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RESEARCH ARTICLE



# On $\tau$-closed totally saturated group formations with Boolean sublattices

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Abstract. In the universe of finite groups the description of $\tau$-closed totally saturated formations with Boolean sublattices of $\tau$-closed totally saturated subformations is obtained. Thus, we give a solution of Question 4.3.16 proposed by A. N. Skiba in his monograph "Algebra of Formations" (1997).

## Introduction

All groups considered are finite. Used notations and terminology are standard (see [1]–[4]). Recall that a formation $\mathfrak{F}$ is called *saturated* if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. It is known [4] that if $\mathfrak{F}$ is a non-empty saturated formation, then $\mathfrak{F} = LF(f)$, i. e., $\mathfrak{F}$ has *a local satellite* $f$.

Every group formation is considered as 0-*multiply saturated* [5]. For $n \geq 1$, a formation $\mathfrak{F} \neq \varnothing$ is called *n-multiply saturated* [5], if it has a local satellite $f$ such that every non-empty value $f(p)$ of $f$ is a $(n-1)$-multiply saturated formation. A formation is called *totally saturated* [5] if it is $n$-multiply saturated for all natural $n$.

Let $\tau$ be a function such that for any group $G$, $\tau(G)$ is a set of subgroups of $G$, and $G \in \tau(G)$. Following [3] we say that $\tau$ is a *subgroup functor* if for every epimorphism $\varphi : A \rightarrow B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

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A group class $\mathfrak{F}$ is called *τ-closed* if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$. The set $l_\infty^\tau$ of all $\tau$-closed totally saturated formations is a complete lattice [3].

A $\tau$-closed totally saturated formation $\mathfrak{F}$ is called $\mathfrak{H}_\infty^\tau$*-critical* (or *a minimal τ-closed totally saturated non-*$\mathfrak{H}$*-formation*) if $\mathfrak{F} \not\subseteq \mathfrak{H}$ but all proper $\tau$-closed totally saturated subformations of $\mathfrak{F}$ are contained in $\mathfrak{H}$.

If $\mathfrak{F}$ and $\mathfrak{M}$ are $l_\infty^\tau$-formations such that $\mathfrak{M} \subseteq \mathfrak{F}$, then $\mathfrak{F}/_\infty^\tau\mathfrak{M}$ denotes the lattice of $l_\infty^\tau$-formations between $\mathfrak{M}$ and $\mathfrak{F}$. In particular, if $\mathfrak{M} = (1)$ is the formation of identity groups, then $L_\infty^\tau(\mathfrak{F})$ denotes the lattice $\mathfrak{F}/_\infty^\tau(1)$.

In this paper we prove the following.

**Theorem 1.** *Let* $\mathfrak{F}$ *and* $\mathfrak{X}$ *be τ-closed totally saturated formations,* $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. *Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_\infty^\tau\mathfrak{F}\cap\mathfrak{X}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F}\cap\mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H}_i|i \in I))$, *where* $\{\mathfrak{H}_i|i \in I\}$ *is the set of all* $\mathfrak{X}_\infty^\tau$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of* $\mathfrak{F}$ *of the form* $(\mathfrak{F}\cap\mathfrak{X}) \vee_\infty^\tau \mathfrak{H}$ *is* $l_\infty^\tau$*-complemented in* $\mathfrak{F}/_\infty^\tau\mathfrak{F}\cap\mathfrak{X}$, *where* $\mathfrak{H}$ *is some* $\mathfrak{X}_\infty^\tau$*-critical subformation of* $\mathfrak{F}$*;*

*4) any* $\mathfrak{X}_\infty^\tau$*-critical subformation of* $\mathfrak{F}$ *has an* $\mathfrak{X}_\infty^\tau$*-complement in* $\mathfrak{F}$.

Note that if in this theorem $\mathfrak{X} = \mathfrak{N}$ and $\tau$ is the trivial subgroup functor (i. e., $\tau(G) = \{G\}$ for all groups $G$) we obtain the main result in [6]. In another special case ($\mathfrak{X} = (1)$ and $\mathfrak{F}$ is soluble) we obtain the main result of Section 4.3 in [3]. In particular, we give a solution of Question 4.3.16 in [3].

## 1. Definitions and Notations

Let $\mathfrak{X}$ be a set of groups. Then $l_\infty^\tau\mathrm{form}\mathfrak{X}$ is the *τ-closed totally saturated formation generated by* $\mathfrak{X}$, i.e., $l_\infty^\tau\mathrm{form}\mathfrak{X}$ is the intersection of all $\tau$-closed totally saturated formations containing $\mathfrak{X}$. If $\mathfrak{X} = \{G\}$, then the formation $l_\infty^\tau\mathrm{form}G$ is called a *one-generated* $\tau$-closed totally saturated formation.

We denote by $\pi(\mathfrak{F})$ the set of prime divisors of orders of groups in $\mathfrak{F}$.

For any two $\tau$-closed totally saturated formations $\mathfrak{M}$ and $\mathfrak{H}$, we write $\mathfrak{M} \vee_\infty^\tau \mathfrak{H} = l_\infty^\tau\mathrm{form}(\mathfrak{M}\cup\mathfrak{H})$.

For any set $\mathfrak{X}$ of groups, we put $\mathfrak{X}_\infty^\tau(p) = l_\infty^\tau\mathrm{form}(G/F_p(G)|G \in \mathfrak{X})$, if $p \in \pi(\mathfrak{X})$, and $\mathfrak{X}_\infty^\tau(p) = \varnothing$ if $p \notin \pi(\mathfrak{X})$.

If $\mathfrak{F}$ is an arbitrary $\tau$-closed totally saturated formation, then the symbol $\mathfrak{F}_\infty^\tau$ denotes the *minimal* $l_\infty^\tau$*-valued local satellite* of $\mathfrak{F}$.

For an arbitrary sequence of primes $p_1, p_2, \dots, p_n$ and any set $\mathfrak{X}$ of groups, the class of groups $\mathfrak{X}^{p_1p_2\cdots p_n}$ is defined as follows:

1) $\mathfrak{X}^{p_1} = (A/F_{p_1}(A)|A \in \mathfrak{X})$;

2) $\mathfrak{X}^{p_1p_2\dots p_n} = (A/F_{p_n}(A)|A \in \mathfrak{X}^{p_1p_2\dots p_{n-1}})$.

A sequence of primes $p_1, p_2, \dots, p_n$ is called *suitable* for $\mathfrak{X}$ if $p_1 \in \pi(\mathfrak{X})$ and for any $i \in \{2, \dots, n\}$ we have $p_i \in \pi(\mathfrak{X}^{p_1p_2\dots p_{i-1}})$.

Let $p_1, p_2, \dots, p_n$ be a suitable sequence for $\mathfrak{F}$. Then the $l_\infty^\tau$-valued local satellite $\mathfrak{F}_\infty^\tau p_1p_2\dots p_n$ is defined as follows:

1) $\mathfrak{F}_\infty^\tau p_1 = (\mathfrak{F}_\infty^\tau(p_1))_\infty$;

2) $\mathfrak{F}_\infty^\tau p_1\dots p_n = (\mathfrak{F}_\infty^\tau p_1\dots p_{n-1}(p_n))_\infty^\tau$.

