

On tame semigroups generated by idempotents with partial null multiplication

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ABSTRACT. Let I be a finite set without 0 and J a subset in $I \times I$ without diagonal elements (i, i) . We define $S(I, J)$ to be the semigroup with generators e_i , where $i \in I \cup 0$, and the following relations: $e_0 = 0$; $e_i^2 = e_i$ for any $i \in I$; $e_i e_j = 0$ for any $(i, j) \in J$. In this paper we study finite-dimensional representations of such semigroups over a field k . In particular, we describe all finite semigroups $S(I, J)$ of tame representation type.

Introduction

We study finite-dimensional representations over a field k of semigroups generated by idempotents.

Let I be a finite set without 0 and J a subset in $I \times I$ without diagonal elements (i, i) . We define $S(I, J)$ to be the semigroup with generators e_i , where $i \in I \cup 0$, and the following relations:

- 1) $e_0 = 0$;
- 2) $e_i^2 = e_i$ for any $i \in I$;
- 3) $e_i e_j = 0$ for any pair $(i, j) \in J$.

The set of all semigroups of the form $S(I, J)$ is denoted by \mathcal{I} . We call $S(I, J) \in \mathcal{I}$ a *semigroup generated by idempotents with partial null multiplication*.

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In this paper we give a criterion for $S(I, J)$ to be of finite representation type and a criterion for a finite $S(I, J)$ to be tame (note that any semigroup $S(I, J)$ of finite type is finite).

1. Formulation of the main results

Throughout the paper, k denotes a field. All vector spaces are finite-dimensional vector spaces over k . Under consideration maps, morphisms, etc., we keep the right-side notation.

Let S be a semigroup and let $M_n(k)$ denotes the algebra of all $n \times n$ matrices with entries in k . A *matrix representation of S (of degree n) over k* is a homomorphism T from S to the multiplicative semigroup of $M_n(k)$. If there is an identity (resp. zero) element $a \in S$, we assume that the matrix $T(a)$ is identity (resp. zero). Since $M_n(k)$ can be considered as the algebra of all linear transformations of any fixed n -dimensional vector space, we can consider representations of the semigroup S in terms of vector spaces and linear transformations. Thus, a *representation of S over k* is a homomorphism φ from S to the multiplicative semigroup of the algebra $End_k U$ with U being a finite-dimensional vector space. Two representation $\varphi : S \rightarrow End_k U$ and $\varphi' : S \rightarrow End_k U'$ are called *equivalent* if there is a linear map $\sigma : U \rightarrow U'$ such that $\varphi\sigma = \varphi'$.

A representation $\varphi : S \rightarrow End_k U$ of S is also denoted by (U, φ) . By the dimension of (U, φ) one means the dimension of U . The representations of S form a category which will be denoted by $rep_k S$ (it has as morphisms from (U, φ) to (U', φ') the maps σ such that $\varphi\sigma = \varphi'$). Since representations $X, Y \in S$ are equivalent iff they are isomorphic as objects of $rep_k S$, we will use both the terms.

In an analogous way we can define representations of the semigroup S over a (not necessarily finite-dimensional) k -algebra Λ ; in this case we must take free Λ -modules of finite rank instead of finite-dimensional vector spaces.

We say that a semigroup is *of finite representation type over k* if it has only finitely many equivalent classes of indecomposable representations (over k), and *of infinite type* if otherwise. Further, we say that a semigroup is *of tame* (respectively, *wild*) *type*, or simply *tame* (respectively, *wild*), if so is the problem of classifying its representations (precise definitions are given below).

Let $S = S(I, J) \in \mathcal{I}$ and $\bar{J} = \{(i, j) \in (I \times I) \setminus J \mid i \neq j\}$. We may assume, without loss of generality, that $I = \{1, 2, \dots, m\}$. With the semigroup $S = S(I, J)$ we associate the quadratic form $f_S(z) : \mathbb{Z}^m \rightarrow \mathbb{Z}$

in the following way:

$$f_S(z) = \sum_{i \in I} z_i^2 - \sum_{(i,j) \in \bar{J}} z_i z_j.$$

We call $f_S(z)$ the quadratic form of the semigroup S .

In this paper we prove the following theorems.

Theorem 1. *A semigroup $S(I, J)$ is of finite representation type over k if and only if its quadratic form is positive (then $S(I, J)$ is finite).*

Theorem 2. *Let $S(I, J)$ be a finite semigroup. Then $S(I, J)$ is tame over k if its quadratic form is nonnegative, and wild if otherwise.*

2. Connections between representations of $S(I, J)$ and representations of quivers

We first recall the notion of representations of a quiver [1].

