_{i_1}, v] \in (\mathfrak{j})$.

(c) Let $m = s+1$. We recall that $\langle e_{i_1}, e_{i_2}\rangle_q = -1$, $\langle e_{i_1}, e_{i_{s+1}}\rangle_q = 1$ and $\langle e_{i_1}, e_{i_j}\rangle_q = 0$ for all $j = 3, \ldots, s$. Applying 1) and the Jacobi identity, it is straightforward to prove the following conditions:

(i) $[v_{i_1}, v] \equiv (-1)^{\varepsilon}[v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_{s+1}}, v_{i_s}, \ldots, v_{i_{k+1}}]$,

(ii) $[v_{i_1}, v] \equiv (-1)^{\varepsilon}[v_{i_1}, [[v_{i_j}, \ldots, v_{i_2}], v_{i_{j+1}}, \ldots, v_{i_{k+1}}]]$, for all $j = 2, \ldots, k, s+1, s \ldots, k+2$.

Without loss of generality, we can assume that in both cases $\varepsilon = 0$. It follows from (i) that there exists a numbering of elements $\{i_2, \ldots, i_{s+1}\}$, such that

$$[v_{i_1}, v] \equiv [v_{i_1}, v_{i_2}, \ldots, v_{i_{s+1}}],$$

where, $[v_{i_2}, \ldots, v_{i_{s+1}}]$ is a root, $\langle e_{i_1}, e_{i_2}\rangle_q = -1$, $\langle e_{i_1}, e_{i_{s+1}}\rangle_q = 1$ and $\langle e_{i_1}, e_{i_j}\rangle_q = 0$ for $j = 3, \ldots, s$. Moreover, it follows from (ii) that the elements $[v_{i_j}, \ldots, v_{i_2}]$ for all $j = 2, \ldots, k, s+1, s \ldots, k+2$, are roots, because otherwise $[v_{i_j}, \ldots, v_{i_2}] \in \mathfrak{j}$ and $[v_{i_1}, v] \in \mathfrak{j}$.

We claim that $\langle e_{i_j}, e_{i_{j+1}}\rangle_q = -1$ for all $j = 2, \ldots, s$. Assume, for the contrary, that there exists $j = 2, \ldots, s$, such that $\langle e_{i_j}, e_{i_{j+1}}\rangle \neq -1$. If $j = s$, then $[v_{i_2}, \ldots, v_{i_{s+1}}]$ is not a root. If $j = 2$, then $[v_{i_2}, v_{i_3}] \in \mathfrak{j}$. Applying the Jacobi identity we get

$$[v_{i_1}, [v_{i_2}, \ldots, v_{i_{s+1}}]] \equiv [v_{i_1}, [v_{i_3}, [v_{i_2}, [v_{i_4}, \ldots, v_{i_{s+1}}]]]]$$

and, by the case 1), $[v_{i_1}, v] \in \mathfrak{j}$. Therefore we can assume that $j = 3, \ldots, s-1$. Since $[v_{i_2}, \ldots, v_{i_{s+1}}]$ is a root, $\langle e_{i_j}, e_{i_{j+1}} + \ldots + e_{i_{s+1}}\rangle_q = -1$ and $\langle e_{i_j}, e_{i_{j+2}} + \ldots + e_{i_{s+1}}\rangle_q \geq -1$. By the bilinearity of $\langle -, -\rangle_q$, Lemma 2.5(c) and our assumptions, we have $\langle e_{i_j}, e_{i_{j+2}} + \ldots + e_{i_{s+1}}\rangle = -1$ and $\langle e_{i_j}, e_{i_{j+1}}\rangle = 0$. By assumptions and (ii), the elements $[v_{i_j}, v_{i_{j-1}}, \ldots, v_{i_2}]$ and $[v_{i_{j+1}}, v_{i_{j+2}}, \ldots, v_{i_{s+1}}]$ are roots. Moreover $\langle e_{i_{j+1}}, e_{i_2} + \ldots + e_{i_{j-1}}\rangle_q = \langle e_{i_{j+1}}, e_{i_2} + \ldots + e_{i_{j-1}} + e_{i_j}\rangle_q = -1$. Set $y = [v_{i_{j-1}}, \ldots, v_{i_2}]$ and $x = [v_{i_{j+2}}, \ldots, v_{i_{s+1}}]$, then