A group $G$ is called *a $\tau$-minimal non-$\mathfrak{H}$-group* (or *an $\mathfrak{H}^\tau$-critical group*) if $G \notin \mathfrak{H}$ but every proper $\tau$-subgroup of $G$ belongs to $\mathfrak{H}$.

A $\tau$-closed totally saturated formation $\mathfrak{F}$ is called *an $l_\infty^\tau$-irreducible* formation if $\mathfrak{F} \neq l_\infty^\tau\text{form}(\cup_{i\in I}\mathfrak{X}_i) = \vee_\infty^\tau(\mathfrak{X}_i|i \in I)$, where $\{\mathfrak{X}_i|i \in I\}$ is the set of all proper $\tau$-closed totally saturated subformations of $\mathfrak{F}$. Otherwise, $\mathfrak{F}$ is called *an $l_\infty^\tau$-reducible* $\tau$-closed totally saturated formation.

Let $\mathfrak{M}$ and $\mathfrak{H}$ be some $\tau$-closed totally saturated subformations of $\mathfrak{F}$, $\mathfrak{X}$ be a class of groups. Then $\mathfrak{H}$ is called *an $\mathfrak{X}_\infty^\tau$-complement* to $\mathfrak{M}$ in $\mathfrak{F}$ if $\mathfrak{F} = l_\infty^\tau\text{form}(\mathfrak{M} \cup \mathfrak{H})$ and $\mathfrak{M} \cap \mathfrak{H} \subseteq \mathfrak{X}$. A subformation of $\mathfrak{F}$ is called *$\mathfrak{X}_\infty^\tau$-complemented* in $\mathfrak{F}$ if it has an $\mathfrak{X}_\infty^\tau$-complement in $\mathfrak{F}$. In addition, the $(1)_\infty^\tau$-complement to $\mathfrak{M}$ in $\mathfrak{F}$ is called *an $l_\infty^\tau$-complement* to $\mathfrak{M}$ in $\mathfrak{F}$, and in this case $\mathfrak{M}$ is called *$l_\infty^\tau$-complemented* in $\mathfrak{F}$. A subformation $\mathfrak{M}$ of $\mathfrak{F}$ is called *complemented* in $\mathfrak{F}$ if $\mathfrak{F} = \text{form}(\mathfrak{M} \cup \mathfrak{H})$ and $\mathfrak{M} \cap \mathfrak{H} = (1)$ for some subformation $\mathfrak{H}$ of $\mathfrak{F}$.

For a set $\pi$ of primes, we use $\mathfrak{N}_\pi$ and $\mathfrak{S}_\pi$ to denote the class of all nilpotent $\pi$-groups and the class of all soluble $\pi$-groups, respectively.

## 2. Used Results

**Lemma 1.** *[7, 8]. Let $\mathfrak{F}$ be a non-soluble $\tau$-closed totally saturated formation. Then $\mathfrak{F}$ has at leat one $\mathfrak{S}_\infty^\tau$-critical subformation.*

**Lemma 2.** *[7, 8]. Let $\mathfrak{F}$ be a $\tau$-closed totally saturated formation. Then $\mathfrak{F}$ is a minimal $\tau$-closed totally saturated non-soluble formation if and only if $\mathfrak{F} = l_\infty^\tau\text{form}G$, where $G$ is a monolithic $\tau$-minimal non-soluble group with a non-abelian minimal normal subgroup $R$ such that $G/R$ is soluble.*

**Lemma 3.** *[7, 8]. Let $G$ be a monolithic group with a non-abelian socle $R$. Then $\mathfrak{F} = l_\infty^\tau\text{form}G$ has a unique maximal $l_\infty^\tau$-subformation $\mathfrak{M} = \mathfrak{S}_{\pi(R)}l_\infty^\tau\text{form}(\{G/R\} \bigcup \mathfrak{X})$, where $\mathfrak{X}$ is the set of all proper $\tau$-subgroups of $G$. In particular, $\mathfrak{S}_{\pi(R)} \subseteq \mathfrak{M} \subset \mathfrak{F}$.*

**Lemma 4.** *[9]. The lattice $l_\infty^\tau$ of $\tau$-closed totally saturated formations is distributive.*

**Lemma 5.** *[10]. For any two $\tau$-closed totally saturated formations $\mathfrak{M}$ and $\mathfrak{F}$ we have*

$$\mathfrak{M} \vee_{\infty}^{\tau} \mathfrak{F}/_{\infty}^{\tau}\mathfrak{M} \simeq \mathfrak{F}/_{\infty}^{\tau}\mathfrak{M} \cap \mathfrak{F}.$$

**Lemma 6.** *[7, 11] The lattice $l_{\infty}^{\tau}$ is algebraic.*

## 3. Main Results

**Lemma 7.** *[7]. Let $\mathfrak{F}$, $\mathfrak{X}$ be $\tau$-closed totally saturated formations such that $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. The formation $\mathfrak{F}$ is an $\mathfrak{X}_{\infty}^{\tau}$-critical formation if and only if either of the following conditions is satisfied:*

*1) $\mathfrak{F} = \mathfrak{N}_p$, where $p \notin \pi(\mathfrak{X})$;*

*2) $\mathfrak{F} = \mathfrak{N}_p\mathfrak{N}_q$ for some different primes $p$ and $q$ in $\pi(\mathfrak{X})$.*

*Proof. Necessity.* Let $\mathfrak{F}$ be an $\mathfrak{X}_{\infty}^{\tau}$-critical formation. Suppose that there exists $p \in \pi(\mathfrak{F})$ such that $p \notin \pi(\mathfrak{X})$. Since $\mathfrak{N}_p \in l_{\infty}^{\tau}$, $\mathfrak{N}_p \subseteq \mathfrak{F} \setminus \mathfrak{X}$, (1) is the unique $l_{\infty}^{\tau}$-subformation of $\mathfrak{N}_p$ and (1) $\subseteq \mathfrak{X}$, we have that $\mathfrak{F} = \mathfrak{N}_p$. So, $\mathfrak{F}$ satisfies 1).

Assume that $\pi(\mathfrak{F}) \subseteq \pi(\mathfrak{X})$. We show that $\mathfrak{F}$ is soluble.

Assume that $\mathfrak{F} \not\subseteq \mathfrak{S}$. Then by Lemma 1, $\mathfrak{F}$ contains at least one $\mathfrak{S}_{\infty}^{\tau}$-critical subformation $\mathfrak{L}$. By Lemma 2, $\mathfrak{L} = l_{\infty}^{\tau}\mathrm{form}L$, where $L$ is a monolithic $\tau$-minimal non-soluble group with a non-abelian minimal normal subgroup $N$ such that group $L/N$ is soluble. It follows from Lemma 3 that $\mathfrak{S}_{\pi} \subset \mathfrak{L}$, where $\pi = \pi(N)$. Since $N$ is non-abelian, we have that $|\pi| \geq 3$. But by hypothesis the formation $\mathfrak{F}$ is an $\mathfrak{X}_{\infty}^{\tau}$-critical formation. Hence $\mathfrak{S}_{\pi} \subseteq \mathfrak{X} \subseteq \mathfrak{N}$, a contradiction. Therefore, $\mathfrak{F}$ is soluble.