Let $Q = (Q_0, Q_1)$ be a (finite) quiver, where Q_0 is the set of its vertices and Q_1 is the set of its arrows $\alpha : x \rightarrow y$.

A representation of the quiver $Q = (Q_0, Q_1)$ over a field k is a pair $R = (V, \gamma)$ formed by a collection $V = \{V_x \mid x \in Q_0\}$ of vector spaces V_x and a collection $\gamma = \{\gamma_\alpha \mid \alpha : x \rightarrow y \text{ runs through } Q_1\}$ of linear maps $\gamma_\alpha : V_x \rightarrow V_y$. A morphism from $R = (V, \gamma)$ to $R' = (V', \gamma')$ is given by a collection $\lambda = \{\lambda_x \mid x \in Q_0\}$ of linear maps $\lambda_x : V_x \rightarrow V'_x$, such that $\gamma_\alpha \lambda_y = \lambda_x \gamma'_\alpha$ for any arrow $\alpha : x \rightarrow y$.

The category of representations of $Q = (Q_0, Q_1)$ will be denoted by $\text{rep}_k Q$.

In an analogous way we can define representations of the quiver Q over a (not necessarily finite-dimensional) k -algebra Λ ; in this case we must take free Λ -modules of finite rank instead of finite-dimensional vector spaces.

A quiver Q is said to be of finite representation type over k if $\text{rep}_k Q$ has only finitely many isomorphism classes of indecomposable representations (over k), and of infinite representation type if otherwise. Further, Q is said to be of tame (respectively, wild) representation type, or simply tame (respectively, wild), if so is the problem of classifying its representations (precise definitions are given below).

Now we proceed to investigate connections between representations of $S(I, J)$ and representations of quivers.

We identify a linear map α of $U = U_1 \oplus \dots \oplus U_p$ into $V = V_1 \oplus \dots \oplus V_q$ with the matrix (α_{ij}) , $i = 1, \dots, p$, $j = 1, \dots, q$, where $\alpha_{ij} : U_i \rightarrow V_j$ are

the linear maps induced by α (then the sum and the composition of maps are given by the matrix rules).

For a finite set X and $Y \subseteq X \times X$, we denote by $Q(X, Y)$ the quiver with vertex set X and arrows $a \rightarrow b$, $(a, b) \in Y$.

Let $S = S(I, J)$, where, as before, $I = \{1, 2, \dots, m\}$. Define the functor F from $\text{rep}_k Q(I, \bar{J})$ to $\text{rep}_k S(I, J)$ as follows. $F = F(I, J)$ assigns to each object $(V, \gamma) \in \text{rep}_k Q(I, \bar{J})$ the object $(V', \gamma') \in \text{rep}_k S(I, J)$, where $V' = \bigoplus_{i \in I} V_i$, $(\gamma'(e_i))_{jj} = \mathbf{1}_{V_j}$ if $i = j$, $(\gamma'(e_i))_{ij} = \gamma_{ij}$ if $(i, j) \in \bar{J}$, and $(\gamma'(e_i))_{js} = 0$ in all other cases. F assigns to each morphism λ of $\text{rep}_k Q(I, \bar{J})$ the morphism $\bigoplus_{i \in I} \lambda_i$ of $\text{rep}_k S(I, J)$.

Proposition 1. *The functor $F = F(I, J) : \text{rep}_k Q(I, \bar{J}) \rightarrow \text{rep}_k S(I, J)$ is full and faithful.*

Proof. It is obvious that the functor F is faithful. It remains to prove that it is full. Let δ be a morphism from $(V, \gamma)F = (V', \gamma')$ to $(W, \sigma)F = (W', \sigma')$. In other words, δ is a linear map of V' into W' such that $\gamma'(e_s)\delta = \delta\sigma'(e_s)$ for $s = 1, \dots, m$. We will consider these equalities as matrix ones (taking into account that $V' = \bigoplus_{i \in I} V_i$ and $W' = \bigoplus_{i \in I} W_i$) and denote by $[s, i, j]$ the scalar equality $(\gamma'(e_s)\delta)_{ij} = (\delta\sigma'(e_s))_{ij}$, induced by the (matrix) equality $\gamma'(e_s)\delta = \delta\sigma'(e_s)$.