$$\begin{aligned}
1 = q(v) &= q(e_y + e_{i_j} + e_{i_{j+1}} + e_x)\\
&= q(e_y + e_{i_j}) + q(e_{i_{j+1}} + e_x) + \langle e_y + e_{i_j}, e_{i_{j+1}} + e_x\rangle_q\\
&= 2 + \langle e_y, e_{i_{j+1}}\rangle_q + \langle e_y, e_x\rangle_q + \langle e_{i_j}, e_{i_{j+1}}\rangle_q + \langle e_{i_j}, e_x\rangle_q\\
&= 2 + (-1) + \langle e_y, e_x\rangle_q + 0 + (-1)\\
&= \langle e_a, e_x\rangle_q.
\end{aligned}$$

It follows

$$\begin{aligned}
q(-e_x + (e_y + e_{i_1})) &= q(e_x) + q(e_y + e_{i_1}) - \langle e_x, e_y + e_{i_1}\rangle_q\\
&= 1 + 1 - \langle e_x, e_y\rangle_q - \langle e_x, e_{i_1}\rangle_q\\
&= 2 - 1 - 1 = 0,
\end{aligned}$$

because $\langle e_x, e_{i_1}\rangle_q = \langle e_{i_{s+1}}, e_{i_1}\rangle_q = 1$. This is a contradiction, because q is positive definite. Finally, we proved that $\langle e_{i_j}, e_{i_{j+1}}\rangle_q = -1$ for all $j = 2, \ldots, s$.

If $\langle e_{i_j}, e_{i_l}\rangle_q \leq 0$ for all $2 \leq j < l \leq s+1$, then $(i_1, i_2, \ldots, i_{s+1})$ is a positive chordless cycle and therefore $[v_{i_1}, v_{i_2}, \ldots, v_{i_{s+1}}] \in \mathfrak{j}$. Indeed, if

$(i_1, i_2, \ldots, i_{s+1})$ is not a positive chordless cycle, then there exists $2 \leq j < l \leq s+1$ such that $l \neq j, j+1$ and $\langle e_{i_j}, e_{i_l}\rangle_q = -1$. Therefore $q(e_{i_2} + \ldots + e_{i_{s+1}}) \leq s - (s-1) + \langle e_{i_j}, e_{i_l}\rangle_q = 0$ and q is not positive definite.

Assume that $\langle e_{i_j}, e_{i_l}\rangle_q > 0$ for some $2 \leq j < l \leq s+1$. Choose j, l such that $2 \leq j < l-1 \leq s+1$ and $l - j$ is minimal with the property $\langle e_{i_j}, e_{i_l}\rangle_q \neq 0$.

If $\langle e_{i_j}, e_{i_l}\rangle_q = -1$, then $q(e_{i_j} + \ldots + e_{i_l}) = 0$. If $\langle e_{i_j}, e_{i_l}\rangle_q = 2$, then $q(e_{i_j} + \ldots + e_{i_l}) = -1$. In both cases q is not positive definite.

Therefore $\langle e_{i_j}, e_{i_l}\rangle_q = 1$. Note that in this case $(i_j, i_{j+1}, \ldots, i_l)$ is a positive chordless cycle and $[v_{i_l}, v_{i_{l-1}}, \ldots, v_{i_{j+1}}]$ is a root. If $l = s+1$, then