Let $h$ is the canonical local satellite of $\mathfrak{X}$. By Theorem 2.5.2 [3, p. 94], $\mathfrak{F} = l_{\infty}^{\tau}\mathrm{form}G$, where $G$ is a group of minimal order in $\mathfrak{F} \setminus \mathfrak{X}$ with the socle $R = G^{\mathfrak{X}}$ such that for all $p \in \pi(R)$ the formation $\mathfrak{F}_{\infty}^{\tau}(p)$ is $(h(p))_{\infty}^{\tau}$-critical. Since by Theorem 1.3.14 [3, p. 33] $\mathfrak{N}_{\infty}^{\tau}(p) = (1)$, we have $h(p) = \mathfrak{N}_p$. Hence, $\mathfrak{F}_{\infty}^{\tau}(p) = l_{\infty}^{\tau}\mathrm{form}(G/F_p(G))$ is an $(\mathfrak{N}_p)_{\infty}^{\tau}$-critical formation. Therefore, $|\pi(\mathfrak{F}_{\infty}^{\tau}(p))| = 1$ and $\mathfrak{F}_{\infty}^{\tau}(p) = \mathfrak{N}_q$, for some prime $q \neq p$. Since $G$ is soluble, it follows that $R$ is a $p$-group and $F_p(G) = R$. Hence, $\pi(G) = \{p, q\}$ and $\mathfrak{F} = \mathfrak{N}_p\mathfrak{N}_q$. Thus, $\mathfrak{F}$ satisfies 2).

*Sufficiency.* Let $\mathfrak{F}$ be a formation satisfying 1) or 2). Then $\mathfrak{F}$ is a hereditary totally saturated formation. Hence, $\mathfrak{F}$ is a $\tau$-closed formation, for any subgroup functor $\tau$. If $\mathfrak{F} = \mathfrak{N}_p$, then (1) is a unique maximal $l_{\infty}^{\tau}$-subformation of $\mathfrak{F}$. But (1) $\subseteq \mathfrak{X} \neq \emptyset$. Hence, $\mathfrak{F}$ is an $\mathfrak{X}_{\infty}^{\tau}$-critical formation.

Let $\mathfrak{F} = \mathfrak{N}_p\mathfrak{N}_q$. Then by Theorem 2.5.3. [3, p. 94] $mathfrakF$ is an $\mathfrak{N}_{\infty}^{\tau}$-critical formation. Since $\mathfrak{N}_{\{p,q\}} \subseteq \mathfrak{X}$, it follows that $\mathfrak{F}$ is a minimal $\tau$-closed totally saturated non-$\mathfrak{X}$-formation. □

**Lemma 8.** *[7]. Let $\mathfrak{F}$ and $\mathfrak{X}$ be $l_\infty^\tau$-formations such that $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. Then $\mathfrak{F}$ has at least one $\mathfrak{X}_\infty^\tau$-critical subformation.*

*Proof.* Assume that $\pi(\mathfrak{F}) \not\subseteq \pi(\mathfrak{X})$ and $p \in \pi(\mathfrak{F}) \setminus \pi(\mathfrak{X})$. Then according to Lemma 6, $\mathfrak{N}_p$ is a required $\mathfrak{X}_\infty^\tau$-critical formation. Now we assume that $\pi(\mathfrak{F}) \subseteq \pi(\mathfrak{X})$, and let $A$ be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{X}$. Then $A$ is a monolithic $\tau$-minimal non-$\mathfrak{X}$-group with the socle $R = A^{\mathfrak{X}}$. Let $p \in \pi(R)$ and $\mathfrak{L} = l_\infty^\tau \mathrm{form} A$. Assume that $R$ is non-abelian. Then by Lemma 3, $\mathfrak{S}_{\pi(R)} \subseteq \mathfrak{L}$. Since $|\pi(R)| \geq 3$, there exists a prime $q \neq p$, $q \in \pi(R)$, such that

$$\mathfrak{M} = \mathfrak{N}_p \mathfrak{N}_q \subset \mathfrak{S}_{\pi(R)} \subset \mathfrak{F}.$$

Since $\mathfrak{N}_{\{p,q\}} \subseteq \mathfrak{X}$, from Lemma 6 it follows that $\mathfrak{M}$ is a required $\mathfrak{X}_\infty^\tau$-critical formation.

Suppose now that $R$ is an abelian $p$-group. Since $R \not\subseteq \Phi(A)$, we have $R = O_p(A) = F_p(A)$ and $A = [R]B$ for some maximal subgroup $B$ in $A$. By Theorem 1.3.14 [3, p. 33],

$$\mathfrak{L}_\infty^\tau(p) = l_\infty^\tau \mathrm{form}(A/F_p(A)) = l_\infty^\tau \mathrm{form} B.$$

Let $q \in \pi(B) \setminus \{p\}$, and $Q$ be a group of prime order $q$. Since $\mathfrak{L}_\infty^\tau(p)$ is totally saturated, $Q \in \mathfrak{L}_\infty^\tau(p)$. Denote by $V$ an exact irreducible $F_p[Q]$-modul, and let $F = [V]Q$. Then

$$F/O_p(F) \simeq Q \in \mathfrak{L}_\infty^\tau(p).$$

Therefore, by Lemma 8.2 [2, p. 78], $F \in \mathfrak{L}$. But

$$\mathfrak{F} = l_\infty^\tau \mathrm{form} F = \mathfrak{N}_p \mathfrak{N}_q.$$

Hence, by Lemma 6, $\mathfrak{F}$ is a required $\mathfrak{X}_\infty^\tau$-critical formation. □

**Lemma 9.** *Let $\mathfrak{X}$, $\mathfrak{M}$ and $\mathfrak{F}$ be $\tau$-closed totally saturated formations such that $\mathfrak{M} \subseteq \mathfrak{X} \subseteq \mathfrak{N}$, and $\mathfrak{F} = \mathfrak{M} \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H}_i | i \in I))$, where $\{\mathfrak{H}_i | i \in I\}$ is some set of $\mathfrak{X}_\infty^\tau$-critical formations. If $\mathfrak{H}$ is an $\mathfrak{X}_\infty^\tau$-critical subformation of $\mathfrak{F}$, then $\mathfrak{H} \in \{\mathfrak{H}_i | i \in I\}$.*

*Proof.* Let $\mathfrak{H}$ be a $\mathfrak{X}_\infty^\tau$-critical subformation of $\mathfrak{F}$. By Lemma 6, $\mathfrak{H}$ satisfies either of the following conditions:

1) $\mathfrak{H} = \mathfrak{N}_p$, where $p \notin \pi(\mathfrak{X})$;

2) $\mathfrak{H} = \mathfrak{N}_p \mathfrak{N}_q$ for some primes $p \neq q$ in $\pi(\mathfrak{X})$.