From an equation $[j, i, j]$ with $j \neq i$ it follows that $\delta_{ij} = 0$, and consequently δ is a diagonal matrix: $\delta = \delta_{11} \oplus \delta_{22} \oplus \dots \oplus \delta_{mm}$. Further, if $\alpha : i \rightarrow j$ is an arrow of the quiver $Q(I, \bar{J})$, then from the equation $[i, i, j]$ we have that $\gamma_\alpha \delta_{jj} = \delta_{ii} \sigma_\alpha$. Consequently, a collection $\bar{\delta} = \{\delta_{ss} \mid s = 1, \dots, m\}$ is a morphism from (V, γ) to (W, σ) . Since $\delta = \delta_{11} \oplus \delta_{22} \oplus \dots \oplus \delta_{mm}$, we have that $\delta = \lambda F$, where $\lambda = \bar{\delta}$, as claimed. \square

Proposition 2. *If the quiver $Q(I, \bar{J})$ has no oriented cycles, then each object of $\text{rep}_k S(I, J)$ is isomorphic to an object of the form $X F(I, J) \oplus (W, 0)$, where X is an object of $\text{rep}_k Q(I, \bar{J})$ (W is a vector space of dimension $d \geq 0$ and $0 : W \rightarrow W$ is the zero map).*

Proof. For simplicity, the quiver $Q(I, \bar{J})$ is denoted by $Q = (Q_0, Q_1)$. The proof will be by induction on m , the case $m = 0, 1$ being trivial.

Now let $m > 1$ and let $R = (U, \varphi)$ be a representation of $S(I, J)$. Fix $s \in Q_0$ such that there is no arrow $i \rightarrow s$; obviously, one can assume that $s = m$. We consider the subsemigroup S' of S generated by e_i , $i \in I' \cup 0$, where $I' = \{1, \dots, m-1\}$. Then $S' = S(I', J')$ with $J' = \{(i, j) \in I \times I \mid i, j \in I'\}$, and $Q' = Q(I', \bar{J}')$ is the full subquiver of Q with vertex set $Q'_0 = I'$.

Denote by $R' = (U, \varphi')$ the restriction of R to S' ($\varphi'(x) = \varphi(x)$ for any $x \in S'$). It follows by induction that $R' \cong \bar{R}' = X' F(I', J') \oplus (W', 0)$,

where X' is a representation of the quiver $Q(I', \bar{J}')$. Let $\bar{R}' = (\bar{U}, \bar{\varphi}')$ and $X' = (V', \gamma')$ with $V' = \{V'_i \mid i \in Q'_0\}$ and $\gamma' = \{\gamma'_\alpha \mid \alpha : i \rightarrow j \text{ runs through } Q'_1\}$. Since $R' \cong \bar{R}'$, there exists a linear map $\sigma : U \rightarrow \bar{U} = V'_1 \oplus V'_2 \oplus \dots \oplus V'_{m-1} \oplus W'$ such that $\varphi'\sigma = \bar{\varphi}'$. Then the representation $R = (U, \varphi)$ is equivalent to the representation $\bar{R} = (\bar{U}, \bar{\varphi})$, where $\bar{\varphi}(e_i) = \bar{\varphi}'(e_i)$ for any $i = 1, \dots, m-1$ and $\bar{\varphi}(e_m) = \varphi(e_m)\sigma$ (because, for $i \neq m$, $\bar{\varphi}'(e_i) = \varphi'(e_i)\sigma = \varphi(e_i)\sigma$, and so $\bar{\varphi}(x) = \varphi(x)\sigma$ for any $x \in S$).

We consider the representation $\bar{R} = (\bar{U}, \bar{\varphi})$ in more detail. We set $V_m = W'$ and consider $\bar{\varphi}$ as a matrix (taking into account that $\bar{U} = V_1 \oplus V_2 \oplus \dots \oplus V_{m-1} \oplus V_m$). For $(p, q) \in J$, we denote by $[p, q, i, j]$ the scalar equality $[\bar{\varphi}(e_p)\bar{\varphi}(e_q)]_{ij} = 0$, induced by the (matrix) equality $\bar{\varphi}(e_p)\bar{\varphi}(e_q) = 0$ (the last equation holds since $e_p e_q = 0$ in $S(I, J)$). It follows from $[m, q, i, q]$ (for any fixed $q \neq m$) that $(\bar{\varphi}(e_m))_{iq} = 0$, and consequently $(\bar{\varphi}(e_m))_{ij} = 0$ for any $(i, j) \in I \times I'$.