$$v \equiv [v_{i_2}, \ldots, v_{i_j}, \ldots, v_{i_l}] \in \mathfrak{j},$$

by the definition. Therefore we can assume that $l < s+1$. If $j = 2$, then

$$\begin{aligned} [v_{i_1}, v] &\equiv [v_{i_j}, [[v_{i_l}, v_{i_{l-1}}, \ldots, v_{i_{j+1}}], [v_{i_{l+1}}, \ldots, v_{i_{s+1}}]] \\ &\equiv [[v_{i_l}, v_{i_{l-1}}, \ldots, v_{i_{j+1}}], [v_{i_j}[v_{i_{l+1}}, \ldots, v_{i_{s+1}}]], \end{aligned}$$

because $[v_{i_j}, [v_{i_l}, v_{i_{l-1}}, \ldots, v_{i_{j+1}}]] \in \mathfrak{j}$. It follows by 1), that $[v_{i_1}, v] \in \mathfrak{j}$, because $\langle e_{i_1}, e_{i_{j+1}} + \ldots + e_{i_l}\rangle_q = 0$. Therefore we can assume that $2 < j < l < s+1$ and

$$[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_j}, [b, y]]]],$$

where $x = [v_{i_{j-1}}, \ldots, v_{i_2}]$, $b = [v_{i_l}, v_{i_{l-1}}, \ldots, v_{i_{j+1}}]$ and $y = [v_{i_{l+1}}, \ldots, v_{i_{s+1}}]$. Since e_x, $e_{i_j} + e_b + e_y$ and $e_x + e_{i_j} + e_b + e_y$ are roots of q, by Lemma 2.5(b′) we have $\langle e_x, e_{i_j} + e_b + e_y\rangle_q = -1$. Consider

$$\begin{aligned} q(-e_y + (e_{i_1} + e_x)) &= q(e_y) + q(e_{i_1} + e_x) - \langle e_{i_1} + e_x, e_y\rangle_q \\ &= 1 + 1 - \langle e_{i_1}, e_y\rangle_q - \langle e_x, e_y\rangle_q \\ &= 1 - \langle e_x, e_y\rangle_q, \end{aligned}$$

because $\langle e_{i_1}, e_y\rangle_q = \langle e_{i_1}, e_{i_{s+1}}\rangle_q = 1$ and $\langle e_{i_1}, e_x\rangle_q = \langle e_{i_1}, e_{i_2}\rangle_q = -1$.

On the other hand

$$[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_j}, [b, y]]]] = -[v_{i_1}, [v_{i_j}, [[b, y], x]]] - [v_{i_1}, [[b, y], [x, v_{i_j}]]].$$

Therefore $\langle e_{i_j}, e_x\rangle_q = -1$, because otherwise by the induction hypothesis we have $[x, v_{i_j}] \in \mathfrak{j}$, and by 1),

$$[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_j}, [b, y]]]] = -[v_{i_1}, [v_{i_j}, [[b, y], x]]] \in \mathfrak{j}.$$

Similarly we have $\langle e_x, e_b\rangle_q = -1$, because otherwise

$$[v_{i_1}, v] \equiv [v_{i_1}, [x, [v_{i_j}, [b, y]]]] \equiv [v_{i_1}, [x, [b, [v_{i_j}, y]]]] \equiv [v_{i_1}, [b, [[y, v_{i_j}], x]]] \in \mathfrak{j},$$

by the case 1).

Finally

$$\langle e_x, e_y\rangle_q = -1 - \langle e_x, e_{i_j} + e_b\rangle_q = -1 - \langle e_x, e_{i_j}\rangle_q - \langle e_x, e_b\rangle_q = 1$$

and $q(-e_y + (e_x + e_{i_1})) = 1 - \langle e_x, e_y\rangle_q = 0$. This is a contradiction, because q is positive definite. This finishes the proof. □

7. Examples and final remarks

In this section we present some examples and remarks that illustrate basic results of this paper.

Theorem 7.1. *If $A = \mathbb{C}Q/I$ is a representation directed $\mathbb{C}$-algebra, such that its Tits form q_A is positive definite, then the map*

$$\Phi : L(q_A, \mathfrak{j}) \to \mathcal{K}(A) \tag{7.2}$$

given by $v_i \mapsto u_i$ is an isomorphism of Lie algebras. Moreover $L(q_A, \mathfrak{j}) \cong G^+(q_A)$.