Assume that $\mathfrak{H}$ satisfies 1). Since $\mathfrak{H} \subseteq \mathfrak{F}$, we have by Corollary 1.3.10 [3, p. 31] that $\mathfrak{H}_\infty^\tau \leq \mathfrak{F}_\infty^\tau$. Therefore, $\mathfrak{H}_\infty^\tau(p) \subseteq \mathfrak{F}_\infty^\tau(p)$. By Theorem 1.3.14 [3, p. 33], we have $\mathfrak{H}_\infty^\tau(p) = (1)$. Hence, $(1) \subseteq \mathfrak{F}_\infty^\tau(p) \neq \varnothing$. By Lemma 4.1.2 [3, p. 152],

$$\mathfrak{F}_\infty^\tau(p) = \mathfrak{M}_\infty^\tau(p) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H}_{i\infty}^\tau(p) | i \in I)).$$

Since $p \notin \pi(\mathfrak{X})$, it follows that $p \notin \pi(\mathfrak{M})$ and $\mathfrak{M}^{\tau}_{\infty}(p) = \varnothing$. Hence,

$$\mathfrak{F}^{\tau}_{\infty}(p) = \vee^{\tau}_{\infty}(\mathfrak{H}^{\tau}_{i\infty}(p) | i \in I).$$

Suppose that $p \notin \pi(\mathfrak{H}_i)$ for all $i \in I$. Then from Theorem 1.3.14 [3, p. 33] it follows that $\mathfrak{H}^{\tau}_{i\infty}(p) = \varnothing$ for all $i \in I$. Therefore, $\mathfrak{F}^{\tau}_{\infty}(p) = \varnothing$, a contradiction. So, there exists $i \in I$ such that $p \in \pi(\mathfrak{H}_i)$. Since $\mathfrak{H}_i$ is an $\mathfrak{X}^{\tau}_{\infty}$-critical formation and $p \notin \pi(\mathfrak{X})$, we see that $\mathfrak{H}_i = \mathfrak{N}_p$. Thus, $\mathfrak{H}_i = \mathfrak{H}$.

Assume that $\mathfrak{H}$ satisfies 2). Then $p, q$ is a suitable sequence for $\mathfrak{H}$ and $\mathfrak{F}$. By Corollary 1.3.10 and Theorem 1.3.14 [3], we obtain that

$$\mathfrak{H}^{\tau}_{\infty}(p) \subseteq \mathfrak{F}^{\tau}_{\infty}(p) \text{ and } \mathfrak{H}^{\tau}_{\infty}p(q) = (1) \subseteq \mathfrak{F}^{\tau}_{\infty}p(q) \neq \varnothing.$$

From Lemma 4.1.2 [3, p. 152] it follows that

$$\mathfrak{F}^{\tau}_{\infty}p(q) = \mathfrak{M}^{\tau}_{\infty}p(q) \vee^{\tau}_{\infty} (\vee^{\tau}_{\infty}(\mathfrak{H}^{\tau}_{i\infty}p(q) | i \in I)).$$

Suppose that $q \in \pi(\mathfrak{M}^{\tau}_{\infty}(p))$. Since $\mathfrak{M}^{\tau}_{\infty}(p)$ is a saturated formation, we have that $\mathfrak{N}_q \subseteq \mathfrak{M}^{\tau}_{\infty}(p)$. By Theorem 1.3.12 [3, p. 32],

$$\mathfrak{N}_p\mathfrak{M}^{\tau}_{\infty}(p) \subseteq \mathfrak{M}.$$

Hence,

$$\mathfrak{H} = \mathfrak{N}_p\mathfrak{N}_q \subseteq \mathfrak{N}_p\mathfrak{M}^{\tau}_{\infty}(p) \subseteq \mathfrak{M} \subseteq \mathfrak{X}.$$

But $\mathfrak{H}$ is an $\mathfrak{X}^{\tau}_{\infty}$-critical formation. We have a contradiction. Therefore, $q \notin \pi(\mathfrak{M}^{\tau}_{\infty}(p))$, $\mathfrak{M}^{\tau}_{\infty}p(q) = \varnothing$ and

$$\mathfrak{F}^{\tau}_{\infty}p(q) = (\vee^{\tau}_{\infty}(\mathfrak{H}^{\tau}_{i\infty}p(q) | i \in I)).$$

If $\mathfrak{H}^{\tau}_{i\infty}p(q) = \varnothing$ for all $i \in I$, then $\mathfrak{F}^{\tau}_{\infty}p(q) = \varnothing$. It is impossible. Therefore, there exists $i \in I$ such that $\mathfrak{H}^{\tau}_{i\infty}p(q) \neq \varnothing$. Hence, $q \in \pi(\mathfrak{H}^{\tau}_{i\infty}(p))$ and $\mathfrak{N}_q \subseteq \mathfrak{H}^{\tau}_{i\infty}(p)$. But by Theorem 1.3.12 [3] we have $\mathfrak{N}_p\mathfrak{H}^{\tau}_{\infty}(p) \subseteq \mathfrak{H}_i$. Therefore,

$$\mathfrak{H} = \mathfrak{N}_p\mathfrak{N}_q \subseteq \mathfrak{N}_p\mathfrak{H}^{\tau}_{i\infty}(p) \subseteq \mathfrak{H}_i.$$

Since $\mathfrak{H}_i$ is an $\mathfrak{X}^{\tau}_{\infty}$-critical formation, we see that $\mathfrak{H}_i = \mathfrak{H}$. □

**Lemma 10.** *я Let $\mathfrak{X}$, $\mathfrak{M}$, $\mathfrak{L}$, and $\mathfrak{F}$ be $\tau$-closed totally saturated formations such that $\mathfrak{X} \subseteq \mathfrak{M} \subseteq \mathfrak{L} \subseteq \mathfrak{F}$. If $\mathfrak{H}$ is an $l^{\tau}_{\infty}$-complement to $\mathfrak{M}$ in $\mathfrak{F}/^{\tau}_{\infty}\mathfrak{X}$, then $\mathfrak{H} \cap \mathfrak{L}$ is an $l^{\tau}_{\infty}$-complement to $\mathfrak{M}$ in $\mathfrak{L}/^{\tau}_{\infty}\mathfrak{X}$.*

*Proof.* Let $\mathfrak{H}_1 = \mathfrak{H} \cap \mathfrak{L}$. Since $\mathfrak{M}$ is $l^{\tau}_{\infty}$-complemented in the lattice $\mathfrak{F}/^{\tau}_{\infty}\mathfrak{X}$ by $\mathfrak{H}$, it follows that $\mathfrak{M} \cap \mathfrak{H} = \mathfrak{X}$ and $\mathfrak{M} \vee^{\tau}_{\infty} \mathfrak{H} = \mathfrak{F}$. From Lemma 4 it follows that

$$\mathfrak{M} \vee^{\tau}_{\infty} \mathfrak{H}_1 = \mathfrak{M} \vee^{\tau}_{\infty} (\mathfrak{H} \cap \mathfrak{L}) = (\mathfrak{M} \vee^{\tau}_{\infty} \mathfrak{H}) \cap (\mathfrak{M} \vee^{\tau}_{\infty} \mathfrak{L}) = \mathfrak{F} \cap \mathfrak{L} = \mathfrak{L}.$$

Besides,

$$\mathfrak{M} \cap \mathfrak{H}_1 = \mathfrak{M} \cap (\mathfrak{H} \cap \mathfrak{L}) = \mathfrak{M} \cap \mathfrak{H} = \mathfrak{X}.$$

But then $\mathfrak{H}_1$ is an $l_\infty^\tau$-complement to $\mathfrak{M}$ in $\mathfrak{L}/_\infty^\tau \mathfrak{X}$. □

**Lemma 11.** *Let $\mathfrak{X}$ and $\mathfrak{F}$ be $\tau$-closed totally saturated formations, $\mathfrak{H}$ be some $\mathfrak{X}_\infty^\tau$-critical subformation of $\mathfrak{F}$. Then $\mathfrak{H}$ has an $\mathfrak{X}_\infty^\tau$-complement in $\mathfrak{F}$ if and only if $\mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$ has an $l_\infty^\tau$-complement in $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$.*