We first consider two special cases: a) $\bar{\varphi}_{mm} = 0$; b) $\bar{\varphi}_{mm} = \mathbf{1} = \mathbf{1}_{V_m}$.

In case a) $(\bar{\varphi})^2 = \bar{\varphi}$ implies $\bar{\varphi} = 0$ and so $\bar{R} = XF(I, J) \oplus (W, 0)$ with $X = (V, \gamma)$, where $V = \{V'_1, \dots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

In case b) an equality $[p, m, p, m]$ for $(p, m) \notin \bar{J}$ implies $(\bar{\varphi})_{pm} = 0$ and so $\bar{R} = XF(I, J) \oplus (W, 0)$ with $X = (V, \gamma)$, where $V = \{V'_1, \dots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

Now we consider the general case. Since $(\bar{\varphi}_{mm})^2 = \bar{\varphi}_{mm}$, there is an invertible map $\nu = (\nu_1, \nu_2) : V_m \rightarrow W_1 \oplus W_2$ such that

$$\bar{\varphi}_{mm}(\nu_1, \nu_2) = (\nu_1, \nu_2) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1} = \mathbf{1}_{W_1}$. Then the representation $\bar{R}' = (\bar{U}, \bar{\varphi}')$ is isomorphic to the representation $\widehat{R}' = (\widehat{U}, \widehat{\varphi}')$, where $\widehat{U} = \widehat{U}_1 \oplus \widehat{U}_2 \oplus \dots \oplus \widehat{U}_{m+1}$ with $\widehat{U}_i = V_i$ for $i = 1, \dots, m-1$, $\widehat{U}_m = W_1$, $\widehat{U}_{m+1} = W_2$, and $\widehat{\varphi}'(e_i) = \bar{\varphi}'(e_i)$ for $i = 1, \dots, m-1$, $(\widehat{\varphi}'(e_m))_{ij} = (\bar{\varphi}'(e_m))_{ij}$ for $(i, j) \in I' \times I'$, $(\widehat{\varphi}'(e_m))_{ij} = 0$ for $i = m, m+1, j \in I'$, $(\widehat{\varphi}'(e_m))_{m, mj} = \mathbf{1} = \mathbf{1}_{W_1}$, $(\widehat{\varphi}'(e_m))_{m, m+1} = 0$, $(\widehat{\varphi}'(e_m))_{m+1, m} = 0$, $(\widehat{\varphi}'(e_m))_{m+1, m+1} = 0$ (for instance, one can take the isomorphism $\beta : \widehat{R}' \rightarrow \bar{R}'$ with $\widehat{\varphi}'(e_i) = \mu^{-1}\bar{R}'\mu$, where $\mu = \mathbf{1}_{U_1} \oplus \dots \oplus \mathbf{1}_{U_{m-1}} \oplus \nu$).

From $(\widehat{\varphi}'(e_i))^2 = \widehat{\varphi}'(e_i)$ it follows that $(\widehat{\varphi}'(e_m))_{i, m+1} = 0$ for any $i \in I'$ (see the partial case a)); (then $(\widehat{\varphi}'(e_m))_{i, m+1} = 0$ for any $i = 1, \dots, m+1$). From the scalar equalities $[p, m, p, m]$ for $(p, m) \notin \bar{J}$ implies $(\widehat{\varphi})_{pm} = 0$ (see the partial case b)). Thus, $\bar{R} = (\widehat{U}, \widehat{\varphi}) \cong R = (U, \varphi)$ has the form $XF(I, J) \oplus (W, 0)$, where $X = (V, \gamma)$ with $V = \{\widehat{U}_i \mid i \in Q_0\}$, $\gamma = \{\gamma_\alpha \mid \alpha : i \rightarrow j \text{ runs through } Q_1\}$ with $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = \widehat{\varphi}(e_m)_{ij}$ for $\alpha \notin Q'_1$ (then $j = m$), and $W = \widehat{W}_{m+1}$, as claimed. \square

Denote by $\text{rep}_k^\circ S(I, J)$ the full subcategory of $\text{rep}_k S(I, J)$ consisting of all objects that have no objects $(W, 0)$, with $W \neq 0$, as direct summands.

We have as an immediate consequence of Propositions 1 and 2 the following statement.