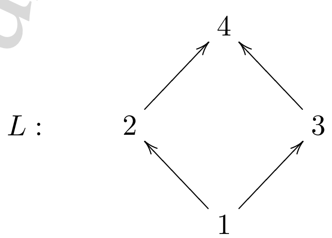
Proof. By Corollary 5.2, the map $\Phi : L(q_A, \mathfrak{r}) \to \mathcal{K}(A)$, given by $\Phi(v_i) = u_i$, is an isomorphism of Lie algebras. By Propositions 6.3 and 6.10, we have $L(q_A, \mathfrak{j}) = L(q_A, \mathfrak{r})$, because q_A is positive definite. The isomorphism $L(q_A, \mathfrak{j}) \cong G^+(q_A)$ follows from Proposition 4.4. □

Remark 7.3. Let A be a representation directed $\mathbb{C}$-algebra and let q_A be its Tits form. It is well-known (see [6]) that q_A is weakly positive. It follows that the set $\mathcal{R}^+_{q_A}$ of positive roots of q_A is finite. Therefore $\dim_{\mathbb{C}} \mathcal{K}(A) = |\mathcal{R}^+_{q_A}|$ is finite. In this case, the subset $\mathfrak{r}$ of $L(q_A)$ is finite, even if q_A is not positive definite. Moreover, we are able to describe an algorithm that constructs the set $\mathfrak{r}$. Indeed, it is enough to develop Definition 3.7 and construct all Weyl roots of q (see [8, Remark 4.15]).

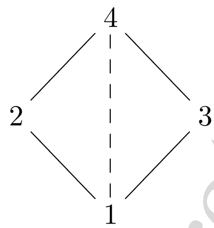
If q_A is positive definite, then $L(q_A, \mathfrak{j}) = L(q_A, \mathfrak{r})$. The set $\mathfrak{j}$ is a minimal set generating the ideal $(\mathfrak{r})$ and $\mathfrak{j}$ is smaller than $\mathfrak{r}$ (see Example 7.4).

If q_A is not positive definite, then $(\mathfrak{j}) \subsetneq (\mathfrak{r})$ in general (see Example 7.5).

Example 7.4. Let L be the following poset



and let KL be the incidence algebra of the poset L (see [15]). It is easy to see that KL is representation directed, q_{KL} is positive definite and $B(q_{KL})$ has the form



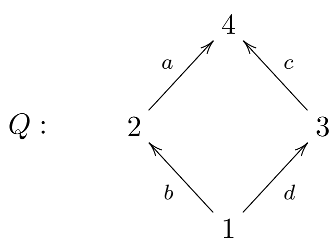
Then

$$\begin{aligned}\mathfrak{j} \;=\; & \{[u_2,u_3],[u_1,u_4],[u_1,[u_1,u_2]],[u_1,[u_1,u_3]],[u_2,[u_1,u_2]],[u_3,[u_1,u_3]],\\ & [u_2,[u_2,u_4]],[u_3,[u_3,u_4]],[u_4,[u_2,u_4]],[u_4,[u_3,u_4]],[u_1[u_2,u_4]],\\ & [u_1,[u_3,u_4]]\}\end{aligned}$$