*Proof.* Let $\mathfrak{M}$ be an $\mathfrak{X}_\infty^\tau$-complement to $\mathfrak{H}$ in $\mathfrak{F}$. Then by definition $\mathfrak{H} \cap \mathfrak{M} \subseteq \mathfrak{X}$ and $\mathfrak{H} \vee_\infty^\tau \mathfrak{M} = \mathfrak{F}$. Put $\mathfrak{M}_1 = \mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$ and $\mathfrak{H}_1 = \mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. Then $\mathfrak{M}_1$ and $\mathfrak{H}_1$ are elements of the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. By Lemma 4,

$$\mathfrak{H}_1 \cap \mathfrak{M}_1 = \mathfrak{H}_1 \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) = (\mathfrak{H}_1 \cap \mathfrak{M}) \vee_\infty^\tau (\mathfrak{H}_1 \cap (\mathfrak{F} \cap \mathfrak{X})) =$$

$$(\mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) \cap \mathfrak{M}) \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) = (\mathfrak{H} \cap \mathfrak{M}) \vee_\infty^\tau (\mathfrak{M} \cap \mathfrak{X}) \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) = \mathfrak{F} \cap \mathfrak{X}.$$

Besides,

$$\mathfrak{H}_1 \vee_\infty^\tau \mathfrak{M}_1 = \mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau \mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) = \mathfrak{F}.$$

Therefore, $\mathfrak{M}_1$ is an $l_\infty^\tau$-complement to $\mathfrak{H}_1$ in the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$.

Conversely, assume that $\mathfrak{H}_1$ has an $l_\infty^\tau$-complement $\mathfrak{M}$ in the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Then $\mathfrak{H}_1 \cap \mathfrak{M} = \mathfrak{F} \cap \mathfrak{X}$ and $\mathfrak{H}_1 \vee_\infty^\tau \mathfrak{M} = \mathfrak{F}$. Hence, by definition, $\mathfrak{M}$ is an $\mathfrak{X}_\infty^\tau$-complement to $\mathfrak{H}_1$ in $\mathfrak{F}$. □

*Proof of Theorem 1.* For an arbitrary $l_\infty^\tau$-formation $\mathfrak{L}$, we denote by $\Omega(\mathfrak{L})$ the set of all its $\mathfrak{X}_\infty^\tau$-critical subformations.

Assume that for $\mathfrak{F}$ Condition 1) is true, and $\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau (\mathfrak{H} | \mathfrak{H} \in \Omega(\mathfrak{F})))$. Assume that $\mathfrak{M} \neq \mathfrak{F}$. Since $\mathfrak{F} \cap \mathfrak{X} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$, $\mathfrak{M}$ is an element of the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Let $\mathfrak{L}$ be an $l_\infty^\tau$-complement to $\mathfrak{M}$ in the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Then $\mathfrak{M} \vee_\infty^\tau \mathfrak{L} = \mathfrak{F}$ and $\mathfrak{M} \cap \mathfrak{L} = \mathfrak{F} \cap \mathfrak{X}$. If $\mathfrak{L} \subseteq \mathfrak{X}$, then $\mathfrak{L} \subseteq \mathfrak{F} \cap \mathfrak{X} \subseteq \mathfrak{M}$ and $\mathfrak{F} = \mathfrak{M} \vee_\infty^\tau \mathfrak{L} = \mathfrak{M}$, which contradicts to our assumption. Therefore, $\mathfrak{L} \not\subseteq \mathfrak{X}$. Hence, by Lemma 8, the formation $\mathfrak{L}$ contains at least one $\mathfrak{X}_\infty^\tau$-critical subformation $\mathfrak{H}$. Since $\mathfrak{H} \subseteq \mathfrak{L} \subseteq \mathfrak{F}$, we have that $\mathfrak{H} \in \Omega(\mathfrak{F}) \subseteq \mathfrak{M}$. But then $\mathfrak{H} \subseteq \mathfrak{L} \cap \mathfrak{M} = \mathfrak{F} \cap \mathfrak{X}$, a contradiction. Hence, $\mathfrak{M} = \mathfrak{F}$.

Now we show that Condition 2) implies Condition 3). Let $\mathfrak{H}_1$ be an $\mathfrak{X}_\infty^\tau$-critical subformation of the formation $\mathfrak{F}$, $\Sigma = \Omega(\mathfrak{F}) \setminus \{\mathfrak{H}_1\}$,

$$\mathfrak{L} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau \mathfrak{H}_1 \quad \text{and} \quad \mathfrak{M} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau (\mathfrak{H} | \mathfrak{H} \in \Sigma)).$$

Then $\mathfrak{L} \vee_\infty^\tau \mathfrak{M} = \mathfrak{F}$. Suppose that $\mathfrak{L} \cap \mathfrak{M} \neq \mathfrak{F} \cap \mathfrak{X}$. Since $\mathfrak{F} \cap \mathfrak{X} \subseteq \mathfrak{L} \cap \mathfrak{M}$, we have $\mathfrak{L} \cap \mathfrak{M} \not\subseteq \mathfrak{F} \cap \mathfrak{X}$, i.e., $\mathfrak{L} \cap \mathfrak{M} \not\subseteq \mathfrak{X}$. Then by Lemma 8, $\mathfrak{L} \cap \mathfrak{M}$ contains some $\mathfrak{X}_\infty^\tau$-critical subformation $\mathfrak{H}_2$. Since $\mathfrak{H}_2 \subseteq \mathfrak{L}$, it follows from Lemma 9 that $\mathfrak{H}_2 = \mathfrak{H}_1$. But $\mathfrak{H}_2 \subseteq \mathfrak{M}$. Hence by Lemma 9, $\mathfrak{H}_2 \in \Sigma$, a contradiction. Thus, $\mathfrak{L} \cap \mathfrak{M} = \mathfrak{F} \cap \mathfrak{X}$. It means that the formation $\mathfrak{L}$ is $l_\infty^\tau$-complemented in the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. So, Condition 3) is true for $\mathfrak{F}$.

Now we assume that for $\mathfrak{F}$ Condition 3) is true. We show that Condition 1) is true. By Lemma 4, the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$ is distributive. Therefore, it is enough to establish that $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$ is a complemented lattice.

Let $\mathfrak{M}$ be an $l_\infty^\tau$-irreducible $\tau$-closed totally saturated subformation of $\mathfrak{F}$, $\mathfrak{M} \not\subseteq \mathfrak{X}$. We prove that $\mathfrak{M}$ is an $\mathfrak{X}_\infty^\tau$-critical formation. Suppose that it is false, and let $\mathfrak{M}_1$ be a maximal $l_\infty^\tau$-subformation in $\mathfrak{M}$. Since $\mathfrak{M}$ is non-$\mathfrak{X}_\infty^\tau$-critical, $\mathfrak{M}_1 \not\subseteq \mathfrak{X}$. Hence, by Lemma 8 the formation $\mathfrak{M}_1$ has at least one $\mathfrak{X}_\infty^\tau$-critical subformation $\mathfrak{H}$. Let $\mathfrak{L} = \mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. Then $\mathfrak{L}$ is an element of the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Let $\mathfrak{R}$ be an $l_\infty^\tau$-complement to $\mathfrak{L}$ in $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Then $\mathfrak{F} = \mathfrak{R} \vee_\infty^\tau \mathfrak{L}$ and $\mathfrak{R} \cap \mathfrak{L} = \mathfrak{F} \cap \mathfrak{X}$. By Lemma 11, $\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}))$ is an $l_\infty^\tau$-complement to $\mathfrak{L}$ in the lattice $\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Therefore,

$$(\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}))) \vee_\infty^\tau \mathfrak{L} = \mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})).$$