Corollary 1. *If the quiver $Q(I, \bar{J})$ has no oriented cycles, then the functor $F = F(I, J)$, viewed as a functor from $\text{rep}_k Q(I, \bar{J})$ to $\text{rep}_k^\circ S(I, J)$, is an equivalence of categories.*

3. Proof of Theorems 1 and 2

In [1] P. Gabriel introduced the quadratic Tits form $q_Q: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ of a quiver $Q = (Q_0, Q_1)$:

$$q_Q(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i \rightarrow j} z_i z_j,$$

where $i \rightarrow j$ runs through Q_1 , and proved that Q is of finite representation type if and only if its Tits form is positive.

The definitions of the quadratic Tits form of a quivers and the quadratic form of a semigroup $S(I, J) \in \mathcal{I}$ immediately imply the following lemma.

Lemma 1. *Let $S = S(I, J) \in \mathcal{I}$ and $Q = Q(I, \bar{J})$. Then the quadratic forms $f_S(z)$ and $q_Q(z)$ coincide.*

Now we prove Theorem 1. In [3] one proves that a semigroup $S(I, J)$ is finite if and only if the quiver $Q(I, \bar{J})$ has no oriented cycles (the Tits form of which are not positive [6]). Then Theorem 1 follows from Corollary 1, Lemma 1 and the above-mentioned Gabriel's results.

Before we begin to prove Theorem 2, we recall precise definitions of tame and wild semigroups (see general definitions in [2]).

For a semigroup S and a k -algebra Λ , we denote by $R_\Lambda(S)$ the set of all representations of S over Λ . By $\mathcal{L}(\Lambda)$ we denote the category of left finite-dimensional (over k) Λ -modules.

Let S be a semigroup and $\Lambda = K_1 = k[x]$. We say that a representation $N = (U, \varphi)$ from $\text{rep}_k S$ is generated by a representation $M = (V, \psi)$ from $R_\Lambda(S)$ if, for some $X \in \mathcal{L}(\Lambda)$, $N \cong M \otimes X = (V \otimes X, \psi \otimes \mathbf{1}_X)$ (the tensor products are considered over Λ).

We assume first that the field k is separable closed. The semigroup S is called *tame* if, for any fixed dimension d , there exist finitely many elements M_i of $R_\Lambda(S)$ such that, up to isomorphism, each indecomposable object of $\text{rep}_k S$ (of the dimension d) is generated by M_i for some

i. Such a set $\{M_i\}$ is called a *parametrizing family of representations of S of dimension d* .

When the field k is not separable closed, the semigroup S is called *tame*, if it is tame over the separable closure \bar{k} of k (in the case of infinite k one can take k itself in place of \bar{k}).

Now we give a definition of wild semigroups.

Let S be a semigroup and $\Lambda = K_2 = k \langle x, y \rangle$ the free associative k -algebra in two noncommuting variables x and y . A representation $M = (V, \psi)$ of S over Λ is said to be *perfect* if it satisfies the following conditions:

- 1) the representation $M \otimes X = (V \otimes X, \psi \otimes \mathbf{1}_X)$ (of S over k) with $X \in \mathcal{L}(\Lambda)$ is indecomposable if so is X ;
- 2) the representations $M \otimes X$ and $M \otimes X'$ are nonisomorphic if so are X and X' .

The semigroup S is called *wild over k* if it has a perfect representation over Λ .

In an analogous way one can define tame and wild quivers; the set of all representations of a quiver Q over an algebra Λ will be denote by $R_\Lambda(Q)$.

Now we prove Theorem 2.

Let $S = S(I, J)$ be a finite semigroup. Then the quiver $Q(I, \bar{J})$ has no oriented cycles (see above). From the papers [4, 5] on tame quivers and the paper [6] on integral quadratic forms it follows that a quiver Q is tame if its Tits form is nonnegative, and wild if otherwise. Then the first part of Theorem 2 follows from Lemma 1, Corollary 1 and the obvious fact that, for $\Lambda = k[x]$, the map $F_\Lambda = F_\Lambda(I, J)$ from $R_\Lambda(Q)$ to $R_\Lambda(S)$, which is defined analogously to the functor $F = F(I, J)$ on objects, “preserves” (from left to right) parametrizing families of any fixed dimension. Analogously, the second part of Theorem 2 follows from Lemma 1, Corollary 1 and the obvious fact that, for $\Lambda = k \langle x, y \rangle$, the map $F_\Lambda = F_\Lambda(I, J)$ from $R_\Lambda(Q)$ to $R_\Lambda(S)$, which is defined analogously to the functor $F = F(I, J)$ on objects, “preserves” (from left to right) perfect representations over Λ .

Theorems 1 and 2 are proved.

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