and $L(q_{KL},\mathfrak{j}) \cong \mathcal{K}(KL)$. Note that

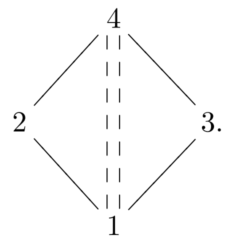
$$\begin{aligned}\mathfrak{r} \;=\; & \{[u_2,u_3],[u_1,u_4],[u_1,[u_1,u_2]],[u_1,[u_1,u_3]],[u_2,[u_2,u_1]],[u_3,[u_3,u_1]],\\ & [u_2,[u_2,u_4]],[u_3,[u_3,u_4]],[u_4,[u_4,u_2]],[u_4,[u_4,u_3]],[u_1[u_2,u_4]],\\ & [u_1,[u_3,u_4]],[u_3,u_2],[u_4,u_1],[u_1,[u_2,u_1]],[u_1,[u_3,u_1]],[u_2,[u_1,u_2]],\\ & [u_3,[u_1,u_3]],[u_2,[u_4,u_2]],[u_3,[u_4,u_3]],[u_4,[u_2,u_4]],[u_4,[u_3,u_4]],\\ & [u_1[u_4,u_2]],[u_1,[u_4,u_3]],[u_1,u_2,u_1,u_3],[u_2,u_2,u_1,u_3],[u_3,u_2,u_1,u_3],\\ & [u_1,u_3,u_1,u_2],[u_2,u_3,u_1,u_2],[u_3,u_3,u_1,u_2],[u_1,u_2,u_3,u_1],\\ & [u_2,u_2,u_3,u_1],[u_3,u_2,u_3,u_1],[u_1,u_3,u_2,u_1],[u_2,u_3,u_2,u_1],\\ & [u_3,u_3,u_2,u_1],[u_4,u_2,u_4,u_3],[u_2,u_2,u_4,u_3],[u_3,u_2,u_4,u_3],\\ & [u_4,u_3,u_4,u_2],[u_2,u_3,u_4,u_2],[u_3,u_3,u_4,u_2],[u_4,u_2,u_3,u_4],\\ & [u_2,u_2,u_3,u_4],[u_3,u_2,u_3,u_4],[u_4,u_3,u_2,u_4],[u_2,u_3,u_2,u_4],[u_3,u_3,u_2,u_4],\\ & [u_1,u_4,u_2,u_3,u_1],[u_2,u_4,u_2,u_3,u_1],[u_3,u_4,u_2,u_3,u_1],[u_4,u_4,u_2,u_3,u_1],\\ & [u_1,u_4,u_3,u_2,u_1],[u_2,u_4,u_3,u_2,u_1],[u_3,u_4,u_3,u_2,u_1],[u_4,u_4,u_3,u_2,u_1],\\ & [u_1,u_1,u_2,u_3,u_4],[u_2,u_1,u_2,u_3,u_4],[u_3,u_1,u_2,u_3,u_4],[u_4,u_1,u_2,u_3,u_4],\\ & [u_1,u_1,u_3,u_2,u_4],[u_2,u_1,u_3,u_2,u_4],[u_3,u_1,u_3,u_2,u_4],[u_4,u_1,u_3,u_2,u_4],\\ & [u_1,u_4,u_2,u_1,u_3],[u_2,u_4,u_2,u_1,u_3],[u_3,u_4,u_2,u_1,u_3],[u_4,u_4,u_2,u_1,u_3],\\ & [u_1,u_4,u_3,u_1,u_2],[u_2,u_4,u_3,u_1,u_2],[u_3,u_4,u_3,u_1,u_2],[u_4,u_4,u_3,u_1,u_2],\\ & [u_1,u_1,u_2,u_4,u_3],[u_2,u_1,u_2,u_4,u_3],[u_3,u_1,u_2,u_4,u_3],[u_4,u_1,u_2,u_4,u_3],\\ & [u_1,u_1,u_3,u_4,u_2],[u_2,u_1,u_3,u_4,u_2],[u_3,u_1,u_3,u_4,u_2],[u_4,u_1,u_3,u_4,u_2]\}\end{aligned}$$

Example 7.5. Consider the following graph



Let $A = \mathbb{C}Q/I$, where $I = (ab, cd)$. The form q_A is not positive definite and $B(q_A)$ has the following form

$B(q_A)$:



Note that $[u_4, [u_3, [u_2, u_1]]] \in (\mathfrak{r})$, but $[u_4, [u_3, [u_2, u_1]]] \notin (\mathfrak{j})$. On the other hand, the algebra A is representation directed and q_A is weakly positive. By Corollary 5.2, $\mathcal{K}(A) \cong L(q_A, \mathfrak{r})$.

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