By Lemma 4,

$$\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) = (\mathfrak{R} \cap \mathfrak{M}) \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}).$$

It means that

$$\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) \subseteq \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}).$$

Since $\mathfrak{L} \subseteq \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$ we have that

$$(\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}))) \vee_\infty^\tau \mathfrak{L} \subseteq \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}).$$

But $(\mathfrak{R} \cap (\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}))) \vee_\infty^\tau \mathfrak{L} = \mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. Hence,

$$\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) \subseteq \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}).$$

The inverse inclusion is obvious. Therefore,

$$\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) = \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}).$$

But by Lemma 5 we have a lattice isomorphism

$$\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})/_\infty^\tau \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) = \mathfrak{M} \vee_\infty^\tau (\mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}))/_\infty^\tau \mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X}) \simeq$$

$$\simeq \mathfrak{M}/_\infty^\tau \mathfrak{M} \cap (\mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) = \mathfrak{M}/_\infty^\tau (\mathfrak{M} \cap \mathfrak{M}_1) \vee_\infty^\tau (\mathfrak{M} \cap \mathfrak{F} \cap \mathfrak{X})) =$$

$$= \mathfrak{M}/_\infty^\tau \mathfrak{M}_1 \cap (\mathfrak{M} \cap \mathfrak{X}) = \mathfrak{M}/_\infty^\tau \mathfrak{M}_1.$$

Therefore, $\mathfrak{M}_1 \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$ is a maximal $\tau$-closed totally saturated subformation of the formation $\mathfrak{M} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. We obtain a contradiction. Hence, $\mathfrak{M}$ is an $\mathfrak{X}_\infty^\tau$-critical formation.

We show now that for any $l_\infty^\tau$-formation $\mathfrak{R}$ in $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$ such that the set of all its $\mathfrak{X}_\infty^\tau$-critical subformations is finite, the following equality is true:

$$\mathfrak{R} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H} | \mathfrak{H} \in \Omega(\mathfrak{R}))). \qquad (\alpha)$$

We shall prove $(\alpha)$ by induction on $|\Omega(\mathfrak{R})|$. If $\mathfrak{R}$ is an $l_\infty^\tau$-irreducible formation, then from above we know that $\mathfrak{R}$ is a $\mathfrak{X}_\infty^\tau$-critical formation, and $(\alpha)$ is true. Let $\mathfrak{R}$ be an $l_\infty^\tau$-reducible formation. Since $\mathfrak{R} \not\subseteq \mathfrak{X}$, we have by Lemma 8 that $\mathfrak{R}$ contains some $\mathfrak{X}_\infty^\tau$-critical formation $\mathfrak{H}$. Let $\mathfrak{H}_1 = \mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. By hypothesis, $\mathfrak{H}_1$ has an $l_\infty^\tau$-complement $\mathfrak{M}$ in the lattice $\mathfrak{F}/_\infty^\tau(\mathfrak{F} \cap \mathfrak{X})$.

By Lemma 11, $\mathfrak{M} \cap \mathfrak{R}$ is a complement to $\mathfrak{H}_1$ in the lattice $\mathfrak{R}/_\infty^\tau(\mathfrak{F} \cap \mathfrak{X})$. Then

$$(\mathfrak{M} \cap \mathfrak{R}) \cap \mathfrak{H}_1 = \mathfrak{F} \cap \mathfrak{X} \text{ and } (\mathfrak{M} \cap \mathfrak{R}) \vee_\infty^\tau \mathfrak{H}_1 = \mathfrak{R}.$$

Since $\mathfrak{H} \not\subseteq \mathfrak{M}$, the number of $\mathfrak{X}_\infty^\tau$-critical subformations of $\mathfrak{M} \cap \mathfrak{R}$ is less than the number of $\mathfrak{X}_\infty^\tau$-critical subformations in $\mathfrak{R}$. Therefore, by induction we can conclude that

$$\mathfrak{M} \cap \mathfrak{R} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{B} | \mathfrak{B} \in \Omega(\mathfrak{M} \cap \mathfrak{R}))).$$

Hence,

$$\mathfrak{R} = (\mathfrak{M} \cap \mathfrak{R}) \vee_\infty^\tau \mathfrak{H}_1 =$$
$$= ((\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{B} | \mathfrak{B} \in \Omega(\mathfrak{M} \cap \mathfrak{R})))) \vee_\infty^\tau (\mathfrak{H} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})) =$$
$$= (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{B} | \mathfrak{B} \in \Omega(\mathfrak{R}))),$$

i.e., $(\alpha)$ is true.

Let now $\mathfrak{M}$ be an $l_\infty^\tau$-subformation of $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$. Assume that

$$\mathfrak{L} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H} | \mathfrak{H} \in \Omega(\mathfrak{F}) \setminus \Omega(\mathfrak{M}))).$$

We show that $\mathfrak{L}$ is an $l_\infty^\tau$-complement to $\mathfrak{M}$ in the lattice $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$.

It is obvious that $\mathfrak{F} \cap \mathfrak{X} \subseteq \mathfrak{M} \cap \mathfrak{L}$. If $\mathfrak{M} \cap \mathfrak{L} \not\subseteq \mathfrak{F} \cap \mathfrak{X}$, then by Lemma 8, $\mathfrak{M} \cap \mathfrak{L}$ has at least one $\mathfrak{X}_\infty^\tau$-critical subformation $\mathfrak{H}$. But then, using Lemma 9, we have that $\mathfrak{H} \in \Omega(\mathfrak{M}) \cap (\Omega(\mathfrak{F}) \setminus \Omega(\mathfrak{M})) = \varnothing$, a contradiction. Hence, $\mathfrak{M} \cap \mathfrak{L} = \mathfrak{F} \cap \mathfrak{X}$.

Let $\mathfrak{F}_1 = \mathfrak{L} \vee_\infty^\tau \mathfrak{M}$. Suppose that $\mathfrak{F}_1 \neq \mathfrak{F}$ and $G$ is a group in $\mathfrak{F} \setminus \mathfrak{F}_1$.

Since $\pi(G)$ is a finite set, by Lemma 7 the set of all $\mathfrak{X}_\infty^\tau$-critical subformations of the formation $\mathfrak{R} = l_\infty^\tau \mathrm{form} G$ is finite. Denote by $\mathfrak{R}_1$ the formation $\mathfrak{R} \vee_\infty^\tau (\mathfrak{F} \cap \mathfrak{X})$. By Lemma 9, the set of all $\mathfrak{X}_\infty^\tau$-critical subformations of the formation $\mathfrak{R}_1$ is finite. Therefore, by ($\alpha$) we have that

$$\mathfrak{R}_1 = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty^\tau (\vee_\infty^\tau(\mathfrak{H}|\mathfrak{H} \in \Omega(\mathfrak{R}))).$$

Since $\Omega(\mathfrak{R}_1) \subseteq \Omega(\mathfrak{F}) = \Omega(\mathfrak{L}) \cup \Omega(\mathfrak{M})$ and $\mathfrak{F} \cap \mathfrak{X} \subseteq \mathfrak{F}_1$, it follows that $\mathfrak{R}_1 \subseteq \mathfrak{F}_1$. Therefore, $G \in \mathfrak{F}_1$, a contradiction. So, $\mathfrak{F} = \mathfrak{F}_1$, and $\mathfrak{F}/_\infty^\tau \mathfrak{F} \cap \mathfrak{X}$ is a complemented lattice. □

In particular, if $\mathfrak{X} = (1)$, from Theorem 1 we deduce the following result.

**Theorem 2.** *Let $\mathfrak{F}$ be a $\tau$-closed totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice $L_\infty^\tau(\mathfrak{F})$ is Boolean;*

*2) $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$;*

*3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.*

*Proof.* By Lemma 7, any $(1)_\infty^\tau$-critical formation $\mathfrak{H}$ has a form $\mathfrak{H} = \mathfrak{N}_p$, where $p$ is a prime. Therefore by Theorem 1,

$$\mathfrak{F} = \vee_\infty^\tau(\mathfrak{N}_p \,|\, p \in \pi(\mathfrak{F})) = \mathfrak{N}_{\pi(\mathfrak{F})}.$$

Thus, Conditions 1) and 2) are equivalent to Conditions 1) and 2) of Theorem 1.

Now we show that any subformation $\mathfrak{N}_p$ of $\mathfrak{F}$ has a complement in $\mathfrak{F}$. By Theorem 1, Condition 2) is equivalent to the following: every subformation $\mathfrak{N}_p$ of $\mathfrak{F}$ has an $l_\infty^\tau$-complement. Let $\mathfrak{M}$ be an $l_\infty^\tau$-complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. Then $\mathfrak{N}_p \vee_\infty^\tau \mathfrak{M} = \mathfrak{F}$ and $\mathfrak{N}_p \cap \mathfrak{M} = (1)$. By Theorem 1.3.16 [3, p. 34], $\mathfrak{F} = \mathrm{form}(\cup_{q\in\pi(\mathfrak{F})}\mathfrak{N}_q\mathfrak{F}_\infty^\tau(q))$. Since $\mathfrak{F} \subseteq \mathfrak{N}$, we have by Theorem 1.3.14 [3, p. 33] that $\mathfrak{F}_\infty^\tau(q) = (1)$. It means that $\mathfrak{F} = \mathrm{form}(\cup_{q\in\pi(\mathfrak{F})}\mathfrak{N}_q)$. Since $\mathfrak{M}$ is contained in $\mathfrak{N}$ and is an $l_\infty^\tau$-formation, we have by Theorem 1.3.16 [3, p. 34] that

$$\mathfrak{M} = \mathrm{form}(\cup_{q\in\pi(\mathfrak{M})}\mathfrak{N}_q) = \mathfrak{N}_{\pi(\mathfrak{F})\setminus\{p\}}.$$

Hence,

$$\mathfrak{F} = \mathrm{form}(\mathfrak{N}_p \cup (\cup_{q\in\pi(\mathfrak{F})\setminus\{p\}}\mathfrak{N}_q)) =$$
$$= \mathrm{form}(\mathfrak{N}_p \cup \mathrm{form}(\cup_{q\in\pi(\mathfrak{F})\setminus\{p\}}\mathfrak{N}_q)) = \mathrm{form}(\mathfrak{N}_p \cup \mathfrak{M}).$$

Thus, $\mathfrak{M}$ is a complement to $\mathfrak{N}_p$ in $\mathfrak{F}$.

Let $\mathfrak{L}$ be a complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. Then $\mathfrak{N}_p \vee \mathfrak{L} = \mathfrak{F}$ and $\mathfrak{N}_p \cap \mathfrak{L} = (1)$. We show that $\mathfrak{L}$ is an $l_\infty^\tau$-complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. Let $\mathfrak{M} = l_\infty^\tau\mathrm{form}\mathfrak{L}$. Suppose that $\mathfrak{M} \not\subseteq \mathfrak{L}$, and let $A$ be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{L}$. Then $A$ is a monolithic group, and $R = \mathrm{Soc(A)} = A^{\mathfrak{L}}$. Since $A \in \mathfrak{N}$, we conclude that $A$ is a $p$-group. If $A \neq R$, then from $A/R \in \mathfrak{L}$ we have

$\mathfrak{N}_p \cap \mathfrak{L} \neq (1)$, a contradiction. It means that $A = R$, and $A$ is a group of order $p$. By Theorem 1.1.5 [3, p. 14], $\pi(\mathfrak{M}) = \pi(\mathfrak{L})$. Therefore, $p \in \pi(\mathfrak{L})$. Since $\mathfrak{L} \subseteq \mathfrak{N}$, we have $\mathfrak{N}_p \cap \mathfrak{L} \neq (1)$, a contradiction. Hence, $\mathfrak{M} = \mathfrak{L}$. Thus, $\mathfrak{L}$ is an $l_\infty^\tau$-complement to $\mathfrak{N}_p$ in $\mathfrak{F}$. □

Theorem 2 gives the answer to Question 4.3.16 [3, p. 178].

In the case when $\tau(G) = S(G)$ is the set of all subgroups of $G$, from Theorem 1 we have the following.

**Corollary 1.** *Let $\mathfrak{F}$ be a hereditary totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice $L_\infty^S(\mathfrak{F})$ is Boolean;*

*2) $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$;*

*3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.*

If $\tau(G) = S_n(G)$ is the set of all normal subgroups of $G$, from Theorem 1 we have

**Corollary 2.** *Let $\mathfrak{F}$ be a normal hereditary totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice $L_\infty^{S_n}(\mathfrak{F})$ is Boolean;*

*2) $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$;*

*3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.*

**Corollary 3.** *[3, p. 177]. Let $\mathfrak{F}$ be a soluble totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice $L_\infty(\mathfrak{F})$ is Boolean;*

*2) $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$;*

*3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.*

Let $\tau$ be a trivial subgroup functor. Then from Theorem 1 we obtain the following.

**Corollary 4.** *Let $\mathfrak{F}$ and $\mathfrak{X}$ be totally saturated formations, $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. Then the following conditions are equivalent:*

*1) the lattice $\mathfrak{F}/_\infty \mathfrak{F} \cap \mathfrak{X}$ is Boolean;*

*2) $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{X}) \vee_\infty (\vee_\infty(\mathfrak{H}_i | i \in I))$, where $\{\mathfrak{H}_i | i \in I\}$ is the set of all $\mathfrak{X}_\infty$-critical subformations of $\mathfrak{F}$;*

*3) every subformation of the form $(\mathfrak{F} \cap \mathfrak{X}) \vee_\infty \mathfrak{H}$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}/_\infty \mathfrak{F} \cap \mathfrak{X}$, where $\mathfrak{H}$ is some $\mathfrak{X}_\infty$-critical subformation of $\mathfrak{F}$;*

*4) any $\mathfrak{X}_\infty$-critical subformation of $\mathfrak{F}$ has an $\mathfrak{X}_\infty$-complement in $\mathfrak{F}$.*

**Corollary 5.** *[12]. Let $\mathfrak{F}$ be a totally saturated formation. Then the following conditions are equivalent:*

*1) $L_\infty^\tau(\mathfrak{F})$ is a complemented lattice;*

*2)* $\mathfrak{F} = \mathfrak{N}_{\pi(\mathfrak{F})}$*;*

*3) the lattice* $L_{\infty}^{\tau}(\mathfrak{F})$ *is Boolean;*

*4) every subformation of the form* $\mathfrak{N}_p$ *in* $\mathfrak{F}$ *is complemented in* $\mathfrak{F}$*.*

In the case when $\mathfrak{X} = \mathfrak{N}$ from Theorem 1 we have

**Corollary 6.** *Let* $\mathfrak{F}$ *be a non-nilpotent* $\tau$*-closed totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_{\infty}^{\tau}\mathfrak{F} \cap \mathfrak{N}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty}^{\tau} (\vee_{\infty}^{\tau}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{N}_{\infty}^{\tau}$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of the form* $(\mathfrak{F} \cap \mathfrak{X}) \vee_{\infty}^{\tau} \mathfrak{H}$ *in* $\mathfrak{F}$ *is* $l_{\infty}^{\tau}$*-complemented in* $\mathfrak{F}/_{\infty}^{\tau}\mathfrak{F} \cap \mathfrak{X}$*, where* $\mathfrak{H}$ *is some* $\mathfrak{N}_{\infty}^{\tau}$*-critical subformations of* $\mathfrak{F}$*.*

*4) every subformation of the form* $\mathfrak{N}_p\mathfrak{N}_q$ *in* $\mathfrak{F}$ *has an* $\mathfrak{N}_{\infty}^{\tau}$*-complement in* $\mathfrak{F}$*.*

**Corollary 7.** *[6]. Let* $\mathfrak{F}$ *be a non-nilpotent totally saturated formation. Then the following conditions are equivalent:*

*1)* $\mathfrak{F}/_{\infty}\mathfrak{F} \cap \mathfrak{N}$ *is a complemented lattice;*

*2) formation* $\mathfrak{F}$ *is soluble, and the lattice* $\mathfrak{F}/_{\infty}\mathfrak{F} \cap \mathfrak{N}$ *is algebraic; furthermore,* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty} (\vee_{\infty}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{N}_{\infty}$*-critical subformations in* $\mathfrak{F}$*;*

*3) the lattice* $\mathfrak{F}/_{\infty}^{\tau}\mathfrak{F} \cap \mathfrak{N}$ *is Boolean.*

*Proof.* By Lemma 7, every $\mathfrak{N}_{\infty}$-critical formation is soluble. Then from Condition 2) of Theorem 1 the formation $\mathfrak{F}$ is soluble. By Lemma 6, the lattice $l_{\infty}^{\tau}$ is algebraic for every subgroup functor $\tau$. Therefore, the lattice $\mathfrak{F}/_{\infty}^{\tau}\mathfrak{F} \cap \mathfrak{N}$ is also algebraic (it is a sublattice of complete algebraic lattice $l_{\infty}^{\tau}$). Applying Theorem 1 and Lemma 4 we conclude that Conditions 1) and 3) are equivalent. □

**Corollary 8.** *Let* $\mathfrak{F}$ *and* $\mathfrak{X}$ *be hereditary totally saturated formations,* $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$*. Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_{\infty}^{S}\mathfrak{F} \cap \mathfrak{X}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{X}) \vee_{\infty}^{S} (\vee_{\infty}^{S}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{X}_{\infty}^{S}$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of the form* $(\mathfrak{F} \cap \mathfrak{X}) \vee_{\infty}^{S} \mathfrak{H}$ *in* $\mathfrak{F}$ *is complemented in* $\mathfrak{F}/_{\infty}^{S}\mathfrak{F} \cap \mathfrak{X}$*, where* $\mathfrak{H}$ *is some* $\mathfrak{X}_{\infty}^{S}$*-critical subformations of* $\mathfrak{F}$*;*

*4) any* $\mathfrak{X}_{\infty}^{S}$*-critical subformation of* $\mathfrak{F}$ *has an* $\mathfrak{X}_{\infty}^{S}$*-complement in* $\mathfrak{F}$*.*

**Corollary 9.** *Let* $\mathfrak{F}$ *and* $\mathfrak{X}$ *be normal hereditary totally saturated formations,* $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$*. Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_{\infty}^{S_n}\mathfrak{F} \cap \mathfrak{X}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{X}) \vee_{\infty}^{S_n} (\vee_{\infty}^{S_n}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{X}_{\infty}^{S_n}$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of the form* $(\mathfrak{F} \cap \mathfrak{X}) \vee_{\infty}^{S_n} \mathfrak{H}$ *in* $\mathfrak{F}$ *is complemented in* $\mathfrak{F}/_{\infty}^{S_n} \mathfrak{F} \cap \mathfrak{X}$*, where* $\mathfrak{H}$ *is some* $\mathfrak{X}_{\infty}^{S_n}$*-critical subformations of* $\mathfrak{F}$*;*

*4) any* $\mathfrak{X}_{\infty}^{S_n}$*-critical subformation of* $\mathfrak{F}$ *has an* $\mathfrak{X}_{\infty}^{S_n}$*-complement in* $\mathfrak{F}$*.*

**Corollary 10.** *Let* $\mathfrak{F}$ *be a non-nilpotent hereditary totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_{\infty}^{S} \mathfrak{F} \cap \mathfrak{N}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty}^{S} (\vee_{\infty}^{S}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{N}_{\infty}^{S}$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of the form* $(\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty}^{S} \mathfrak{H}$ *in* $\mathfrak{F}$ *is complemented in* $\mathfrak{F}/_{\infty}^{S} \mathfrak{F} \cap \mathfrak{N}$*, where* $\mathfrak{H}$ *is some* $\mathfrak{N}_{\infty}^{S}$*-critical subformations of* $\mathfrak{F}$*;*

*4) every subformation of the form* $\mathfrak{N}_p \mathfrak{N}_q$ *in* $\mathfrak{F}$ *has an* $\mathfrak{N}_{\infty}^{S}$*-complement in* $\mathfrak{F}$*.*

**Corollary 11.** *Let* $\mathfrak{F}$ *be a non-nilpotent normal hereditary totally saturated formation. Then the following conditions are equivalent:*

*1) the lattice* $\mathfrak{F}/_{\infty}^{S_n} \mathfrak{F} \cap \mathfrak{N}$ *is Boolean;*

*2)* $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty}^{S_n} (\vee_{\infty}^{S_n}(\mathfrak{H}_i | i \in I))$*, where* $\{\mathfrak{H}_i | i \in I\}$ *is the set of all* $\mathfrak{N}_{\infty}^{S_n}$*-critical subformations of* $\mathfrak{F}$*;*

*3) every subformation of the form* $(\mathfrak{F} \cap \mathfrak{N}) \vee_{\infty}^{S_n} \mathfrak{H}$ *in* $\mathfrak{F}$ *is complemented in* $\mathfrak{F}/_{\infty}^{S_n} \mathfrak{F} \cap \mathfrak{N}$*, where* $\mathfrak{H}$ *is some* $\mathfrak{N}_{\infty}^{S_n}$*-critical subformations of* $\mathfrak{F}$*;*

*4) every subformation of the form* $\mathfrak{N}_p \mathfrak{N}_q$ *in* $\mathfrak{F}$ *has an* $\mathfrak{N}_{\infty}^{S_n}$*-complement in* $\mathfrak{F}$*.*

## References

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